Passivity-Based Model Reference Robust Control for a Class of Nonlinear Systems with Input and State Measurement Delays

Erick J. Rodríguez-Seda, Pedro O. López-Montesinos, Dušan M. Stipanović, and Mark W. Spong

Abstract—This paper presents a model reference robust control framework that guarantees asymptotic stability of a class of nonlinear systems with arbitrarily large constant input and state measurement delays as well as uncertain system parameters. The proposed control framework combines the use of a modified wave scattering transformation to achieve overall stability and to enforce state convergence of the time delay nonlinear system to the desired state independently of unknown initial conditions, transmission losses, and computational errors. A numerical example illustrates the effectiveness of the proposed controller.

I. INTRODUCTION

Time delays may naturally appear in state and output feedback control systems, a phenomenon well known to cause performance degradation and, in many cases, instability [1]. Common reasons to the origin and development of time delays in a control loop include: slow process reactions to control stimuli [2], [3], appreciable controller-to-plant separation distances [4], complex communication protocols and network topologies [5], limited update rates of control algorithms [6], and large sampling times of sensing mechanisms [7], [8]. For instance, the transmission of control signals to remote systems via Internet may lead to relatively large time-varying delayed control inputs due to possible network congestion and the physical separation between controller and plant [9], [10]. Similarly, in the localization process of indoor-operated unmanned vehicles, vision-based tracking systems based on complex image recognition techniques may induce control loop latencies up to 1 second [11]. These latencies may further increase due to the aggregation of propagation delays in the transmission channels if the localization process is carried off-board [12]. In all the above scenarios, it is critical to design control laws conformed to time delay models that guarantee stability and good performance of the entire system.

A. Related Work

The design of control laws and stability assessment criteria for time delay systems has sparked considerable research interest over the last several decades, leading to a rich set of documented results for systems with linear dynamics [1], [8], [13]. Yet, we can assert that: 1) in contrast to the linear case, work on robust stability and control of nonlinear systems with input and state measurement delays has received less attention; and 2) the limited number of published control frameworks for nonlinear systems with delayed control are predominantly constricted to plants with linear norm-bounded nonlinearities, known constant delays, or well-known dynamics. Some recent examples include a model reference adaptive control that enforces stability and state tracking of a known-structured nonlinear system with bounded nonlinearities and known input delays [14]; a backstepping design technique that guarantees stability of a class of well-defined nonlinear systems with arbitrarily large, known, constant delays [15]; and a delayed output feedback control that achieves asymptotic stability for a family of systems with linear norm-bounded-in-state and in-control nonlinearities [16]. While all these examples solve the stability issue induced by delays for specific sets of systems, the control ideas presented therein are not easily adapted to more general classes of uncertain nonlinear systems.

Recently, the control of a larger class of nonlinear plants with delayed output feedback was addressed through the use of the scattering transformation, well known in the teleoperation field [17]. In references [18]–[20], it was shown that the utilization of the scattering transformation to couple the plant with the controller can stabilize passive and non-passive, static-output-feedback-stabilizable plants independently of any constant delays and uncertain system parameters. Later, the stability of interconnected systems satisfying an inequality condition of small $L_2$-gain (or an akin output strictly passive inequality [21], [22]) was established through the use of the scattering transformation and extended to time-varying delays [23]. Similarly, the use of the scattering transformation to stabilize passive systems has been explored for losses in the controller-to-plant communication channel [24]. Although the aforementioned efforts are applicable to a broad class of nonlinear systems with delays; they either address systems with no measurement delays, since instantaneous local state values are required to construct the scattering transformation outputs [18]–[20], [22], [24], or their avail is conditional to a small $L_2$-gain requirement [21], [23]. In this paper, we aim to extend similar results on stability to a class of exponentially stable nonlinear systems with delays in the local control loop.

B. Contributions

Following the research line of [18]–[24], we propose the design of a Model Reference Robust Control (MRRC)
framework that combines the use of the wave-based scattering transformation to guarantee asymptotic stability of a class of nonlinear systems with dynamic uncertainties and arbitrarily large input and state measurement constant delays. The proposed control law assumes that the unforced (i.e., zero input control) system is exponentially stable\(^1\) in order to establish delay-independent stability\(^2\) of the controlled system. The design of the controller is constituted by two parts: a linear reference model and a scattering transformation block. The first is designed according to the desired input-to-output properties that the delayed system must mimic, while the latter is used to stabilize the delayed coupling between plant and controller. In addition, the outputs of the scattering transformation are passively modified to enable explicit full state tracking between controller (i.e., reference model) and plant independently of dissimilar and unknown initial conditions as well as losses in the transmission lines; a recurring problem with scattering transformation based techniques. The overall framework is then validated via a numerical example.

To the best of our knowledge, nonlinear systems with explicit state measurement delays have not been addressed from a passivity-based scattering transformation perspective.

II. Preliminaries

We consider a class of uncertain, affine-in-control nonlinear systems with input and state measurement delays described by

\[
\begin{align*}
\dot{x}(t) &= f(x(t), \dot{x}(t)) + G(x(t), \dot{x}(t))u(t) \\
y(t) &= x(t)
\end{align*}
\]  

(1)

where \(x(t), \dot{x}(t) \in \mathbb{R}^n\) are the state vectors (e.g., positions and velocities), \(y(t) \in \mathbb{R}^n\) is the output vector, \(f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n\) is a continuous function with \(f(x(t), 0) = 0\), and \(G : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n\) is, without loss of generality, a continuous, positive-definite, symmetric matrix. The control input \(u(t) \in \mathbb{R}^n\) is a delayed output/state feedback law depending on \(x(t-T_1-T_2)\) and \(\dot{x}(t-T_1-T_2)\), where \(T_1 \geq 0\) and \(T_2 \geq 0\) correspond to the measurement and plant-to-controller delay and the controller-to-plant delay, respectively. A pictorial representation of the system and controller is given in Fig. 1.

In what follows, we will assume that the time delay nonlinear system (1) satisfies the following stability property. For simplicity, we will omit time dependence of signals when necessary to avoid cluttering of equations.

**Assumption 2.1:** There exists a radially-unbounded storage function \(V(\dot{x}) : \mathbb{R}^n \rightarrow [0, \infty)\), a positive constant \(\rho\), and a positive-definite, constant matrix \(\Lambda \in \mathbb{R}^{n \times n}\) such that

\[
V(\dot{x}) \leq -\rho x^T \dot{x} + x^T \Lambda u.
\]

(2)

\(^1\)For time delay control frameworks based on exponential stability of the unforced system or similar assumptions, see [1], [14], [15].

\(^2\)The design of the control gains is independent of the delay. However, if the delayed input to the system is uncertain or can not be recovered from the output, the total round trip delay must be known. More details on this are provided in Section VII-A.

Although the previous assumption may seem slightly restrictive, it is a fact that many linear and nonlinear systems exhibit such property. For instance, the existence of a radially-unbounded Lyapunov function \(V\) satisfying (2) is always guaranteed if (1) is either an exponentially stable linear system or a dissipative nonlinear Lagrangian system (under some mild conditions). Moreover, as the next lemma will show, exponentially stable, affine-in-control systems independent of \(x(t)\) conform to Assumption 2.1, at least, locally.

**Lemma 2.1:** Consider the nonlinear affine-in-control system in (1). Suppose that \(f = f(\dot{x}(t))\) is a continuously differentiable function and \(G\) is, without loss of generality, a positive-definite, constant, diagonal matrix. Furthermore, assume that the origin \(f(0) = 0\) for the unforced system is locally exponentially stable for all \(\|\dot{x}\| \leq \zeta\), i.e., for \(u(t) = 0, \exists k, \lambda > 0\) such that

\[
\|\dot{x}(t)\| \leq k \|\dot{x}(0)\| e^{-\lambda t}, \quad \forall t \geq 0, \|\dot{x}(0)\| \leq \zeta.
\]

Then, there exists a positive constant \(\zeta < \zeta\) such that for all \(\|\dot{x}\| < \zeta\) Assumption 2.1 holds.

**Proof:** Let \(f(\dot{x})\) be a continuous differentiable function, \(G\) a diagonal, positive-definite matrix, and let \(f(0)\) be exponentially stable for all \(\|\dot{x}\| \leq \zeta\) and \(u(t) \equiv 0\). Then, the last assumption implies that the Jacobian matrix

\[
A = \frac{\partial f}{\partial \dot{x}}|_{x=0}
\]

is Hurwitz. Therefore, for any positive-definite, symmetric matrix \(Q\), there exists an unique positive-definite, symmetric matrix \(P\) that satisfies the following Lyapunov equation [25]

\[
PA + A^TP = -Q.
\]

(5)

Then, using \(V(\dot{x}) = \dot{x}^TP\dot{x}\) as our Lyapunov function and taking its time derivative along the trajectories of the system yields

\[
\dot{V}(\dot{x}) = \dot{x}^TPf(\dot{x}) + f^T(\dot{x})P\dot{x} + 2\dot{x}^TPG\dot{u} = \dot{x}^T(PA + A^TP)\dot{x} + 2\dot{x}^TPG\dot{u}
\]

\[
= -\dot{x}^TQ\dot{x} + 2\dot{x}^TPf(\dot{x}) + 2\dot{x}^TPG\dot{u}
\]

(6)

where \(f(\dot{x}) = f(\dot{x}) - A\dot{x}\). Now, applying the Mean Value Theorem [26], we have that for any \(\dot{x}\), there is a constant \(z\) for which

\[
f(\dot{x}) = f(0) + \frac{\partial f}{\partial \dot{x}}(z)\dot{x} = \frac{\partial f}{\partial \dot{x}}(z)\dot{x}.
\]

(7)

Therefore, \(h(\dot{x}) = \frac{\partial f}{\partial \dot{x}}(z) - A\dot{x}\). Moreover, since \(\frac{\partial f}{\partial \dot{x}}\) is continuous, we have that for any \(\epsilon > 0\), \(\exists \zeta < \zeta\) such that \(\|h(\dot{x})\| < \epsilon \|\dot{x}\|\) for all \(\|\dot{x}\| < \zeta\). Consequently, choosing \(\zeta\) such that \(\epsilon < \rho/(2\gamma)\), where \(\gamma\) and \(\rho\) are the smallest and largest eigenvalues of \(Q\) and \(P\), respectively, yields that

\[
\dot{V}(\dot{x}) \leq -\frac{\rho}{2} \dot{x}^T \dot{x} + 2\epsilon \dot{x}^TP\dot{x} + 2\dot{x}^T P G u, \quad \forall \|\dot{x}\| \leq \zeta
\]

(8)

which confirms the claim for \(\rho = \gamma - 2\epsilon \gamma > 0\) and \(\Lambda = (1/2)PG > 0\), since \(G\) is positive-definite and diagonal. ■
Controller System

\[ y(t) = y(t - T_2) \]
\[ u(t) = y(t) \]
\[ r_m(t) = y(t - T_1) \]

Fig. 1. Nonlinear plant and controller with input and state measurement delays. The signals \( r_m(t), y(t), \) and \( y(t) \) represent the input and output of the controller and the output of the system, respectively.

**Remark 2.1:** The maximum bound on the stability region \( \zeta \) for (8) is attainable when \( Q = I \), where \( I \) is the \( n \times n \) identity matrix [27].

**Remark 2.2:** Analogous results to Lemma 2.1 can be easily derived for exponentially stable systems with \( f = f(x(t), x(t)) \) and \( G = G(x(t)) \). For such systems we can always construct a positive-definite Lyapunov function \( V = x^T P(x)x \) which locally satisfies (2) with some matrix \( \Lambda(x) = (1/2)P(x)G(x) \). Then, the problem evolves to find a positive-definite, symmetric matrix \( P(x) = HGG^{-1}(x) \) that satisfies its Lyapunov equation for some positive-definite, constant matrix \( H \) and some positive-definite, symmetric matrix \( G(x) \).

In addition, Assumption 2.1 is directly linked to the concept of output strict passivity. In fact, the reader can easily verify that Assumption 2.1 holds whenever (1) is output strictly passive [25], where the passivity property is defined as follows.

**Definition 2.1:** [28] A system with input \( u \) and output \( y \) is said to be passive if

\[ \int_0^t u^T y d\theta \geq -\kappa + \nu \int_0^t u^T u d\theta + \rho \int_0^t y^T y d\theta \quad (9) \]

for some nonnegative constants \( \kappa, \nu, \) and \( \rho \). Moreover, it is said to be lossless if equality persists and \( \nu = \rho = 0 \), input strictly passive if \( \nu > 0 \), and output strictly passive if \( \rho > 0 \).

### III. Control Framework

In this section, we introduce a passivity-based MRRC framework for a class of time delay nonlinear systems satisfying Assumption 2.1. The control framework is comprised of two elements: the reference model and the wave-based scattering transformation block. The reference model is designed according to some ideal input-to-output (or equivalently, input-to-state) properties that we would like the time delay nonlinear plant to mimic. The scattering transformation is designed such that the delayed transmission lines between controller and plant are passified. The passivation of the transmission lines will then be exploited to guarantee delay-independent stability of the overall system.

**A. Reference Model**

We construct, for simplicity, an asymptotically stable linear reference model as

\[ \dot{x}_m(t) = A_m x_m(t) + u_m(t) + r_m(t) \]
\[ y_m(t) = x_m(t) \quad (10) \]

where \( x_m(t), x_m(t) \in \mathbb{R}^n \) are the state vectors, \( y_m(t) \in \mathbb{R}^n \) is the output vector, \( u_m(t) \in \mathbb{R}^n \) is the control input, and \( A_m \in \mathbb{R}^{n \times n} \) is, without loss of generality, a negative-definite, diagonal matrix. The reference signal \( r_m(t) \in \mathbb{R}^n \) is given by

\[ r_m(t) = K_d(x_d - x_m(t)) \quad (11) \]

where \( K_d \in \mathbb{R}^{n \times n} \) is a positive-definite, constant matrix and \( x_d \in \mathbb{R}^n \) is the desired state constant vector.

**B. Scattering Transformation**

If the reference model and the time delay nonlinear system are to be directly coupled through their delayed outputs \( y_m(t - T_2) \) and \( y(t - T_1) \), it can be shown that the communication channel may act as a nonpassive coupling element (i.e., may generate energy), potentially leading the system to instability [17]. In order to passify the communication channel and avoid instability, we propose the use of the wave-based scattering transformation. The wave variables \( w_m(t) \) and \( v(t) \), and the new control inputs \( u_m(t) = -\tau_m(t) \) and \( u(t) = \tau(t) \) are computed as

\[ \tau_m(t) = b e_m(t) + K_m e_m(t) \quad (12) \]
\[ w_m(t) = \sqrt{\frac{2}{b}} \tau_m(t) - v_m(t) \quad (13) \]
\[ x_{md}(t) = \frac{1}{b} \left( \tau_m(t) - \sqrt{2b} v_m(t) \right) \quad (14) \]
\[ e_m(t) = x_m(t) - x_{md}(t) \quad (16) \]

for the reference model and

\[ \tau(t) = \sqrt{2b} w(t) \quad (17) \]
\[ v(t) = w(t) - \sqrt{2b} x(t) \quad (18) \]

for the nonlinear system; where the wave impedance \( b \) is a positive constant, \( K_m \) is a symmetric, positive-definite matrix, and

\[ v_m(t) = v(t - T_1) \quad (19) \]
\[ w(t) = w_m(t - T_2). \quad (20) \]

The implementation of the scattering transformation and the reference model is schematized in Fig. 2.

The importance of the scattering transformation lies on the passivation of the communication channel independently of any large, constant round-trip delay. To demonstrate this statement, let us verify that the communication channel is rendered passive, that is, that the output (generated) energy of the communication channel is bounded by the input (admitted) energy. Manipulating (12)-(20) we can easily...
show that the net power flow in the coupling channel is given by
\[
\dot{x}_{md}^T \Lambda \tau_m - \dot{x}_m^T \Lambda (\tau - b \dot{x})
\]
\[
= \frac{1}{2} \left( w_m^T \Lambda w_m - w_m^T \Lambda w + v_T \Lambda v - v_{md}^T \Lambda v_{md} \right). \tag{21}
\]
Then, integrating (21) with respect to time yields
\[
\int_0^t (\dot{x}_{md}^T \Lambda \tau_m - \dot{x}_m^T \Lambda (\tau - b \dot{x})) \, d\theta = \frac{1}{2} \int_{t-T_2}^{t} w_m^T \Lambda w_m \, d\theta + \frac{1}{2} \int_{t-T_2}^{t} v_T \Lambda v \, d\theta \geq 0 \tag{22}
\]
which confirms the passivity claim independently of the size of \( T_1 \) and \( T_2 \).

**Remark 3.1:** The definition of the scattering transformation proposed here differs from its typical implementation [17]-[24] in the sense that the current (i.e., undelayed) states of the nonlinear plant are unaccessible to the system, and therefore, can not be used when computing the transformation variables. As a consequence, all the scattering transformation variables are computed at the same location in the network (see Fig. 2) as opposed to their conventional bisected (or mirror) implementation.

**IV. Stability Analysis**

Having established the control framework and the passivation of the communication channel, we now proceed to demonstrate asymptotic stability of (1) independently of arbitrarily large input and state measurement delays.

**Theorem 4.1:** Consider the time delay nonlinear system (1) coupled to the reference model (10) via the scattering transformation (12)-(20). Suppose that Assumption 2.1 holds and let \( b < \rho / \lambda \), where \( \lambda > 0 \) is the largest eigenvalue of \( \Lambda \). Then, we have the following results.

i. All signals \( x_m(t), x(t), e_m(t), \dot{x}_m(t), \dot{x}(t), \dot{e}_m(t), \dot{x}_m(t) \), and \( \dot{x}(t) \) are bounded \( \forall t \geq 0 \) and the velocities \( \dot{x}_m(t), \dot{e}_m(t), \dot{x}_m(t) \), and \( \dot{x}(t) \) converge to zero.

ii. The error signals \( e_m(t) \) and \( x_m(t) - x_d \) converge asymptotically to zero.

**Proof:** Suppose that \( \exists V(\dot{x}) \geq 0 \) such that (2) holds for some constant \( \rho > 0 \) and matrix \( \Lambda > 0 \) and consider the following Lyapunov candidate function \( S(t) = S(x_m(t), e_m(t), x_m(t), \dot{x}(t)) \) given by
\[
S = V(\dot{x}) + \frac{1}{2} (x_d - x_m)^T \Lambda K_d (x_d - x_m)
\]
\[
+ \frac{1}{2} \dot{x}_m^T \Lambda \dot{x}_m + \frac{1}{2} e_m^T \Lambda e_m
\]
\[
+ \frac{1}{2} \int_0^t (x_{md}^T \Lambda \tau_m - x_m^T \Lambda (\tau - b \dot{x})) \, d\theta. \tag{23}
\]
Taking the time derivative of (23) yields
\[
\dot{S} \leq - \rho \dot{x}^T \Lambda \dot{x} + \dot{x}_m^T \Lambda K_d (x_d - x_m)
\]
\[
+ \dot{x}_m^T \Lambda (A_m x_m - \tau_m + K_d (x_d - x_m))
\]
\[
+ e_m^T \Lambda K e_m + x_{md}^T \Lambda \tau_m - x_m^T \Lambda (\tau + b \dot{x})^T \dot{x}
\]
\[
\leq - \rho \dot{x}^T \Lambda \dot{x} + \dot{x}_m^T \Lambda K_d (x_d - x_m)
\]
\[
+ \dot{x}_m^T \Lambda (A_m x_m - \tau_m + K_d (x_d - x_m))
\]
\[
+ e_m^T \Lambda K e_m + x_{md}^T \Lambda \tau_m - x_m^T \Lambda (\tau + b \dot{x})^T \dot{x}
\]
\[
\leq - \rho \dot{x}^T \Lambda \dot{x} + \dot{x}_m^T \Lambda K_d (x_d - x_m)
\]
\[
+ \dot{x}_m^T \Lambda (A_m x_m - \tau_m + K_d (x_d - x_m))
\]
\[
+ e_m^T \Lambda K e_m + x_{md}^T \Lambda \tau_m - x_m^T \Lambda (\tau + b \dot{x})^T \dot{x}.
\]
Now, in order to prove the second statement, let us rewrite equation (27) as
\[ \tau_m(t) = \frac{b}{2} (\dot{x}_m(t) + \ddot{x}_m(t - T)) + \frac{K_m}{2} \int_{t-T}^{t} \dot{e}_m(\theta)d\theta - b\dot{x}(t - T_1). \] (30)

Since all signals at the right-hand side of (30) vanish as \( t \to \infty \), we have that \( \tau_m \to 0 \), which implies that \( e_m \to 0 \).

Then, computing the time derivative of (27) yields that
\[ \dot{\tau}_m(t) = \frac{b}{2} (\ddot{x}_m(t) + \ddot{x}_m(t - T)) + \frac{K_m}{2} (\dot{e}_m(t) + \dot{e}_m(t - T)) - b\ddot{x}(t - T_1) \] (31)

and due to the fact that all signals at the right-hand side of the above equation are once again bounded, we obtain that \( \dot{\tau}_m(t) \) is also bounded. Likewise, by taking the time derivative at both sides of (10), we can easily verify that \( \ddot{x}_m \in \mathcal{L}_\infty \). Then, since \( \int_{0}^{T} \dot{x}_m(\theta)d\theta \to -\dot{x}_m(0) < \infty \) as \( t \to \infty \), we can apply Barbacat’s Lemma [25] and conclude that \( \dot{x}_m \to 0 \).

Finally, using (10) and the convergence results for \( x_m, \ddot{x}_m, \) and \( \tau_m \), we obtain that \( \tau_m \to 0 \Rightarrow \dot{x}_m \to \dot{x}_d \to 0 \) which completes the proof.

The above theorem just established global asymptotic stability and boundedness of the states for any time delay nonlinear system that globally satisfies Assumption 2.1. If Assumption 2.1 does not hold globally, then local conclusions on convergence and stability can be easily claimed.

Remark 4.1: Note that the design of the control parameters is independent of the size of the delay. That is, no information about the delay is required to design \( b, K_m, \) and (10).

V. FULL STATE CONVERGENCE

Theorem 4.1 establishes convergence of the reference model to the desired state, i.e., \( x_m(t) \to x_d \). Yet, it does not guarantee full state tracking convergence between the reference model and the nonlinear plant, i.e., \( x(t) \to x_m(t) \).

To illustrate the ability of the proposed MRRC framework to achieve full state tracking, let us consider equation (28).

Under steady-state conditions, Theorem 4.1 concludes that both \( \ddot{x}_m(t) \) and \( e_m(t) \) vanish and \( x_{md}(t) \to x_m(t) \to x_d \) as \( t \to \infty \). Therefore, taking the limit as \( t \to \infty \) of (28) yields
\[ x_m(t) - x_{md}(0) = x(t) - x(0) \] (32)

which means that \( x(t) \to x_m(t) \to x_d \) if \( x_{md}(0) = x(0) \). Thus, the proposed control framework guarantees state convergence of the time delay nonlinear system to the desired state if the initial conditions of the system and the controller are matched.

In practice, the initial conditions of the time delay system are generally uncertain. Consequently, the matching of the initial states between controller and plant might be unfeasible. Moreover, differentiation techniques to compute the states \( \dot{x}(t) \) (e.g., velocities) necessary for the scattering transformation are typically subjected to numerical errors while the transmission of state information via the communication lines may as well suffer from losses; two conditions that may cause state drifts between the reference model and the plant even when initial conditions are matched. Motivated by these practical limitations, we propose a compensation technique that modifies the scattering transformation output \( v_m \) such that full state convergence can be explicitly enforced independently of dissimilar initial conditions and transmission losses. The compensation technique preserves the passivity of the communication channel and hence, stability of the system is guaranteed.

A. Design of State Tracking Compensator

To preserve passivity of the transmission lines, we must first ensure that the new scattering transformation output \( v_m \) does not violate the passivity condition in (22). This means that the energy in the transmission line, given by
\[ E_i(t) = \int_{0}^{t} (v_i^2(\theta - T) - v_{im}^2(\theta))d\theta, \quad \forall i \in \{1, \cdots, n\}, \] (33)
must never become negative, where \( v_i \) is the \( i \)th scalar component of the output vector \( v \) and \( v_{im} \) is the \( i \)th element of \( v_m \). According to this requirement, we propose to modify the previous transmission equation (19) for \( v_{im} \) as
\[ v_{im}(t) = \text{sign}(\ddot{v}_{im}(t)) \min(|\ddot{v}_{im}(t)|, |v_i(t - T_1)|) \] (34)
if \( E_i(t) \leq \sigma_i \), and
\[ v_{im}(t) = v_i(t - T_1) + \phi_i(t) (x_i(t) - x_i(t - T_1)) \] (35)
otherwise, where
\[ \ddot{v}_{im}(t) = v_i(t - T_1) + \frac{\gamma_i (1 - e^{-\delta_i \sigma_i})}{\alpha_i (||x_{im}(t)|| + ||x(t - T_1)||)} \] (36)
and
\[ \phi_i(t) = \frac{\gamma_i (1 - e^{-\delta_i E_i(t)})}{\alpha_i (||x_{im}(t)|| + ||x(t - T_1)||)} + 1 \] (37)
for some positive constants \( \sigma_i, \gamma_i, \delta_i, \) and \( \alpha_i \).

We now proceed to establish convergence of the system state vector \( x(t) \) to the desired state \( x_d \) under the utilization of the above compensator.

Theorem 5.1: Consider the time delay nonlinear system (1) coupled to the reference model (10) via the scattering transformation (12)-(18), (20) and full state compensator (35), (37). Suppose that Assumption 2.1 holds and let \( b < \rho/\bar{X} \), where \( \bar{X} > 0 \) is the largest eigenvalue of \( \Lambda \). Furthermore, assume that \( \exists t_0 \geq 0 \) such that \( E_i(t) > \sigma_i \) for all \( i \in \{1, \cdots, n\} \) and \( t \geq t_0 \). Then, we have the following results.

i. All signals \( x_m(t), e_m(t), \ddot{x}_m(t), x(t), w_m(t), w(t), v(t), \) and \( \tau(t) \) are bounded \( \forall t \geq t_0 \) and the velocities \( \dot{x}_m(t), e_n(t), \) and \( \dot{x}(t) \) converge to zero.

ii. If \( f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}, G, \frac{\partial G}{\partial x}, \) and \( \frac{\partial G}{\partial x} \) are bounded for \( x(t) \in \mathcal{L}_\infty \) and \( K_m = b^2I \), then the error signals...
\( e_m(t), x_m(t) - x_d, \) and \( x_m(t) - x(t) \) converge asymptotically to zero.

**Proof:** Assume that there exists a constant \( t_0 \geq 0 \) such that \( E_i(t) > \sigma_i, \forall i \in \{1, \ldots, n\}, t \geq t_0 \) and consider once again the candidate Lyapunov function given in (23). Since \( E_i(t) > \sigma_i, S(t) \) is positive-definite for \( (x_m(t) - x_d, e_m(t), x_m(t), \dot{x}(t)) \neq (0, 0, 0, 0) \). Taking the time derivative of (23), yields (24), from which we automatically obtain that \( x_m, x_{md}, e_m, \dot{x}_m, \ddot{x} \in L_\infty \). Then, solving (17) via the transmission equation (20), it is easy to show that

\[
\tau(t) = 2r_m(t - T_2) - v_m(t - T_2)
\]

(38)

and substituting (10) into (40) give us that the time derivative of (39) yields

\[
\dot{\tau}(t) = b \ddot{x}_m(t - T_2) + b^2 e_m(t - T_2)
\]

(40)

from which we can conclude that \( \dot{\tau} \) is also bounded.

Next, suppose that \( f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}, G, \frac{\partial G}{\partial x}, \) and \( \frac{\partial G}{\partial x} \) are all bounded for \( x(t) \in L_\infty \). This implies that \( \ddot{x} \) and \( \dot{x} \) are also bounded. Then, since \( \int_0^t \ddot{x}(\theta) d\theta \to -\dot{x}(0) \) as \( t \to \infty \), we can apply Barbalat’s Lemma and conclude that \( \dot{x} \to 0 \).

Hence, \( \tau \to 0 \) (from (11)), which implies that \( e_m \to 0 \) (from (39)) and \( v \to 0 \) (from (18) and (17)). Similarly from (12), we obtain that \( \tau_m \to 0 \).

Now, manipulating the equations of the scattering transformation it is easy to obtain that

\[
2b \dddot{x}_{md}(t) = b (\dddot{x}_m(t) - \dddot{x}_m(t - T)) + 2b \dddot{x}(t - T_1)
\]

(42)

where \( \Phi(t) \) is a diagonal matrix with \( i \)th entries given by \( \phi_i(t) \). Then, since \( (1 - e^{-\delta_i T_1}) \in (1 - e^{-\delta_i \sigma_i}) \) \( \forall t \), we obtain that \( \Phi(t) \) is both lower and upper bounded by some finite, positive constants (from (37)). Therefore, \( (x_{md}, \dddot{x}_m, \dot{x}, \dddot{e}_m) \to (0, 0, 0, 0) = x_m(t) - x(t - T_1) = 0 \) which for finite delay \( T_1 \) implies that \( x_m(t) - x(t) \to 0 \) as \( t \to \infty \). This last result and the fact that \( x_m(t) - x(t - T_1) \) is uniformly continuous, further imply that \( x_m(t) - x(t - T_1) \) is bounded. Likewise, from (42) we obtain that \( x_{md} \) and \( \dot{e}_m \) are also bounded. Then, by taking the time derivative of (41) we can show that

\[
\dddot{\tau}(t) = b A_m^2 \dddot{x}_m(t - T_2) + b^2 (A_m + bI) \dddot{e}_m(t - T_2)
\]

(43)

is bounded. Finally, using Barbalat’s Lemma and noting that \( \int_0^T \dddot{\tau}(\theta) d\theta \to 0 \) \( \forall t \geq 0 \), we can conclude that \( \dddot{\tau}(t) \to 0 \), which further implies that \( x_m(t) \to 0 \) (from (40)). Then, from the definition of the reference model we obtain that \( r_m(t) \to 0 \Rightarrow x_d - x_m(t) \to 0 \), and the proof is complete.

**Remark 5.1:** Theorem 5.1 guarantees state convergence between reference model and plant as long as \( E_i(t) > \sigma_i \) \( \forall t \geq t_0 \). In the case this last requirement does not hold, no further conclusion besides boundedness of \( x_m(t) - x(t) \) can be claimed. In order to guarantee that \( E_i(t) > \sigma_i \) \( \forall t \geq t_0 \), we must prevent the system from reaching steady-state before the compensation task is consumed. Therefore, it is appealing to investigate properties on \( r_m(t) \) that will guarantee state convergence before all signals in the system reach a steady-state value.

**VI. Simulations**

As a mean of validation, we modeled the response of a two degrees-of-freedom, revolute-joint planar manipulator with measurement and input delays employing the proposed control framework. The dynamics of the manipulator is given by (1) with

\[
f(x, \dot{x}) = -M^{-1}(x)C(x, \dot{x})\dot{x} - M^{-1}(x)N\dot{x}
\]

(44)

where \( x = [x_1, x_2]^T \) and \( \dot{x} = [\dot{x}_1, \dot{x}_2]^T \) represent the angular positions and velocities, respectively, for the first and second links. The positive-definite inertia matrix \( M \) and the centrifugal and Coriolis matrix \( C \) are constructed as

\[
M(x) = \begin{bmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{bmatrix}, C(x, \dot{x}) = \begin{bmatrix}
h\dot{x}_2 & h(\dot{x}_1 + \dot{x}_2) \\
-h\dot{x}_1 & 0
\end{bmatrix}
\]

where \( m_1 = (52.72 + 5.85 \cos(x_2))10^{-2}\text{kgm}^2 \), \( m_2 = (3.27 + 2.92 \cos(x_2))10^{-2}\text{kgm}^2 \), \( m_3 = (3.27, 10^{-2}\text{kgm}^2 \), and \( h = (-2.92 \sin(x_2))10^{-2}\text{kgm}^2 \). The diagonal matrix \( N \) corresponds to the coefficient matrix of viscous friction and is equal to \( 0.5I(\text{kgm}^2/s) \).

From the above mathematical representation (44), it is easy to show (using \( V(x) = \frac{1}{2}x^T M(x)x \geq 0 \)) that (44) satisfies Assumption 2.1 with \( \Lambda = I \) and \( \rho = 0.5 \). Therefore, in order to guarantee stability with the proposed MRMC framework, we choose \( b = 0.3 < \rho \). The reference model and the reference control law are finally designed according to (10) and (11) with

\[
A_m = \begin{bmatrix}
-10 & 0 \\
0 & -10
\end{bmatrix}, K_d = \begin{bmatrix}
4 & 0 \\
0 & 4
\end{bmatrix}
\]

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A. Simulation Results

Two different experiments were simulated while performing the same control task. The control task consisted on driving the nonlinear manipulator from an unknown initial position $x(0) = [x_1(0), x_2(0)]^T = [-\frac{\pi}{2}, -\frac{\pi}{2}]^T(\text{rad})$ to the desired configuration $x_d = [x_{d1}, x_{d2}] = [\pi, \pi]^T(\text{rad})$ when the interconnection delays between controller and plant were $T_1 = 1.0s$ and $T_2 = 0.5s$.

In the first simulated experiment, the proposed MRRC framework was applied with no state compensator. The design parameters for the scattering transformation were chosen as $K_m = b^2I = 0.09I$ while initial state values for the controller were set to zero, i.e., $x_m(0) = x_{md}(0) = 0$. The response of the system is depicted in Fig. 3. As can be noticed from both plots, the state of the reference model converged to the desired configuration and the position error between the reference model and the time delay system stabilized at a constant value. Yet, the error did not converge to zero as both reference model and nonlinear systems were initialized at different configurations, i.e., $x_m(t) - x(t) = x_m(0) - x(0) = [\frac{\pi}{2}, \frac{\pi}{2}]^T$.

We next simulated the response of the overall system to the control task employing the full state compensator. The parameters for the scattering transformation and the initial states for the controller were designed as in the previous simulated experiment. The parameters for the state compensator were chosen as $\gamma = [2, 10]^T$, $\delta = [2, 10]^T$, $\sigma = [\frac{1}{1000}, \frac{1}{1000}]^T$, and $\alpha = [6, 25]^T$. The position response for the proximal ($i = 1$) and distal ($i = 2$) links are illustrated in Fig. 4. Both plots evidence that the time delay nonlinear plant was able to track the motion of the reference system and converged to the desired configuration even when the initial conditions for controller and nonlinear plant were different and the delays were considerably large.

Finally, Fig. 5 illustrates the evolution of the compensator’s energy. Note that after a few seconds, the system built enough energy to modify the scattering transformation outputs such that position convergence, as claimed by Theorem 5.1, was achieved.

VII. Final Remarks

A. Unknown Round-Trip Delay

The proposed controller implicitly assumes that the delayed wave variable $w_m(t-T)$ is known or that, at least, can be decoded from the output $y(t)$ of the system. Specifically, we assumed that $w_m(t-T)$ is available when computing (18) and (19), i.e.,

$$v_m(t) = v(t-T) = w_m(t-T) - \sqrt{2b}x(t-T).$$

In case that $w_m(t-T)$ is unknown or unavailable, knowledge of the round-trip delay value is required to reconstruct $w_m(t-T)$. In a near future, we aim to assess the sensitivity of the system to uncertainties in the delay and accordingly derive sufficient stability conditions.

It is worth mentioning that knowledge of the delay or $w_m(t-T)$ is not required to design the parameters for the reference model, scattering transformation, and state compensator.

B. Conclusions

In this paper, a novel MRRC framework was designed to guarantee asymptotic stability of a class of nonlinear systems with constant input and state measurement delays. The proposed controller combines the use of the scattering transformation to passify the coupling between plant and controller independently of the delay and introduces a state tracking compensator that enforces full state convergence for unknown initial conditions, computational errors, and transmission losses. The effectiveness of the proposed controller was validated via simulation results.

REFERENCES

Fig. 4. Response of proximal and distal links with the proposed MRRC framework and full state compensator when $T_1 = 1.0\,\text{s}$ and $T_2 = 0.5\,\text{s}$.

Fig. 5. Available energy for compensation.


