Stability Analysis for Interconnected Piecewise Linear Planar Systems

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Abstract

Stability analysis of interconnected piecewise linear planar systems is provided. The connected piecewise linear system consists of a pair of piecewise linear planar systems interconnected through the system mode. The stability analysis is predicated on the negativity of the integral of the radial growth rate and the existence of periodic orbits in a planar torus space. Finally, an illustrative example is provided to demonstrate efficacy of the proposed approach.

1. Introduction

Stabilization problem of piecewise linear and affine systems has been attracting much attention in the literature (see, for example, [1–4]). In particular, even 2-dimensional piecewise linear systems have rich characteristics. Based on the results given in [5], in this paper we develop a method of determining stability of a class of interconnected piecewise linear planar systems. The connected piecewise linear system consists of a pair of 2 piecewise linear planar systems interconnected through the system mode. The stability analysis is predicated on the negativity of the integral of the radial growth rate and the existence of periodic orbits in a planar torus space. Finally, we show a illustrative examples to demonstrate efficacy of the proposed approach.

The notation used in this paper is fairly standard. Specifically, \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}^n \) denotes the set of \( n \times 1 \) real column vectors, \( \mathbb{R}_+^n \) denotes the nonnegative orthant of \( \mathbb{R}^n \), \( \mathbb{N} \) denotes the set of natural numbers, \( S^1 \triangleq [0, 2\pi) \) denotes the set of circumference, and \( \mathbb{T}^2 \triangleq S^1 \times S^1 \) denotes the 2-dimensional torus. Furthermore, we write \((\cdot)^T\) for transpose and \( \| \cdot \| \) for the Euclidean vector norm.

2. Mathematical Preliminaries

In this section we introduce notation, several definitions, and some key results concerning 2-dimensional linear dynamical systems that are necessary for developing the main results of the paper. Specifically, consider the planar linear system given by

\[
\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad t \geq 0,
\]

where \( x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^2 \) is the state vector and \( A \in \mathbb{R}^{2 \times 2} \). Furthermore, consider the polar form \((r, \theta)\) of the coordinate \((x_1, x_2)\) as shown in Figure 2.1, where \( r \) is the distance of \( x \) from the origin and \( \theta \) is the angle (phase) from the positive \( x_1 \)-axis in the counterclockwise direction.

2.1. Rotational Direction of Trajectories

The rotational direction of the trajectory of (1) at \( x \sim (r, \theta) \) can be determined by examining the sign of \( \frac{d\theta}{dt} \), that is, \( \frac{d\theta}{dt} > 0 \) (resp., \( \frac{d\theta}{dt} < 0 \)) implies that the trajectory of (1) is moving in the counterclockwise (resp., clockwise) direction at \( x \). In fact, since

\[
\frac{d\theta}{dt} = \frac{1}{\|x\|^2} \det \begin{bmatrix} x_1 & \dot{x}_1 \\ x_2 & \dot{x}_2 \end{bmatrix} = \frac{1}{r^2} \det \begin{bmatrix} \eta(\theta) \\ A\eta(\theta) \end{bmatrix},
\]

where \( \eta(\theta) \triangleq [\cos \theta, \sin \theta]^T \), the rotational direction of the trajectory of (1) at \( x \) can be determined by examining the sign of \( \det[\eta(\theta), A\eta(\theta)] \). It is important to note that the sign of \( \det[\eta(\theta), A\eta(\theta)] \) depends solely on...
leads us to determine stability of (1). Of course, if the real part of the complex conjugate eigenvalue of $A$ is negative (resp., positive), then (7) is negative (resp., positive). Note that $\theta_0$ in (7) is arbitrary.

Next, the following lemma characterizes the duration when the trajectories travel from $\theta = 0$ to $\theta = 2\pi$.

**Lemma 2.1.** Consider the linear system given by (1). Then the time (period) $T$ for the trajectories of (1) to travel from the phase $\theta = \theta_0$ to $\theta = \theta_0 + 2\pi$ is given by

$$T = 2\int_0^\pi \frac{d\theta}{\det[\eta(\theta), A\eta(\theta)]}. \quad (8)$$

**Proof.** The proof is immediate from (2) and

$$T = \int_{\theta_0}^{\theta_0 + 2\pi} \frac{dt}{d\theta}, \quad \int_{\theta_0}^{\theta_0 + 2\pi} \frac{1}{\det[\eta(\theta), A\eta(\theta)]} d\theta = 2 \int_0^\pi \frac{1}{\det[\eta(\theta), A\eta(\theta)]} d\theta. \quad (11)$$

Note that the period $T$ does not depend on the initial position of the trajectory.

Finally, we present one of the ways to check whether the linear system (1) is stable.

**Proposition 2.1.** Consider the linear system given by (1), where $A$ has complex conjugate eigenvalues and satisfies $\det[\eta(\theta), A\eta(\theta)] > 0$, $\theta \in [0, 2\pi]$. Furthermore, let

$$\gamma \triangleq \int_0^{2\pi} \rho(\theta) d\theta. \quad (12)$$

Then the following statements hold:

- If $\gamma < 0$, then the zero solution to (1) is globally exponentially stable;
- If $\gamma = 0$, then the zero solution to (1) is marginally stable and the trajectory of (1) constitutes a closed orbit;
- If $\gamma > 0$, then the zero solution to (1) is unstable.

This variable $\gamma$ represents how far the trajectory is going to be from the origin when the trajectory travels and makes one round from a point with phase $\theta_0$ back to another point with the same phase $\theta_0$. Note that $\gamma$ does not depend on $\theta_0$. 

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3. Problem Formulation

In this section, we consider the piecewise linear system $G$ given by

$$
\dot{x}(t) \triangleq \begin{bmatrix} A_i^{(A)} & 0 \\ 0 & A_i^{(B)} \end{bmatrix} x(t), \quad \text{if } x(t) \in D_i,
$$

$$
x(0) = x_0, \quad t > 0, \tag{13}
$$

where $x(t) \triangleq [x_A(t), x_B(t)]^T \in \mathbb{R}^4$ is the state vector, $x_A(t) \triangleq [x_{A,1}(t), x_{A,2}(t)]^T \in \mathbb{R}^2$, $x_B(t) \triangleq [x_{B,1}(t), x_{B,2}(t)]^T \in \mathbb{R}^2$, $A_i^{(A)}, A_i^{(B)} \in \mathbb{R}^{2 \times 2}$, $i = 1, \ldots, k$, are the matrices that we are allowed to assign to the domain $D_i$, and $k$ is the number of domains (modes) which the entire state space is partitioned into. We call the boundaries of the domains switching surfaces. Hence, when the trajectory reach a switching surface, the system mode of (13) changes. Furthermore, we assume that each of the subspaces, $x_A$ and $x_B$, is partitioned by semi-infinite straight lines so that the domains $D_i$, $i = 1, \ldots, k$, are given by

$$
D_i = \left\{ x \in \mathbb{R}^4 : \begin{bmatrix} C_i^{(A)} & 0 \\ 0 & C_i^{(B)} \end{bmatrix} x \leq 0 \right\}, \tag{14}
$$

where $C_i^{(A)}, C_i^{(B)} \in \mathbb{R}^{2 \times 2}$, $i = 1, \ldots, k$, characterizes the slopes of the semi-infinite straight lines. Note that there is a possibility that $C_i^{(j)} = C_j^{(i)}$ for some $i, j \in \{1, \ldots, k\}, i \neq j$. Moreover, $D_i$, $i = 1, \ldots, k$, are assumed to satisfy

$$
\bigcup_{i=1}^k D_i = \mathbb{R}^4,
$$

$$
\text{int}(D_i) \cap \text{int}(D_j) = \emptyset, \quad i, j = 1, \ldots, k, \quad i \neq j. \tag{15}
$$

The piecewise linear system $G$ can be seen as two piecewise linear planar systems $\dot{x}_A(t) = A_i^{(A)} x_A(t)$ and $\dot{x}_B(t) = A_i^{(B)} x_B(t)$, which are interconnected through the system mode so that the system mode $i$ of the two subsystems at time $t$ is determined by the conditions

$$
\begin{bmatrix} C_i^{(A)} & 0 \\ 0 & C_i^{(B)} \end{bmatrix} x \leq 0, \quad i = 1, \ldots, k. \tag{16}
$$

In addition, for simplicity of exposition we have the following assumption on the system matrices $A_i^{(A)}, A_i^{(B)}$, $i = 1, \ldots, k$.

**Assumption 3.1.** The sign of $\det[y, A_i^{(\phi)} y]$, characterizing the rotational direction, is positive at any point of $y \in \mathbb{R}^2$ for each matrix $\phi \in \{A, B\}$, $i = 1, \ldots, k$.

In the following section, we construct a framework to check asymptotic stability of the system $G$.

4. Stability Analysis

As discussed in Section 2.3, stability of the planar linear system (1) can be determined by checking the sign of the integral (5). In this section we apply and extend the same idea to the interconnected piecewise linear system given by (13) which has any periodic orbit on the subspace of the polar angles. For stability analysis of the piecewise linear system $G$, we consider the polar form transformation. Then, we characterize the periodic orbits on the subspace of the polar angles. Finally we check the stability of the system with the idea of the radial growth rate.

4.1. Polar Form

In this section we consider a polar form transformation for the connected planar system. For each $x_A$ and $x_B$, the the subspace of piecewise linear system $G$ given by (13), we apply the polar form transformation such that

$$
x_A = \begin{bmatrix} r_A \cos \theta_A \\ r_A \sin \theta_A \end{bmatrix}, \quad x_B = \begin{bmatrix} r_B \cos \theta_B \\ r_B \sin \theta_B \end{bmatrix}. \tag{17}
$$

Then the piecewise linear system $G$ can be described with $r_A$, $r_B$, $\theta_A$, and $\theta_B$, such that

$$
\begin{align}
\dot{r}_A(t) &= r_A(t) f_i^{(A)}(\theta_A(t), \theta_B(t)), \\
r_A(0) &= \|x_A(0)\|, \quad t \geq 0, \tag{18}
\end{align}
$$

$$
\begin{align}
\dot{r}_B(t) &= r_B(t) f_i^{(B)}(\theta_A(t), \theta_B(t)), \\
r_B(0) &= \|x_B(0)\|, \tag{19}
\end{align}
$$

$$
\begin{align}
\dot{\theta}_A(t) &= g_i^{(A)}(\theta_A(t), \theta_B(t)), \\
\theta_A(0) &= \tan^{-1}\left(\frac{x_{A,2}(0)}{x_{A,1}(0)}\right), \tag{20}
\end{align}
$$

$$
\begin{align}
\dot{\theta}_B(t) &= g_i^{(B)}(\theta_A(t), \theta_B(t)), \\
\theta_B(0) &= \tan^{-1}\left(\frac{x_{B,2}(0)}{x_{B,1}(0)}\right), \tag{21}
\end{align}
$$

where

$$
\begin{align}
f_i^{(A)}(\theta_A) &\triangleq \eta^T(\theta_A) A_i^{(A)} \eta(\theta_A), \tag{22}
\end{align}
$$

$$
\begin{align}
f_i^{(B)}(\theta_B) &\triangleq \eta^T(\theta_B) A_i^{(B)} \eta(\theta_B), \tag{23}
\end{align}
$$

$$
\begin{align}
g_i^{(A)}(\theta_A) &\triangleq \det[\eta(\theta_A), A_i^{(A)}], \tag{24}
\end{align}
$$

$$
\begin{align}
g_i^{(B)}(\theta_B) &\triangleq \det[\eta(\theta_B), A_i^{(B)}], \tag{25}
\end{align}
$$

$$
\begin{align}
s(\theta_A, \theta_B) &\triangleq i \quad \text{for} \quad \left[\begin{array}{c} \eta(\theta_A) \\ \eta(\theta_B) \end{array}\right] \in D_i, \quad i = 1, \ldots, k. \tag{26}
\end{align}
$$

Note that $\dot{\theta}_A(t)$ and $\dot{\theta}_B(t)$ depend solely on $\theta_A(t)$ and $\theta_B(t)$ but $r_A(t)$ nor $r_B(t)$ and we call the self-contained subsystem (20), (21) $G_0$. 

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4.2. Periodic Orbits on the Subspace of Polar Angles

To analyze the orbits of the system given by (18)–(21), we consider the orbits after infinite time on the subspace of polar angles $\theta_A$ and $\theta_B$. Specifically, we consider the system $G^g$ given by (20) and (21), which is a part of the system $G$ given by (18)–(21). Note that the domain of each mode of the subsystems is lattice-shaped (Figure 4.1). Furthermore, from Assumption 3.1, $\theta_A(t)$ and $\theta_B(t)$ are positive for all $t \geq 0$ and hence the system $G^g$ possesses no equilibrium point. However, there is a possibility that some periodic orbits on the subspace of the polar angles exist.

First, to obtain the periodic orbit(s) on the subspace of the polar angles, we consider the impact maps $M_i : B_i^- \rightarrow B_i^+$, $i = 1, \ldots, k$, where $B_i^- \subset \mathbb{T}^2$ is the inward boundary of mode $i$ and $B_i^+ \subset \mathbb{T}^2$ is the outward boundary of mode $i$. (See Figure 4.2.)

**Definition 4.1.** The functions $M_i : B_i^- \rightarrow B_i^+$, $i = 1, \ldots, k$, are the maps from $(\theta_{A,i}^a, \theta_{B,i}^a)$ to $(\theta_{A,i}^b, \theta_{B,i}^b)$ along the trajectories of the system $G^g$ given by (20), (21), where $(\theta_{A,i}^a, \theta_{B,i}^a)$ is the state that goes into the domain of mode $i$ and $(\theta_{A,i}^b, \theta_{B,i}^b)$ is the state that escapes from the same domain.

The following lemma is needed to obtain the impact maps $M_i$, $i = 1, \ldots, k$.

**Lemma 4.1.** For the trajectory of the system $G^g$ given by (20) and (21), let the initial state of the trajectory be given by $(\theta_A(0), \theta_B(0)) = (\theta_{A,i}^a, \theta_{B,i}^a)$, let the state at $t = T$ be given by $(\theta_A(T), \theta_B(T)) = (\theta_{A,i}^b, \theta_{B,i}^b)$, and let the mode $\sigma(\theta_A(t), \theta_B(t)) = i$ be constant at $t \in (0, T)$. Then it follows that

$$\phi(A_i^a, \theta_{A,i}^a) - \phi(A_i^a, \theta_{A,i}^a) = \phi(A_i^b, \theta_{B,i}^b) - \phi(A_i^b, \theta_{B,i}^b),$$

where

$$\phi(A, \theta) \triangleq \frac{\tan^{-1} \left( \frac{(A_{22} - A_{11})/2 - A_{12} \tan(\theta)}{\sqrt{\det(GA - A^T G)}} \right)}{\sqrt{\det(GA - A^T G)}},$$

and $A_{ij}$ is the $(i, j)$ element of $A$.

**Proof.** From (20) and (21), the trajectories of $\theta_A(t)$, $\theta_B(t)$ on $t \in (0, T)$ satisfies

$$1 = \frac{1}{g_i^A(\theta_A)} \frac{d\theta_A}{dt} = \frac{1}{g_i^B(\theta_B)} \frac{d\theta_B}{dt},$$

and the integrals of the equations from $t = 0$ to $t = T$ yield

$$T = \int_{\theta_{A,i}^a}^{\theta_{A,i}^b} \frac{1}{g_i^A(\theta_A)} d\theta_A = \int_{\theta_{B,i}^a}^{\theta_{B,i}^b} \frac{1}{g_i^B(\theta_B)} d\theta_B.$$

Therefore, it follows that

$$\int_{\theta_{A,i}^a}^{\theta_{A,i}^b} \frac{1}{g_i^A(\theta_A)} d\theta_A = \int_{\theta_{B,i}^a}^{\theta_{B,i}^b} \frac{1}{g_i^B(\theta_B)} d\theta_B.$$  

Finally, calculating the integrals of the equation, we get (27). \qed

We can obtain the impact map $M_i$ by solving (27). Here, the orbit repeats crossing the line $\theta_B = 0$ infinitely many times because $g_i^B(\theta_B)$ is strictly positive for $i = 1, \ldots, k$, and $\theta_B \in \mathbb{T}$. The Poincaré map $F_A$ on the Poincaré section $\theta_B = 0$ is defined as follows.

**Definition 4.2.** The function $F_A : [0, 2\pi) \rightarrow [0, 2\pi)$ is the map $\theta_A \mapsto F_A(\theta_A)$ when the trajectory of the system $G^g$ moves from $(\theta_A, 0)$ to $(F_A(\theta_A), 2\pi)$.

Using composition of the impact maps $M_i$ in order of the mode that trajectory traverses, we can derive the Poincaré map $F_A$. If the Poincaré map $F_A$ has one or more fixed points, then the system $G^g$ has one or more closed orbits and we can derive the closed orbits from the fixed points of $F_A$. Finally, we note the relation between the periodic orbits and the fixed points of $F_A^m$ that is the $m$th iterate of $F_A$. (For example, $F_A^m(\cdot) = F_A(F_A(\cdot))$.)
Lemma 4.2. Consider the system $G^\theta$ given by (20) and (21), where $\dot{\theta}_A > 0$ and $\dot{\theta}_B > 0$, and the Poincaré map $F_A$ given in Definition 4.2. If there exists $m \in \mathbb{N}$ such that the map $F^m_A$ has $p \in \mathbb{N}$ fixed points, then system $G^\theta$ has $p/m$ periodic orbit(s).

Note that $p/m$ is integer. Furthermore, if the map $F^m_A$, $m \in \mathbb{N}$, has no fixed point, then the system $G^\theta$ has no periodic orbit.

4.3. Stability Analysis with Radial Growth Rate

In this section, we present stability analysis for the system $G$ given by (13) which has periodic orbit(s) on the subspace of the polar angles.

First, we provide convergence analysis for the trajectory $x(t)$ such that the initial phase $(\theta^0_A, \theta^0_B)$ is on the periodic orbit on the subspace of the polar angles. Now, with the radial growth rates

$$
\begin{align*}
\rho_i^{(A)}(\theta) &\triangleq \frac{\eta^T(\theta)A_i(A)\eta(\theta)}{\det[\eta(\theta), A_i(A)^T\eta(\theta)]}, \\
\rho_i^{(B)}(\theta) &\triangleq \frac{\eta^T(\theta)A_i(B)\eta(\theta)}{\det[\eta(\theta), A_i(B)^T\eta(\theta)]},
\end{align*}
$$

we derive the lemma of the convergence analysis.

Lemma 4.3. Consider the system $G$ given by (13). Assume that there exists a periodic orbit in the subspace $G^\theta$ and the initial phase $(\theta^0_A, \theta^0_B)$ is on the periodic orbit. If

$$
\gamma_A \triangleq \sum_{j=1}^l \gamma_j^A < 0, \quad \gamma_B \triangleq \sum_{j=1}^l \gamma_j^B < 0,
$$

are satisfied, where $\gamma_j^A, \gamma_j^B$ are given by

$$
\begin{align*}
\gamma_j^A &\triangleq \int_{\theta_{j-1}^A}^{\theta_j^A} \rho_i^{(A)}(\theta) d\theta, \\
\gamma_j^B &\triangleq \int_{\theta_{j-1}^B}^{\theta_j^B} \rho_i^{(B)}(\theta) d\theta,
\end{align*}
$$

and $l$ is the number of the switches in a period of the periodic orbit, then $x(t) \to 0$ as $t \to \infty$.

Proof. Since $(\theta_0^A, \theta_0^B)$ is a point on the periodic orbit on the plane $(\theta_A, \theta_B)$, there exists $T > 0$ that satisfies

$$
\theta_A(T + t) = \theta_A(t), \quad \theta_B(T + t) = \theta_B(t), \quad t \geq 0.
$$

We set $T_{\min}$ to the smallest $T$ that satisfies (37). Hence, since we can choose $i > 0$ and $\alpha \in \mathbb{N}_0$ such that $i < T_{\min}$ and $t = \tilde{t} + \alpha T_{\min}$ for $t > 0$,

$$
\begin{align*}
\rho_A(t) &= \rho_A(\tilde{t} + \alpha T_{\min}) = \left( \sum_{j=1}^l \gamma_j^A \right) r_A(\tilde{t}), \\
\rho_B(t) &= \rho_B(\tilde{t} + \alpha T_{\min}) = \left( \sum_{j=1}^l \gamma_j^A \right) r_B(\tilde{t}).
\end{align*}
$$

Therefore, since $\alpha \to \infty$ as $t \to \infty$, if the condition (33) satisfied then

$$
||x(t)|| = \sqrt{r_A^2(t) + r_B^2(t)} \to 0, \quad (40)
$$
as $t \to \infty$.

Note that there exist $a_A > 0$, $a_B > 0$, and $T > 0$, such that $x_A(t) \leq a_A e^{\gamma_A t/T}$ and $x_B(t) \leq a_B e^{\gamma_B t/T}$ so that $\gamma_A$ and $\gamma_B$ denote the convergence rate of $x_A(t)$ and $x_B(t)$. Finally, we get the the way of global asymptotic stability analysis for the trajectory $x(t)$ of the system $G$ given by (13) which has periodic orbit(s) on the subspace of polar angles $(\theta_A, \theta_B)$.

Theorem 4.1. Consider the system $G$ given by (13). Assume that there exists a periodic orbit in the subspace $G^\theta$. If $\gamma_A, \gamma_B$ satisfy (33) for each periodic orbit on the subspace of polar angles, then the system is globally exponentially stable.

Proof. For any initial state $x_0$, $(\theta_A(t), \theta_B(t))$ converges to any periodic orbit on the subspace of the polar angles. Since there exist $m_A$ and $p$ such that the map $F^m_A$ defined by Definition 4.2 has fixed points and $(\theta_A(t), \theta_B(t))$ converges to $p$th periodic orbit on the subspace of the polar angles, $\tilde{\gamma}_A(t)$ defined by

$$
\tilde{\gamma}_A(t) \triangleq \int_{\theta_A(t)}^{\theta_A(t)+2m_A \pi} \frac{1}{r_A(t)} \frac{dr_A(t)}{d\theta_A(t)} d\theta_A(t),
$$

converges to $\gamma_A$ given by (33) as $t \to \infty$, and also $\gamma_B(t)$ does. Therefore, if (33) holds, then all trajectories converge to zero exponentially.

5. Illustrative Numerical Example

In this section we present a numerical example to demonstrate the utility of the framework proposed in the paper. Specifically, consider the interconnected piecewise linear system (13) with $A_i^{(A)}, A_i^{(B)}$, $i = 1, \ldots, 16,$
Figure 5.1: Map $F_A$, $F_A^2$, and $F_A^3$ given by

\[
\begin{align*}
A_1^{(A)} &= A_5^{(A)} = A_9^{(A)} = A_{13}^{(A)} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \\
A_2^{(A)} &= A_6^{(A)} = A_{10}^{(A)} = A_{14}^{(A)} = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}, \\
A_3^{(A)} &= A_7^{(A)} = A_{11}^{(A)} = A_{15}^{(A)} = \begin{bmatrix} -4 & -8 \\ 3 & -3 \end{bmatrix}, \\
A_4^{(A)} &= A_8^{(A)} = A_{12}^{(A)} = A_{16}^{(A)} = \begin{bmatrix} -1 & -4 \\ 5 & -1 \end{bmatrix}, \\
A_1^{(B)} &= A_2^{(B)} = A_3^{(B)} = A_4^{(B)} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \\
A_5^{(B)} &= A_6^{(B)} = A_7^{(B)} = A_8^{(B)} = \begin{bmatrix} -1 & -2 \\ 3 & -1 \end{bmatrix}, \\
A_9^{(B)} &= A_{10}^{(B)} = A_{11}^{(B)} = A_{12}^{(B)} = \begin{bmatrix} 0 & -2 \\ 3 & 0 \end{bmatrix}, \\
A_{13}^{(B)} &= A_{14}^{(B)} = A_{15}^{(B)} = A_{16}^{(B)} = \begin{bmatrix} -1 & -2 \\ 1 & -1 \end{bmatrix},
\end{align*}
\]

where there are 16 switching surfaces (actual values $C_i^{(A)}$ and $C_i^{(B)}$, $i = 1, \ldots, 16$, are omitted).

From Lemma 4.2 and Figure 5.1(c), there exist 2 ($= 6/3$) periodic orbits because the map from $θ$ to $F_3^n(θ)$ has 6 fixed points. Now the convergence rates of $x_A(t)$ and $x_B(t)$ with respect to the 1st closed orbit are calculated as

\[ \gamma_A = -0.358, \quad \gamma_B = -0.374, \quad (42) \]

and those with respect to the 2nd closed orbit are

\[ \gamma_A = -0.468, \quad \gamma_B = -2.088. \quad (43) \]

Thus, since all of the convergence rates are strictly negative, it follows from Theorem 4.1 that the zero solution of the connected piecewise linear planar system is globally exponentially stable (Figure 5.2).

6. Conclusion

In this paper we propose the way to analyze the stability of connected piecewise linear planar systems. First, we search all periodic orbit on phase space using poincaré map. Second, we obtain impact sequence for each periodic orbit. Finally, we check the convergence for each periodic orbit using radial growth rate.

Future extensions include the stability analysis for general multi-dimensional piecewise linear systems.

References


