Variable Structure Adaptive Backstepping Control for a Class of Unknown Switched Linear Systems

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Abstract—In this paper, we consider adaptive control of switched linear systems with unknown parameters and unknown switching signals. Variable structure (VS) adaptive backstepping control with tuning function for relative degree one case is discussed. The parametrization of tuning function scheme leads the switched system into a non-switched one with a step change in the input channel and thus the stability properties are derived independent of the switching signal. Moreover, a VS adaptive backstepping control to switched linear systems with relative degree one and two are also proposed. Output error convergence and signal boundedness properties are derived by multiple Lyapunov functions. With the proposed VS adaptive backstepping controller, if the switching signal satisfies the average dwell time conditions, then the output error will converge to a designed residue set and all signals will be bounded. Some simulation results are provided to validate the analysis.

I. INTRODUCTION

In this paper, we consider the control of switched linear systems with unknown parameters and unknown switching signals. We use output feedback adaptive controller to achieve output tracking and signal boundedness. Since stability of switched systems is more involved than non-switched systems, the controller design becomes challenging especially when we have no information about the switching signals. Existing results of adaptive control for time varying systems is one of the directions to our problem. For example, in [1], the authors proposed a robust adaptive law with projection to show signal boundedness of the time varying systems with unknown parameters. However, when considering time varying system, we usually assume that the parameter’s variations should be slow, smooth, or parameterizable. For switched systems, there are inevitably abrupt changes in plant parameters and thus conventional adaptive control techniques can hardly handle such situations so as to meet the stabilization purpose.

Variable structure control has good performance to parameter variations and matched uncertainties. In [2] and [3], we propose a variable structure model reference adaptive control (VS MRAC) scheme for unknown switched linear systems and obtain a sufficient condition for output error convergence and signal boundedness in terms of the dwell time of switching signals. In this paper, we apply the output feedback VS adaptive backstepping controller proposed in [4] to switched systems and show that under the average dwell time condition ([5]), all signals are bounded and output error will converge to zero or to a designed residue set. Multiple Lyapunov functions (MLFs) ([6]) are used as the main tool for stability analysis. In addition, VS adaptive backstepping control with tuning function design proposed in [7] is discussed for the relative degree one case. We shown that output tracking error is converged to zero independent of the switching signals.

This paper is organized as follows. Notations and problem formulation are given in section II. In section III, tuning function design is discussed for relative degree one case. Moreover, a VS adaptive backstepping control of switched linear systems with relative degree one and two are also proposed in section III. Stability analysis using MLFs are given and we show signal boundedness and tracking error convergence for systems with switching signals satisfying some average dwell time condition. In section IV, numerical simulation results are presented and the conclusions are given in section V.

II. PROBLEM STATEMENT

A. Notations

The switched systems we consider in this paper is denoted as
\[ \dot{x} = f_\sigma(t)(x), \quad \sigma(t) \in \mathbb{P} = \{1, 2, \ldots, P\} \]
which consists of subsystems \( f_i(x), \quad i = 1, 2, \ldots, P \), and the piecewise constant switching signal \( \sigma : [0, \infty) \rightarrow \mathbb{P} \). The states are continuous with the jumped vector fields. We denote the switching time instants and indices of active subsystems by
\[ \{(T_1, \sigma(T_1)), \ldots, (T_r, \sigma(T_r)), \ldots\} \]

For convenience, let \( T_0 = t_0 \) be the initial time and \( \sigma(T_0) \) be the initial active subsystem. Throughout this paper, the switching signal is assumed to be right continuous, \( i. e., \lim_{t \rightarrow T^+} \sigma(t) = \sigma(T) \). The switching signals are nonzero, which means that the number of switchings will be finite in any finite time interval, and the switching durations \( \tau_r = T_{r+1} - T_r > 0 \) for all positive integers \( r \). Moreover, the switching signals will not stop switching after finite switch. Let \( N_s(t, t_0) \) denote the number of switchings of the switching signal \( \sigma \) during the time interval \((t_0, t)\). We call the switching signal \( \sigma \) has average dwell time \( \tau_a \) if, given

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any time interval \((t_0, t)\), it satisfies
\[
N_\sigma(t, t_0) - N_0 \leq \frac{t - t_0}{\tau_a}, \tag{3}
\]
where \(N_0\) is a given positive constant. Complete and detailed classifications of switching signals are introduced in [8].

In this paper, \(|x|\) denotes the Euclidean norm of a scalar or a vector \(x \in \mathbb{R}^n\) and \(|\cdot|\) is the induced norm of a matrix. 
\(e_i\) denotes the vector having identity in the \(i\)th component and zeros for others. For example, \(e_1 = [1, 0, ..., 0]^T\).

### B. Problem Statement

Consider the SISO switched linear system in observer canonical form
\[
\begin{align*}
\dot{x}_p &= A_{p\sigma}x_p + b_{p\sigma}u \\
y_p &= c^Tx_p = [1, 0, ..., 0]x_p
\end{align*}
\tag{4}
\]
where \(u \in \mathbb{R}, b_{p\sigma} \in \mathbb{R}^n, x_p \in \mathbb{R}^n\),
\[
A_{p\sigma} = \begin{bmatrix}
-a_1^2 & 1 & 0 & \cdots & 0 \\
-a_2^2 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-a_{n-1}^2 & 0 & 0 & \cdots & 1 \\
-a_n^2 & 0 & 0 & \cdots & 0
\end{bmatrix}, \quad b_{p\sigma} = \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}
\tag{5}
\]
and \(\sigma : [0, \infty) \rightarrow \mathbb{P}\) is the switching signal that governs the switching sequence of the switched system.

This switched system has the corresponding input-output representation
\[
y_p = W_{p\sigma}(t)(s)[u] = Z_{p\sigma}(t)(s)[u], \tag{6}
\]
where \(Z_{p\sigma}(s) = b_m s^n + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0\) and \(R_{p\sigma}(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n\). Here, \(a_i\) denote a parameter of the \(i\)th subsystem. To avoid confusion, we will use \((z_1)^i\) to represent \(z_1\) to the \(i\)th power. The transfer functions \(W_{p\sigma}(s), i \in \mathbb{P}\), are strictly proper and parameters of them are all unknown. Only the input and output can be measured, and we do not know when the plant switchings occur. Note that in (4), we assume that the output channel is fixed. For a switched system with controller canonical form, there will be output “jumps” at switching time instants due to parameter switches. Thus it is difficult to apply output feedback control for switched systems in controller canonical form.

Given a reference signal \(y_m\), the control purpose is to design the output feedback control \(u\) such that all signals in the switched system are bounded and the output of switched plant tracks the reference output as well as possible, i.e., making the output error \(z_1 = y_p - y_m\) as small as possible. For this tracking problem of switched systems, we do not have the information about the time instants at which the switchings occurred nor the knowledge of the active subsystem. The following assumptions are made:

- (A1) For all the transfer functions \(W_{p\sigma}, i \in \mathbb{P}, R_{p\sigma}(s)\) is of order \(n\) and \(Z_{p\sigma}(s)\) is of order \(m\), which means that \(W_{p\sigma}\) has relative degree \(N \equiv n - m\);
- (A2) The first \(N\) derivatives of the reference signal \(y_m(t)\) are known and bounded.
- (A3) All the plants are completely controllable and observable;
- (A4) \(W_{p\sigma}\) is minimum phase for all \(i \in \mathbb{P}\);
- (A5) The signs of \(b_{m}\) are all positive for all \(i \in \mathbb{P}\);
- (A6) All the unknown parameters lie in a known compact set, which implies that the upper bound of unknown parameters is known.

### III. VS ADAPTIVE BACKSTEPPING CONTROL OF SWITCHED LINEAR SYSTEMS

#### A. Tuning Function Design with Relative Degree One

For switched systems (4) with \(N = 1\), the adaptive backstepping control using tuning function is appealing since the unknown parameters are parameterized in the input channel. Details of tuning function design is introduced in [13]. Now we design a VS adaptive backstepping controller using tuning function design to the switched linear systems with relative degree one. First, an adaptive observer using K-filters is used to estimate the state. Rewrite system (4) as
\[
\begin{align*}
\dot{x}_p &= A_{p\sigma}x_p + F(y, u)^{T}\theta^\sigma \\
y &= C^Tx_p,
\end{align*}
\tag{7}
\]
where
\[
A = \begin{bmatrix}
0 & I_{n-1} \\
0 & 0 & \ddots & 0 \\
& \vdots & \ddots & \ddots \\
& & \ddots & \ddots & 0 \\
& & & \ddots & \ddots & 0 \\
& & & & \ddots & \ddots & 0 \\
& & & & & \ddots & \ddots & 0 \\
& & & & & & \ddots & \ddots
\end{bmatrix}, \quad \theta^\sigma = \begin{bmatrix}
b^{\sigma}_m \\
b^{\sigma}_0 \\
b^{\sigma}_1 \\
a^{\sigma}_1 \\
& \ddots \\
& & \ddots \\
& & & \ddots & \ddots \\
& & & & \ddots & \ddots \\
& & & & & \ddots & \ddots \\
& & & & & & \ddots & \ddots \\
& & & & & & & \ddots & \ddots \\
& & & & & & & & \ddots & \ddots \\
& & & & & & & & & \ddots & \ddots \\
& & & & & & & & & & \ddots & \ddots
\end{bmatrix}
\]
and
\[
F(y, u)^T = \begin{bmatrix}
0_{(N-1)\times(m+1)} \\
I_{(m+1)\times(m+1)}^{-1} - I_{n\times n}
\end{bmatrix} \in \mathbb{R}^{n\times(n+m+1)}.
\]

Choose a vector \(k = [k_1, \cdots, k_n]^T\) such that \(A_0 = A - k\sigma I_{T}\) is Hurwitz. Then there exists \(P = P^T > 0\) such that
\[
A_0^TP + PA_0 = -I. \tag{8}
\]
Define the following filters
\[
\begin{align*}
\dot{\xi} &= A_0\xi + ky \\
\dot{\Omega}^T &= A_0\Omega^T + F(y, u)^T,
\end{align*}
\tag{9}
\]
and let the state estimation be
\[
\hat{x} = \xi + \Omega^T\theta^\sigma, \tag{10}
\]
then the state estimation error \(\epsilon = x - \hat{x}\) will vanish exponentially with \(\epsilon = A_0\epsilon\) and
\[
x = \hat{x} + \epsilon = \xi + \Omega^T\theta^\sigma + \epsilon. \tag{11}
\]
From the special structure of \( F(y, u)^T \), we can denote \( \Omega^T = [v_m, \ldots, v_1, v_0, \Xi] \), where \( v_j \in \mathbb{R}^n, j = 0, 1, \ldots \) are vectors that can be obtained from a filter of input \( u \), and \( \Xi \) can be obtained from a filter of output \( y \). The implementation of K-filters are summarized in the following ([13]):

\[
\begin{align*}
\dot{\eta} &= A_0 \eta + e_n y \\
\lambda &= A_0 \lambda + e_n u \\
\Omega^T &= [v_m, \ldots, v_0, \Xi] \\
v_j &= (A_0)^j \lambda, \ j = 0, \ldots, m \\
\Xi &= -(A_0)^n \eta, \ A_0 \eta, \eta \\
\xi &= -(A_0)^n \eta
\end{align*}
\]

Suppose that relative degree of \( W_{pi} \) is one, \( \forall \in \mathbb{P} \), that is, \( N = 1 \). Then,

\[
\dot{x}_1 = x_2 - a_1^T y + b_0^T u = x_2 - ye_1^T a^\sigma + b_m u.
\]

By (11), \( x_2 \) can be represented by

\[
x_2 = \xi_2 + [v_{m,2}, v_{m-1,2}, \ldots, v_{0,2}, \Xi_{(2)}] \theta^\sigma + \epsilon_2.
\]

Here the subindex 2 in the right hand side denotes the second component of a column vector. For example, \( v_{m,2} \) denotes the second component of \( v_m \).

Thus, we have

\[
\begin{align*}
\dot{x}_1 &= \xi_2 + [v_{m,2}, v_{m-1,2}, \ldots, v_{0,2}, \Xi_{(2)} - ye_1^T a^\sigma] \theta^\sigma + \epsilon_2 + b_m^T u \\
&= \xi_2 + \omega^T \theta^\sigma + \epsilon_2 + b_m^T u,
\end{align*}
\]

where \( \omega = [v_{m,2}, v_{m-1,2}, \ldots, v_{0,2}, \Xi_{(2)} - ye_1^T a^\sigma] \).

Define the output error \( \tilde{z}_1 = y_p - y_m \), then

\[
\dot{\tilde{z}_1} = \xi_2 + \omega^T \theta^\sigma + \epsilon_2 + b_m^T u - \tilde{y}_m.
\]

Design the control law as

\[
u = \hat{\gamma} \hat{\alpha}_1,
\]

where \( \hat{\gamma} \) is the estimation of \( \gamma^\sigma = 1/(b_m^\sigma) \) and

\[
\hat{\alpha}_1 = -c_1 z_1 - d_1 z_1 - \xi_2 - \omega^T \hat{\theta} + \tilde{y}_m,
\]

where \( \hat{\omega} = [|\omega_{1}|, \ldots, |\omega_{n}|]^T \). The estimate parameters are decided by the variable structure adaptive laws similar to that in [7]:

\[
\begin{align*}
\hat{\theta}_i &= sgn(z_1) \hat{\theta}_i, \quad \hat{\theta}_i > |\theta_i^\sigma|, \\
\hat{\gamma} &= -\gamma sgn(b_m^\sigma) \gamma sgn(\hat{\alpha}_1 z_1), \quad \hat{\gamma} > 1/|b_m^\sigma|,
\end{align*}
\]

for \( i = 1, 2, \ldots, n + m + 1 \), and \( \sigma \in \mathbb{P} \). Here \( \omega_i \) stands for the \( i \)th component of the regressor vector \( \omega \). Now we can discuss the stability properties of the closed-loop system. Consider a Lyapunov function \( V = \frac{1}{2} z_1^2 + \frac{1}{2\alpha} \epsilon^T P \epsilon \), then the time derivative of \( V \) is

\[
\dot{V} = -c_1 z_1^2 + \omega^T \hat{\theta} z_1 - b_m^\sigma \hat{\gamma} \hat{\alpha}_1 z_1 - \frac{1}{4d_1} \epsilon^T \epsilon
\]

\[
- d_1 z_2^2 + z_1 \epsilon_2 - \frac{1}{4d_1} \epsilon^T \epsilon
\]

\[
\leq -c_1 z_1^2 + \omega^T \hat{\theta} z_1 - b_m^\sigma \hat{\gamma} \hat{\alpha}_1 z_1 - \frac{1}{4d_1} \epsilon^T \epsilon
\]

\[
= -c_1 z_1^2 + \sum_{i=1}^{2n} (\theta_i^\sigma \omega_{zi} - \hat{\theta}_i |\omega_{zi}|)
\]

\[
- (b_m^\sigma \hat{\gamma} \hat{\alpha}_1 z_1 + |b_m^\sigma| \hat{\gamma} |\hat{\alpha}_1 z_1|) - \frac{1}{4d_1} \epsilon^T \epsilon
\]

\[
\leq -c_1 z_1^2 - \frac{1}{4d_1} \epsilon^T \epsilon
\]

(19)

Note that \( \hat{\gamma} = \gamma - \hat{\gamma}, \hat{\theta} = \theta - \hat{\theta} \). We use (15) and

\[
\dot{z}_1 = \xi_2 + \omega^T \theta^\sigma + \epsilon_2 - b_m^\sigma \hat{\gamma} \hat{\alpha}_1 + \tilde{y}_m
\]

to obtain the first equality in (19). The first inequality in (19) is obtained by using the completion of square and thus

\[
-d_1 z_1^2 + z_1 \epsilon_2 - \frac{1}{4d_1} \epsilon^T \epsilon = -d_1 (z_1 - \frac{1}{2d_1} \epsilon_2)^2 - \frac{1}{4d_1} (\epsilon_1^2 + \epsilon_2^2 + \cdots + \epsilon_n^2) \leq 0.
\]

From (19) we know that \( z_1 \) and \( \epsilon \) converge to zero exponentially and \( x_1 = y \) is bounded. The boundedness of the output is decided by the variable structure adaptive law. The parametrization in tuning function scheme leads to the proof of Theorem 10.6 in [13].

Now we conclude the result for tuning function design:

**Theorem 1.** For switched system (4) which satisfy (A1)-(A6) with \( N = 1 \) and using the VS adaptive tuning function designed controller (16), all signals in the closed loop system are bounded and the tracking error \( \tilde{z}_1 \) will converge to zero independent of the switching signals.

**Remark 1.** For relative degree one case, given any non-zero switching signals \( \sigma \), the output will converge to the reference output exponentially and all signals in the switched system are bounded. This is also a good property is attributed to the parametrization of the K-filter and the variable structure adaptive law. The parametrization in tuning function scheme leads the switched system to a non-switched one with a step change in the input channel and thus the stability properties are derived independent of the switching signal. In [7], the output feedback VS adaptive backstepping control with tuning function design is also proposed for non-switched systems. However, to the authors’ best knowledge, the VS adaptive backstepping design using tuning function for \( N \geq 2 \) case is still lacking in the literatures.

**Remark 2.** In [14], the adaptive control problem of switched systems is also considered and the authors propose a tuning function design controller with a modified leakage type robust adaptive law instead of the variable structure design here. The advantage of robust adaptive law is that it does not need the upper bound of the unknown parameters, which is necessary in our VS design. However, the robust adaptive law approach can only achieve error boundedness with the following from [14]:

\[
\|z_1(t)\| \leq c_1 \|x(t_0)\| e^{-\alpha(t-t_0)} + c_2 \int_{t_0}^{t} e^{-\alpha(t-\tau)} \|\phi(\tau)\| d\tau,
\]

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where φ is a signal related to the adaptive gains and a pre-specified estimate parameter \( a^* \). The VS adaptive design here can force the error converge to a residual set whose size can be specified.

B. VS Design of Switched Linear Systems with Relative Degree One

Now we will employ the transformation proposed in [10] to design another VS adaptive backstepping controller. Let \( z_1 = y_p - y_m, z_2 = x_2, \ldots, z_n = x_n \), then we can represent (4) as

\[
\dot{z} = Ax + b_{pr}u - a^*y_m - e_1 \dot{y}_m - a^*z_1, \quad (20)
\]

where

\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 
\end{bmatrix}.
\]

Define a stable filter \( \lambda(s) = (s + \lambda_1) \cdots (s + \lambda_{N-1}) \) and let \( \eta_1 = \frac{1}{\lambda(s)}u \). Then we can rewrite (20) as

\[
\dot{z} = Ax - a^*y_m - e_1 \dot{y}_m - a^*z_1 + d^*\eta_1, \quad (21)
\]

where \( d^* = [d_1^*, \ldots, d_N^*]^T \) is derived from

\[
(b_m^* s^n + \cdots + b_0^*)\lambda(s) = d_1^* s^{n-1} + \cdots + d_n^*.
\]

Note that \( d_i^* = b_m^* \) and \( d_i^* s^{n-1} + \cdots + d_n^* \) are Hurwitz polynomials for all \( i \in \mathbb{P} \).

Define \( \zeta = [\zeta_1, \ldots, \zeta_n]^T \) with \( \zeta_1 = z_2 - \frac{a_2^*}{a_1^*}z_1, \quad \zeta_2 = z_3 - \frac{a_3^*}{a_2^*}z_1, \ldots, \quad \zeta_{N-1} = z_n - \frac{a_n^*}{a_{n-1}^*}z_1 \), then we can transform the system to the following backstepping based error dynamics:

\[
\dot{z}_1 = \zeta_1 + \frac{a_2^*}{a_1^*}z_1 - a_1^*y_m - \dot{y}_m - a_1^*z_1 + d_1^*\eta_1 \quad (22),
\]

\[
\dot{\zeta} = \Lambda_\sigma \zeta + (\beta_\sigma + \Psi_\sigma)z_1 + \Psi_{\sigma_{r}}(y_m, \dot{y}_m, d^*) \quad (23)
\]

where

\[
\Lambda_\sigma = \begin{bmatrix}
-\frac{d_2^*}{d_1^*} & 1 & \cdots & 0 \\
-\frac{d_3^*}{d_2^*} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{d_{n+1}^*}{d_n^*} & 0 & \cdots & 0 \\
-\frac{d_n^*}{d_n^*} & 0 & \cdots & 0
\end{bmatrix}, \quad \beta_\sigma = \begin{bmatrix}
\frac{d_1^*}{d_1^*}a_1^* - a_2^*
\frac{d_2^*}{d_1^*}a_1^* - a_3^*
\vdots
\frac{d_n^*}{d_{n-1}^*}a_{n-1}^* - a_n^*
\end{bmatrix},
\]

\[
\Psi_\sigma = \begin{bmatrix}
\frac{d_1^*}{d_1^*}(\dot{y}_m + a_1^*y_m) - a_2^*y_m \\
\vdots \\
\frac{d_n^*}{d_n^*}(\dot{y}_m + a_1^*y_m) - a_n^*y_m
\end{bmatrix}, \quad \text{and}
\]

\[
\Psi_{\sigma_{r}} = \begin{bmatrix}
\frac{d_1^*}{d_1^*}(\dot{y}_m + a_1^*y_m) - a_2^*y_m \\
\vdots \\
\frac{d_n^*}{d_n^*}(\dot{y}_m + a_1^*y_m) - a_n^*y_m
\end{bmatrix}.
\]

Since the upper bound of parameters are known, we can assume that upper bound of \( \Lambda_\sigma, \beta_\sigma, \Psi_\sigma \) and \( \Psi_{\sigma_{r}} \) are known.

The control \( u \) is defined by

\[
u = (s + \lambda_1)(s + \lambda_2) \cdots (s + \lambda_{N-1})\eta_1,
\]

and we will design \( u \) from the virtual control \( \eta_1 \) using backstepping. First we consider \( N = 1 \) case, then \( u = \eta_1 \).

Define an estimate model for \( \zeta \) by \( \hat{\zeta} = \Lambda_\sigma \hat{\zeta} + \Psi_{\sigma_{r}}\), and let \( \hat{\zeta} = \zeta - \hat{\zeta} \), then

\[
\hat{\zeta} = \Lambda_\sigma \hat{\zeta} + (\beta_\sigma + \Psi_\sigma)z_1.
\]

Now we employ the idea from [4], [9] and [10] to design the controller.

Consider the multiple Lyapunov functions \( V_{1i} \) for \( i \in \mathbb{P} \). Let \( V_{1i} = \frac{1}{2}(z_i^2 + \zeta_i^2) + \zeta_i^T P_i \zeta_i \) where \( P_i > 0 \) satisfies the Lyapunov equation

\[
A_i^T P_i + P_i A_i = -Q = -(q_0 + \rho \epsilon_1)I < 0, \quad (24)
\]

for given positive constants \( q_0, \rho \) and \( \epsilon_1 \). Then

\[
V_{1i} = z_1 z_1 + (q_0 + \rho \epsilon_1)|\zeta|^2 + 2 \zeta_i^T P_i (\beta_i + \Psi_i) z_1 \quad (25)
\]

From (22), we can design \( u = \eta_1^* \) with

\[
\eta_1^* = -sgn(d_i^*)[v_{11}(t)z_1 + v_{11}(z_1)v_{12}(t)], \quad (26)
\]

where

\[
v_{11}(t) \geq \frac{1}{|d_i^*|} \left( \frac{|d_i^*|^2}{|d_1^*|^2} + \frac{1}{2} + \frac{\|P_i\|(||\beta_i| + |\Psi_i||)^2}{q_0} + \rho \epsilon_1 \right),
\]

and

\[
v_{12}(t) \geq \frac{1}{|d_i^*|} \left( \frac{|\zeta|^2}{|d_1^*|^2} + (q_0 + \rho \epsilon_1)|\zeta|^2 \right), \quad \text{for all } \sigma(t) \in \mathbb{P}.
\]

Note that the upper bound of \( |\zeta|^2 \) can be obtained by comparison lemma. The value of \( ||P_i|| \) can be specified to satisfy (24) accompanied with the different values of \( q_0, \epsilon_1, \) or \( \rho \). Thus,

\[
\dot{V}_{1i} \leq z_1 \left[ |\dot{\zeta}| + |a_1^*z_1| + |\dot{y}_m| + |a_1^*y_m| - |d_1^*|v_{12}^2 \right] + \left( |d_1^*| - |d_i^*|v_{11}(z_1)^2 \right) - (q_0 + \rho \epsilon_1)|\zeta|^2 \quad (27)
\]

\[
\leq \left( |d_1^*| - |d_i^*|v_{11}(z_1)^2 \right) + \frac{1}{2} + \frac{\|P_i\|(||\beta_i| + |\Psi_i||)^2}{q_0}(z_1^2) \quad (27)
\]

\[
\leq \frac{1}{2} + \frac{\|P_i\|(||\beta_i| + |\Psi_i||)^2}{q_0} z_1 - \sqrt{q_0} |\zeta|^2 - \rho \epsilon_1 |\zeta|^2 \quad (27)
\]

\[
\leq -\rho \epsilon_2 (z_1^2 + |\zeta|^2) \quad (27)
\]

\[
\leq -mV_{1i},
\]

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where

\[ 0 < m \leq \frac{\rho \varepsilon_2}{\lambda_{\max}(P_i)}, \tag{28} \]

for some \( \varepsilon_2 > 0 \).

From inequality (27) and the fact that \( \forall i, j, \in \mathbb{P}, V_{1i} \leq MV_{ij}, \) where

\[ M \geq \max_{i \in \mathbb{P}} \frac{\lambda_{\max}(P_i)}{\min_{i \in \mathbb{P}} \lambda_{\min}(P_j)} \geq 1, \tag{29} \]

However, there are state jumps at \( T_k \) (see definition of \( \zeta \)), thus we need to guarantee boundedness of the state in addition to the dwell time condition. We know that \( \zeta(T_k) \leq \zeta(T_{k+1}) + K \zeta_1 \), where \( K \) is a constant related to \( \left( ||\frac{d}{dt} x ||, \frac{d}{dt} \lambda \right) \). Due to the state jump, there will be an additional term in the stability analysis of each interval and thus we shall employ the results of input-to-state stability (ISS) of switched systems (Theorem 3.1, [11]) to conclude that if the average dwell time \( \tau_0 \geq \frac{\ln M}{m} \), then all signals are bounded and output error \( z_1 \) converges to zero. For detail, see the stability analysis in [2], [5], and [11].

For relative degree two case, \( u = (s + \lambda_1) \eta_1 \) and we must use a smooth \( \eta_1 \) to construct \( u \). Let

\[ \eta_1 = \begin{cases} \eta_1^* & \text{if } |z_1| > \Delta, \\ \eta_1^s & \text{otherwise}, \end{cases} \tag{30} \]

where \( \Delta \) is any given positive constant and \( \eta_1^s \) is a function to make \( \eta_1 \) smooth. Before we design the controller for relative degree two, we discuss the properties of the smooth controller for relative degree one case.

**Theorem 2.** For \( N = 1 \), if the switched systems (4) which satisfy (A1)-(A6) have average dwell time \( \tau_0 \geq \frac{\ln M}{m} \) for some positive constant \( M \) and \( m \), then with the smooth controller (30), all signals in the closed loop system are bounded and the tracking error \( z_1 \) will converge to the designed set \([-\Delta, \Delta] \).

**Proof.** We discuss the stability property with \( |z_1| > \Delta \) and \( |z_1| \leq \Delta \) case. If \( |z_1| > \Delta \), then from analysis above, if the average dwell time condition is satisfied, we know that \( z_1 \) will decrease until \( |z_1| \leq \Delta \) and all signals are bounded. We need to show signal boundedness when \( |z_1| \leq \Delta \). For \( |z_1| \leq \Delta \), we consider the Lyapunov function candidate \( V_{11\Delta} = \zeta^T P_i \zeta \). Then we know that \( V_{11\Delta} \leq M' V_{11\Delta} \), where

\[ M' \geq \max_{i \in \mathbb{P}} \frac{\lambda_{\max}(P_i)}{\min_{i \in \mathbb{P}} \lambda_{\min}(P_j)}, \]

Form analysis in subsection III. A, we have

\[ \dot{V}_{11\Delta} \leq -\rho \varepsilon_1 |\zeta|^2 + \frac{\frac{1}{2} + ||P_i||(|\beta| + |\Psi_i|)^2}{q_0} (z_1)^2 \]

\[ \leq -m' V_{11\Delta} + C_1, \tag{31} \]

where \( C_1 \) is a positive constant and

\[ 0 < m \leq m' \leq \frac{\rho \varepsilon_1}{\lambda_{\max}(P_i)}. \]

Thus from the results of ISS of switched systems, the system is bounded when \( |z_1| \leq |\Delta| \).

Hence we can conclude that if the average dwell time \( \tau_0 \geq \frac{\ln M}{m} \geq \frac{\ln M'}{m'} \), all signals are bounded and \( z_1 \) will converge to the designed set \([-\Delta, \Delta] \).

Now we design the controller for \( N = 2 \). We want to design \( u \) such that \( \eta_1 \) is close to \( \eta_{1d} \). Let \( \tilde{\eta}_1 = \eta_{1d} - \eta_1 \), then

\[ \dot{\tilde{\eta}}_1 = \frac{\partial \eta_{1d}}{\partial z_1} \dot{z}_1 - \lambda_1 \tilde{\eta}_1 + \lambda_1 \eta_{1d} - u. \tag{32} \]

Let the multiple Lyapunov function

\[ V_{21} = \frac{1}{2} \dot{z}_1^2 + \zeta^T P_i \zeta + (\tilde{\eta}_1)^2, \quad \forall i \in \mathbb{P}. \]

Then, using similar idea as the design in \( N = 1 \) case, we can design

\[ u^* = \eta_2^* = \lambda_1 \eta_{1d} + v_{21}(t) \tilde{\eta}_1 + v_{22}(t) sgn(\tilde{\eta}_1), \tag{33} \]

with properly chosen \( v_{21}(t) \) and \( v_{22}(t) \) such that when \( |z_1| > \Delta \),

\[ \dot{V}_{21} \leq -(\rho - 1)\varepsilon_2 (|z_1|^2 + |\zeta|^2) - \lambda_1 (\tilde{\eta}_1)^2 \leq -m_2 V_{21}, \tag{34} \]

for some positive \( m_2 \). When \( |z_1| \leq \Delta \), the Lyapunov function candidate \( V_{21\Delta} = \zeta^T P_i \zeta + (\tilde{\eta}_1)^2 \) satisfies

\[ \dot{V}_{21\Delta} \leq -m_2 V_{21\Delta} + C_2. \tag{35} \]

Thus, using similar argument in the proof of Theorem 2, we can obtain the following results:

**Theorem 3.** For \( N = 2 \), if the switched systems (4) which satisfy (A1)-(A6) have average dwell time \( \tau_0 \geq \frac{\ln M}{m} \) for some positive constant \( M_2 \) and \( m_2 \), then with the controller (33), all signals in the closed loop system are bounded and the tracking error \( z_1 \) will converge to the designed set \([-\Delta, \Delta] \).

**Remark 3.** In [2] and [3], we have proposed a VS MRAC controller for switched linear systems. The main distinction for stability properties of these two types of controller is that using backstepping design, we have more freedom to adjust the convergence rate which in turn affects the average dwell time \( \tau_0 \). This can be seen from (24), in which we can choose \( Q \) arbitrarily for every \( P_i \), and thus the convergence rate is affected. Although \( P_i \) is affected by \( \Lambda_i \), which is not a designed term, the freedom of choosing \( Q \) arbitrarily still helps us to improve the average dwell time condition. In VS MRAC design, we use the SPR condition and MKY lemma ([12]) in stability analysis, and thus the convergence rate depends strictly on \( \Lambda_i, P_i, \) and \( b_{i\sigma} \). Hence, the average dwell time condition derived in VS MRAC design may be more conservative than VS backstepping design. The price of VS adaptive backstepping design is the additional complexity of the nonlinear controller.

**IV. SIMULATION RESULTS**

In this section, we present some simulation results of the VS adaptive backstepping controller to switched systems. We
consider the relative degree one case. Let \( P = \{1, 2\} \) and given the second order plants

\[
A_{p1} = \begin{bmatrix} 5 & 1 \\ -6 & 0 \end{bmatrix}, \quad b_{p1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\]

\[
A_{p2} = \begin{bmatrix} 11 & 0 \\ -10 & 0 \end{bmatrix}, \quad b_{p2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]

The initial condition is \( x_0 = [2, 0]^T \). The reference output \( y_m(t) = 2 \sin 5t \). For the VS adaptive backstepping control, choose \( \Delta = 0.5, \Delta_{\eta_1} = 0.5, \epsilon_1 = 1, q_0 = 1 \). From (30) and (26), design \( \eta_{1d} = -[(50z_1) + (60 + 15|z_1|)]\tanh(2\Delta) \). The performance of the VS adaptive backstepping control is shown in Fig. 1. We can see that the switching signal does not have much effects on the tracking error compared with the non-switched system. In this example, the matrices \( \Lambda_i \) are \( 1 \times 1 \) and thus there exists common Lyapunov functions for this case. For the VS adaptive controller with tuning function design, choose \( c_1 = d_1 = 10 \) and \( k = [2, 2]^T \). The VS adaptive law is chosen as \( \dot{\theta} = [1, 3, 15, 15]^T \) and \( \dot{\gamma} = 2 \). In Fig. 2, tracking error is shown and the switching signal is the same as that in Fig. 1. The error converge to zero exponentially with some transient response.

![Fig. 1. Tracking error of non-switched and switched systems using smooth VS adaptive controller](image1)

![Fig. 2. Tracking error of non-switched and switched systems using VS tuning function design](image2)

V. CONCLUSIONS

In this paper, the variable structure adaptive backstepping controller is applied to unknown switched linear systems. Error convergence and signal boundedness are shown by conditions in terms of the average dwell time of the switching signals. The tuning function design is discussed for relative degree one case, and the output error convergence is independent of the switching signals. The VS adaptive backstepping control here employs the smooth approximation to solve the differentiability problem which is usually occurred in VS design. Hence the output error converges to a designed residue set rather than zero, if the average dwell time conditions hold. Moreover, the VS adaptive backstepping design has more freedom on the dwell time condition than VS MRAC design since the SPR condition and MKY lemma are not used in the stability analysis.

Tuning function design has appealing feature for switched systems, however, the VS design for cases with relative degree greater than one is still lacking. In the future works, we will investigate the VS based tuning function design for higher relative degree cases. Relaxing the assumptions as what is already done in conventional adaptive control theory for non-switched systems is another topic of our future research.

REFERENCES


