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Abstract

We present a method to identify the parameters of a state space model for a max-plus-linear discrete event system from input-output sequences. The approach is based on recasting the identification problem as an optimization problem over the solution set of an extended linear complementarity problem. Recently, we have shown that such a problem can be solved much more efficiently than previously by using a mixed integer programming approach. The resulting algorithm allows us to identify a state space model of a max-plus-linear discrete event system from input-output data. This method works for both structured and fully parameterized state space identification. In addition, we also obtain an estimate of the state sequence.

1 Introduction

1.1 Overview

A discrete event system (DES) is a dynamic, asynchronous system, where the state transitions are initiated by events that occur at discrete time instants. Typical examples of DES are flexible manufacturing systems, telecommunication networks, parallel processing systems, traffic control systems, and logistic systems. There exist many different modeling and analysis frameworks for DES such as Petri nets, finite state machines, automata, languages, process algebra, computer models, etc. In this paper we consider the class of DES with synchronization but no concurrency. Such DES can be described by models that are "linear" in the max-plus algebra [1], and therefore, they are called max-plus-linear (MPL) DES.

One of the main advantages of an analytic max-plus-algebraic model of a DES is that it allows us to derive some properties of the system (such as the steady state behavior) fairly easily, whereas in some cases brute force simulation might require a large amount of computation time [1]. In addition, the analytic model can be used in a model-based control setting to compute optimal input sequences for a given DES. In [8] we have developed a model predictive control (MPC) approach for MPL DES that uses a state space model of the DES to predict the future behavior of the system. MPC uses a moving horizon approach in which the model of the system is regularly updated as new measurements become available. Hence, an efficient on-line identification procedure is required.

If we want to use a model for control and other purposes, we have to be able to determine the parameters of the model. Most identification methods for MPL DES use a transfer function approach [3, 10]. One could argue that an identified transfer function description can be transformed into a state space model, but then the connection with the physical structure is usually lost[1]. Furthermore, compared to transfer functions state space models have certain advantages: they explicitly take the initial state of the system into account, they can reveal "hidden" behavior such as unobservable unstable modes, the extension from SISO to MIMO is more intuitive and elegant for state space models, and the analysis is often easier. In addition, the MPC framework of [8] requires a state space model. Therefore, we focus on state space identification for MPL DES.

In [13, 14] a state space identification method has been derived in which the internal structure of the system is assumed to be completely known and the state is assumed to be measurable. In this paper, we assume that only input-output sequences are available (i.e., we do not require measurements of the (full) state of the system[2]). In addition, our method can also be used for fully parameterized state space identification (i.e., when the internal structure of the system is not known). Furthermore, the proposed identification method also yields an estimate of the state sequence.

1.2 Max-plus-linear discrete event systems

Addition and maximization are the basic operations of the max-plus algebra. Due to the analogies between conventional algebra and max-plus algebra [1], these operations are also called max-plus-algebraic addition and multiplication, and denoted by $\oplus$ and $\otimes$ respectively:

$$x \oplus y = \max(x,y) \quad \text{and} \quad x \otimes y = x + y$$

for $x, y \in \mathbb{R}_\varepsilon \overset{\text{def}}{=} \mathbb{R} \cup \{-\infty\}$. Define $\varepsilon = -\infty$. Note that ε is the zero element for $\oplus$ (i.e., $x \oplus \varepsilon = x$ for all $x \in \mathbb{R}_\varepsilon$) and that it is absorbing for $\otimes$ (i.e., $x \otimes \varepsilon = \varepsilon$ for all $x \in \mathbb{R}_\varepsilon$).

For matrices $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ and $C \in \mathbb{R}_\varepsilon^{n \times p}$ we can extend the

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[1]For MPL DES the physical layout of the system is often clearly recognizable in the structure of the state space matrices (see also Section 3).

[2]Incomplete or no measurements of the state could occur in systems where not all internal starting and finishing times are monitored, or communicated to the control center. Note, however, that our approach can also take partial or full state measurements into account (see Remark 2.3).

definition of $\oplus$ and $\otimes$ as follows:

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$

$$(A \otimes C)_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} = \max_{k=1,\ldots,n}(a_{ik} + c_{kj}) \ .$$

DES with synchronization but no concurrency can be modeled using the operations maximization (corresponding to synchronization: a new operation starts as soon as all preceding operations have been finished) and addition (corresponding to durations: the finishing time of an operation equals the starting time plus the duration). This leads to a description that is "linear" in the max-plus algebra [1]:

$$x(k) = A \otimes x(k-1) \oplus B \otimes u(k) \quad (1)$$
$$y(k) = C \otimes x(k) \quad (2)$$

with $A \in \mathbb{R}_{\varepsilon}^{n \times n}$, $B \in \mathbb{R}_{\varepsilon}^{n \times m}$, $C \in \mathbb{R}_{\varepsilon}^{l \times n}$, and with n the system order, and m and l the number of inputs and outputs. Systems that can be described by the model (1)–(2) are called time-invariant max-plus-linear (MPL) DES. The index k is called the event counter. The state $x(k)$ typically contains the time instants at which the internal events occur for the kth time, $u(k)$ contains the time instants at which the input events occur for the kth time, and $y(k)$ contains the time instants at which the output events occur for the kth time.

1.3 The Extended Linear Complementarity

The Extended Linear Complementarity Problem (ELCP) arose from our research on DES and hybrid systems, and is defined as follows [5]:

Given $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{q \times n}$, $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$ and $\phi_1, \ldots, \phi_m \subseteq \{1, \ldots, p\}$, find $x \in \mathbb{R}^n$ such that

$$Ax \geqslant c \quad (3)$$
$$Bx = d \quad (4)$$
$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i = 0 \ . \quad (5)$$

The set $\{x \in \mathbb{R}^n \mid Ax \geqslant c, Bx = d\}$ is called the feasible set of the ELCP (3)–(5). The surplus variable $\mathrm{s}^+(i, x)$ of the ith inequality of $Ax \geqslant c$ is defined as $\mathrm{s}^+(i, x) = (Ax - c)_i$. Condition (5) represents the *complementarity condition* of the ELCP and can be interpreted as follows. Since $Ax \geqslant c$, all the terms in (5) are nonnegative. Hence, (5) is equivalent to $\prod_{i \in \phi_j}(Ax - c)_i = 0$ for $j = 1, \ldots, m$. So each set ϕ_j corresponds to a group of inequalities in $Ax \geqslant c$, and in each group at least one inequality should hold with equality (i.e., the corresponding surplus variable is equal to 0).

In general, the solution set of the ELCP consists of the union of a subset of the faces of the polyhedron defined by (3)–(4). In [5] we have developed an algorithm to find a parametric representation of the *entire* solution set of an ELCP. However, the computation time and the memory storage requirements of this algorithm increase exponentially as the size of

the ELCP increases, which makes this approach intractable for medium or large-scale ELCPs. However, recently we have shown the following proposition, which transforms an ELCP into a mixed integer linear programming problem, and which allows us to solve much larger instances of the ELCP than was possible previously (see also [7]):

Theorem 1.1 *If the residues of the ELCP (3)–(5) are bounded from above (i.e., there exist a diagonal matrix $D_{\mathrm{upp}} \in \mathbb{R}^{p \times p}$ such that $\mathrm{s}^+(i, x) \leqslant (D_{\mathrm{upp}})_{ii}$ for $i = 1, \ldots, p$), then any solution of the following mixed integer linear feasibility problem:*

$$\delta \in \{0, 1\}^p, \ x \in \mathbb{R}^n \quad (6)$$
$$0 \leqslant Ax - c \leqslant D_{\mathrm{upp}} \delta \quad (7)$$
$$Bx = d \quad (8)$$
$$\sum_{i \in \phi_j} \delta_i \leqslant \#\phi_j - 1 \qquad for \ j = 1, \ldots, m, \quad (9)$$

where $\#\phi_j$ denotes the number of elements of the set ϕ_j, yields a solution of the ELCP and vice versa.

Remark 1.2 A *sufficient* condition for the surplus variables of the inequalities of the ELCP to be bounded is that the feasible set of the ELCP is bounded.

In general, upper bounds for the surplus variables over the feasible set can be computed efficiently using a linear programming (LP) problem:

$$(D_{\mathrm{upp}})_{ii} = \max_{Ax - c \geqslant 0}(Ax - c)_i \quad \text{for } i = 1, \ldots, p.$$

If any of these LP problems yields an unbounded objective function, then the ELCP does not have a bounded feasible set and then the condition of Theorem 1.1 does not hold.
If we know upper bounds x_{upp} and lower bounds x_{low} for the components of x, e.g., as a consequence of physical or other constraints or because of additional information that is available, then we can even more efficiently compute upper bounds for the surplus variables over the feasible set as

$$(D_{\mathrm{upp}})_{ii} = (A^+ x_{\mathrm{upp}} - A^- x_{\mathrm{low}} - c)_i \quad \text{for } i = 1, \ldots, p, \quad (10)$$

with A^+ and A^- defined by $(A^+)_{ij} = \max(a_{ij}, 0)$ and $(A^-)_{ij} = \max(-a_{ij}, 0)$ respectively (so $A = A^+ - A^-$). $\diamond$

2 State space identification of MPL DES from input-output behavior

2.1 Problem statement

Suppose that for a given MPL DES of the form (1)–(2) we have an input-output sequence $\{(u_k, y(k)\}_{k=1}^{N}$, and that we want to identify the system matrices A, B, and C from this sequence. We make the standard assumption of system identification that all modes of the system are observable and that the input-output sequence is sufficiently rich to capture all the relevant information about the system (see also [14]). For the sake of simplicity of notation we assume that the given system is SISO. Note, however, that the extension to the MIMO case is straightforward.

2.2 Solution

First, we suppose that the internal structure of the system that we want to identify is known. In that case we know which entries of the system matrices contain process and/or transportation times and which entries are equal to ε (see, e.g., [1] or Section 3 for an illustration). This implies that both the size and the ε-structure[3] of A, B and C are known.

Now we consider the following estimation model:

$$\hat{x}(k) = \hat{A} \otimes \hat{x}(k-1) \oplus \hat{B} \otimes u(k) \tag{11}$$

$$\hat{y}(k) = \hat{C} \otimes \hat{x}(k) \tag{12}$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{x}(k)$ and $\hat{y}(k)$ are the estimates of respectively A, B, C, $x(k)$ and $y(k)$. Furthermore, if — based on physical constraints or insight, or additional information — we know hard upper and lower bounds for (some of) the entries of $\hat{A}$, $\hat{B}$, $\hat{C}$, or $\hat{x}(0)$, we may add the constraints

$$A_{\min} \leqslant \hat{A} \leqslant A_{\max} \qquad B_{\min} \leqslant \hat{B} \leqslant B_{\max} \tag{13}$$

$$C_{\min} \leqslant \hat{C} \leqslant C_{\max} \qquad x_{\min}(0) \leqslant \hat{x}(0) \leqslant x_{\max}(0) \tag{14}$$

where unknown bounds correspond to $-\infty$ and ∞ entries in $A_{\min}$, $B_{\min}$, $C_{\min}$, $x_{\min}(0)$ and $A_{\max}$, $B_{\max}$, $C_{\max}$, $x_{\max}(0)$. The ε-structure of the system matrices[4] can be specified by setting the corresponding entries of $A_{\max}$, $B_{\max}$, $C_{\max}$ (and of $A_{\min}$, $B_{\min}$, $C_{\min}$) equal to $\varepsilon = -\infty$. Define

$$\hat{x}_{\mathrm{tot}} = \begin{bmatrix} \hat{x}(0) \\ \vdots \\ \hat{x}(N) \end{bmatrix}, \quad \hat{y}_{\mathrm{tot}} = \begin{bmatrix} \hat{y}(1) \\ \vdots \\ \hat{y}(N) \end{bmatrix}, \quad y_{\mathrm{tot}} = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix}.$$

Since we want to minimize the difference between the measured and the estimated output, we define the following *MPL state space identification problem* to determine the optimal estimates of the system matrices:

$$\min_{\hat{A}, \hat{B}, \hat{C}, \hat{x}_{\mathrm{tot}}, \hat{y}_{\mathrm{tot}}} \| \hat{y}_{\mathrm{tot}} - y_{\mathrm{tot}} \|_2^2$$

subject to (11)–(12) for $k = 1, \ldots, N$ and (13)–(14).

Remark 2.1 A given input-output behavior of an MPL system can be represented by several choices of the system matrices A, B, C. This implies that the solution of the MPL state space identification problem defined above will not be unique since in this problem only the input-output behavior is taken into account. A very important difference between conventional linear systems and MPL systems is that for linear systems the set of all equivalent state space realizations can be characterized via similarity transformations, whereas this does not hold for MPL systems (see, e.g., [6]). Furthermore, to the authors' best knowledge there are currently no

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[3]The ε-structure of a matrix details which entries are equal to ε and which entries are finite, without giving specific values for the latter ones.

[4]When solving the optimization problem that corresponds to the MPL state space identification problem, the ε entries specified by the ε-structure of the system matrices can be removed immediately since these entries do not contribute to the left-hand side of (11)–(12).

analytic characterizations of the set of all equivalent state space realizations of an MPL system.

In conventional system identification this uniqueness issue is usually addressed by selecting a canonical form for the state space model [11], or by using a projection in the parameter space [12]. In the MPL setting there are no canonical forms due to the lack of a characterization of the set of all equivalent state space realizations, and only few results are available in connection with projections in the max-plus algebra (see, e.g., [4]). Nevertheless, by imposing the ε-structure of the system matrices we can decrease the degree of non-uniqueness. Moreover, we often know bounds on the entries of the system matrices based on physical insight or additional information (cf. (13)–(14)). Furthermore, in on-line adaptive applications we could add a term to the objective function that minimizes the deviation between the current and the new estimates. $\diamond$

Proposition 2.2 *The constraints of the MPL state space identification problem (i.e., (11)–(12) for $k = 1, \ldots, N$ and (13)–(14)) can be recast as an ELCP.*

Proof: The proof is similar to the reasoning made in Section 5.2 of [8]. Basically, the proof boils down to the fact that for $\alpha, \beta, \gamma \in \mathbb{R}$ the constraint $\max(\alpha, \beta) = \gamma$ is equivalent to the ELCP $\alpha \leqslant \gamma$, $\beta \leqslant \gamma$, $(\gamma - \alpha)(\gamma - \beta) = 0$. ∎

One approach for solving the MPL state space identification problem consists in computing the parametrized solution set of the ELCP (cf. [5]) corresponding to (11)—(14) and then optimizing the objective function over the parameters of this solution set. However, except for small-sized problems the computational requirements of this approach are too high. Therefore, we will use Theorem 1.1 to reformulate the ELCP as a mixed integer feasibility problem and combine that with the objective function to obtain a mixed integer quadratic programming problem, for which recently efficient solvers have been developed [2, 9].

In order to be able to apply Theorem 1.1 we have to show that the surplus variables of the ELCP are bounded from above over the feasible set. We will do this by showing that the variables of the ELCP are bounded (cf. (10)). Note that we only have to consider the non-ε entries of $\hat{A}$, $\hat{B}$ and $\hat{C}$ since the ε entries do not contribute to the expressions for the entries of the state and output vector (cf. Footnote 4).

Since we assume that all relevant dynamics of the system are captured by the given input-output sequence (i.e., there are no unobservable modes), and since all finite entries of the estimated system matrices should be nonnegative (due to their physical interpretations as sums or maximums of process times), it follows from (11)–(12) that

$$\hat{x}(k) \geqslant \hat{x}(k-1), \quad \hat{x}(k) \geqslant u(k), \quad \hat{y}(k) \geqslant u(k) \tag{15}$$

for $k = 1, \ldots, N$. In general, (13)–(14) do not define finite upper and lower bounds for all finite entries of the system matrices (recall that some entries of $A_{\min}$, $B_{\min}$, $C_{\min}$, $A_{\max}$,

$B_{\max}$, and $C_{\max}$ may be infinite). However, since the finite entries of the system matrices correspond to (nested) sums and maxima of process times and transportation times (see also Section 3), these entries are nonnegative. Furthermore, if we define the maximum dwelling time in the system as $d_{\max} = \max_{k=1,\ldots,N}(y(k) - u(k))$, then

$$0 \leqslant \hat{a}_{i_A j_A}, \hat{b}_{i_B j_B}, \hat{c}_{i_C j_C} \leqslant d_{\max}$$

for all $(i_A, j_A) \in \mathscr{I}_f(A), (i_B, j_B) \in \mathscr{I}_f(B), (i_C, j_C) \in \mathscr{I}_f(C)$, where $\mathscr{I}_f(A) = \{(i,j) \,|\, a_{i,j} \neq \varepsilon\}$. So all the finite entries of $\hat{A}, \hat{B}$ and $\hat{C}$ are bounded.

Since the maximum dwelling time is $d_{\max}$, we have

$$\hat{y}(k) \leqslant u(1) + k d_{\max} \quad \text{for } k = 1, \ldots, N. \quad (16)$$

In combination with (15) this implies that the entries of $\hat{y}_{\text{tot}}$ are also bounded from above and from below.

Since we assume that are no unobservable modes, and since the maximum dwelling time in the system is $d_{\max}$, we have[5]

$$\hat{x}(k) \leqslant u(1) + (k + n - 1) d_{\max} \quad \text{for } k = 0, \ldots, N.$$

In general $x(0)$ may contain ε-entries. However, if we replace these entries by $u(1) - (N+1)d_{\max}$ then the input-output behavior of the system does not change. Hence, the entries of $\hat{x}(0)$ can also be bounded from below. In combination with (15) this implies that $\hat{x}(k)$ is bounded from below for $k = 0, 1, \ldots, N$. Hence, $\hat{x}_{\text{tot}}$ is bounded.

So, since all variables of the ELCP that corresponds to the constraints of MPL state space identification problem are bounded, the surplus variables of the ELCP are also bounded (cf. Remark 1.2 and (10)). Hence, the ELCP can be recast as a mixed integer feasibility problem. In combination with the quadratic objective function this results in a mixed integer quadratic programming problem of the form

$$\min_{\hat{z}, \delta} \|\hat{y}_{\text{tot}} - y_{\text{tot}}\|_2^2$$

$$\text{subject to } F\hat{z} + G\delta \leqslant h \text{ and } \delta \in \{0, 1\}^q,$$

for appropriately defined matrices F, G and a vector h, where the vector $\hat{z}$ contains the (non-ε) entries of $\hat{A}, \hat{B}, \hat{C}, \hat{x}_{\text{tot}}$, and $\hat{y}_{\text{tot}}$. This optimization problem can be solved using, e.g., a branch-and-bound algorithm [2, 9].

2.3 System matrices with unknown structure

If we do not know the system's internal structure, we have to determine a system order $\hat{n}$ and adapt the lower bounds derived above so that all entries of $\hat{A}, \hat{B}, \hat{C}$ are finite. To the authors' best knowledge there currently exist no efficient methods to determine a good estimate of the minimal system order of an MPL DES from input-output data in the

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[5]The extra factor $(n-1)d_{\max}$ with respect to (16) is introduced since in general not every state is directly coupled to the output. However, since we assume that there are no unobservable modes, the influence of each state component can be noticed at the output after at most $n-1$ event steps or processing cycles.

case of noise or model structure mismatch. Therefore, we first assume that a reasonable estimate of the system order is available (e.g., based on physical insight or based on partial knowledge about the system). If such an estimate is not available, we can determine an upper bound for the system order and then use a binary or enumerative search procedure in combination with the above identification approach to obtain a system order that yields the best trade-off between fit of the output and size of the system matrices.

Similar to the reasoning made in Section 2.2 it can be shown that in this case we can also use the transformation into a mixed integer optimization problem to obtain an estimated state space model of the given system.

Remark 2.3 In the preceding sections we have assumed that the states could not measured (e.g., since not all internal starting and finishing times are monitored, measured or communicated to the control center). However, if partial (or even full) state measurements are available, they can easily be incorporated into the MPL state space identification problem by adding an extra term $\|\hat{x}_{\text{meas,tot}} - x_{\text{meas,tot}}\|^2$ to the objective function, where $x_{\text{meas,tot}}$ contains all measured state components in the event horizon $[1, N]$ and is defined similarly to y_{tot}, and $\hat{x}_{\text{meas,tot}}$ contains the corresponding components of $\hat{x}_{\text{tot}}$. Note that we have $\hat{x}_{\text{meas,tot}} = D_m \hat{x}_{\text{tot}}$ for an appropriately defined "selector" matrix D_m that has exactly one 1 entry on each row, and 0 entries elsewhere. Hence, this augmented MPL state space identification problem can also be recast as a mixed integer quadratic programming problem. ◇

2.4 An alternative approach without state estimates

If the state measurements are not available or not used, we may as well eliminate the state estimates from (11)–(12). This results in

$$\hat{y}_{\text{tot}} = \begin{bmatrix} \hat{C} \\ \hat{C} \otimes \hat{A} \\ \vdots \\ \hat{C} \otimes \hat{A}^{\otimes N} \end{bmatrix} \otimes \hat{x}(0) \oplus \qquad (17)$$

$$\begin{bmatrix} \hat{C} \otimes \hat{B} & \mathcal{E} & \ldots & \mathcal{E} \\ \hat{C} \otimes \hat{A} \otimes \hat{B} & \hat{C} \otimes \hat{B} & \ldots & \mathcal{E} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{C} \otimes \hat{A}^{\otimes N-1} \otimes \hat{B} & \hat{C} \otimes \hat{A}^{\otimes N-2} \otimes \hat{B} & \ldots & \hat{C} \otimes \hat{B} \end{bmatrix} \otimes u_{\text{tot}}$$

where $\hat{A}^{\otimes k} = \hat{A} \otimes \hat{A} \otimes \ldots \otimes \hat{A}$ (k times) denotes the kth max-plus-algebraic matrix power of $\hat{A}$, $\mathcal{E}$ is the max-plus-algebraic zero matrix (i.e., $(\mathcal{E})_{ij} = \varepsilon$ for all i, j), and $u_{\text{tot}} = \begin{bmatrix} u^T(1) & u^T(2) & \ldots & u^T(N) \end{bmatrix}^T$.

Now we consider the following alternative MPL state space identification problem:

$$\min_{\hat{A}, \hat{B}, \hat{C}, \hat{x}(0), \hat{y}_{\text{tot}}} \|\hat{y}_{\text{tot}} - y_{\text{tot}}\|_2^2$$
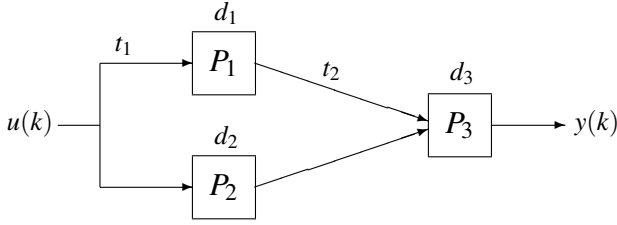
$$\text{subject to (17) and (13)–(14).}$$

Figure 1: A simple manufacturing system.

The constraints of this problem can also be recast as an ELCP, and it can thus also be solved using a mixed integer quadratic programming method. The ELCP of the alternative MPL state space identification problem will have less variables but more equations than the ELCP of the original MPL state space identification problem. In general, it is quite difficult to make hard claims which of the ELCPs can be solved most efficiently. This is a topic for future research.

3 Worked example

Consider the production system of Figure 1. This manufacturing system consists of three processing units: P_1, P_2 and P_3, and works in batches (one batch for each finished product). Raw material is fed to P_1 and P_2, processed and sent to P_3, where assembly takes place. Note that each input batch of raw material is split into two parts: one part of the batch goes to P_1 and the other part goes to P_2.

The processing times for P_1, P_2 and P_3 are respectively d_1, d_2 and d_3 time units. We assume that it takes t_1 time units for the raw material to get from the input source to P_1, and t_2 time units for a finished product of P_1 to get to P_3. The other transportation times are assumed to be negligible. At the input of the system and between the processing units there are buffers with a capacity that is large enough to ensure that no buffer overflow occurs. A processing unit can only start working on a new product if it has finished processing the previous one. We assume that each processing unit starts working as soon as all parts are available.

Now we write down the max-plus-algebraic state space model of this DES. First, we determine $x_1(k)$, i.e., the time instant at which processing unit P_1 starts working for the kth time. If we feed raw material to the system for the kth time, then this raw material is available at the input of processing unit P_1 at time $t = u(k) + t_1$. However, P_1 can only start working on the new batch of raw material as soon as it has finished processing the current, i.e. the $(k-1)$th, batch. Since the processing time on P_1 is d_1 time units, the $(k-1)$th intermediate product will leave P_1 at time $t = x_1(k-1) + d_1$. Since P_1 starts working on a batch of raw material as soon as the raw material is available and the current batch has left the processing unit, this implies that we have

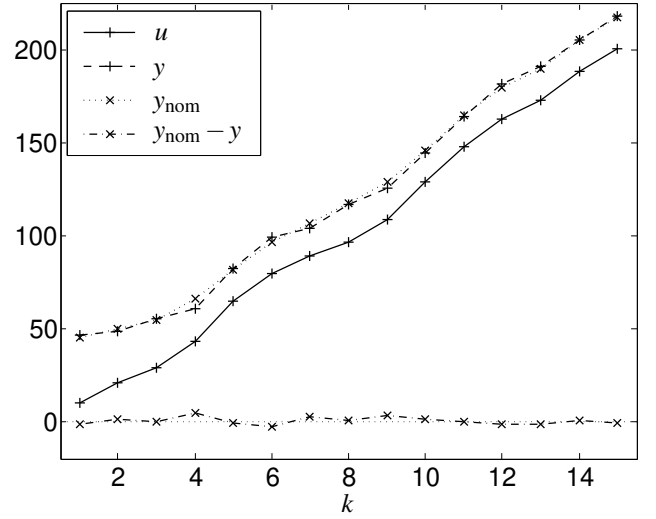$$x_1(k) = \max\left(x_1(k-1) + d_1, u(k) + t_1\right) .$$

Using a similar reasoning we find

Figure 2: The input u, the nominal output y_{nom}, and the measured output signal y used for the identification.

$$x_2(k) = \max\left(x_2(k-1) + d_2, u(k)\right)$$
$$x_3(k) = \max\left(x_1(k) + d_1 + t_2, x_2(k) + d_2, x_3(k-1) + d_3\right)$$
$$= \max\left(x_1(k-1) + 2d_1 + t_2, x_2(k-1) + 2d_2,\right.$$
$$\left. x_3(k-1) + d_3, u(k) + \max(d_1 + t_1 + t_2, d_2)\right)$$
$$y(k) = x_3(k) + d_3 .$$

If we rewrite the above evolution equations as an MPL state space model of the form (1)–(2), we obtain

$$x(k) = \begin{bmatrix} d_1 & \varepsilon & \varepsilon \\ \varepsilon & d_2 & \varepsilon \\ 2d_1 + t_2 & 2d_2 & d_3 \end{bmatrix} \otimes x(k-1) \oplus \quad (18)$$
$$\begin{bmatrix} t_1 \\ 0 \\ \max(d_1 + t_1 + t_2, d_2) \end{bmatrix} \otimes u(k)$$
$$y(k) = \begin{bmatrix} \varepsilon & \varepsilon & d_3 \end{bmatrix} \otimes x(k) . \quad (19)$$

Assume that the nominal values of the processing and transportation times are given by $d_1 = 10, d_2 = 11, d_3 = 5, t_1 = 1, t_2 = 1$, and that the actual process and transportation times are corrupted by zero-mean Gaussian noise with a standard deviation of 1 for d_1 and d_2, 0.5 for d_3, and 0.25 for t_1 and t_2. Let $x(0) = \begin{bmatrix} 0 & 6 & 35 \end{bmatrix}^T$ and[6] $N = 15$. We have chosen the following input signal:

$$\{u(k)\}_{k=1}^N = 10, 21, 29, 43, 65, 80, 89, 97, 109,$$
$$129, 148, 163, 173, 189, 201 .$$

The corresponding output signals for nominal and perturbed processing and transportation times are plotted in Figure 2.

Now we use the transformation into mixed integer quadratic programming problem to obtain estimates of the system matrices. First, we explicitly impose the ε-structure of the system matrices. This yields

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[6]In practice, much higher values will be used for N, but in order not to overload the plots, we have selected this value.
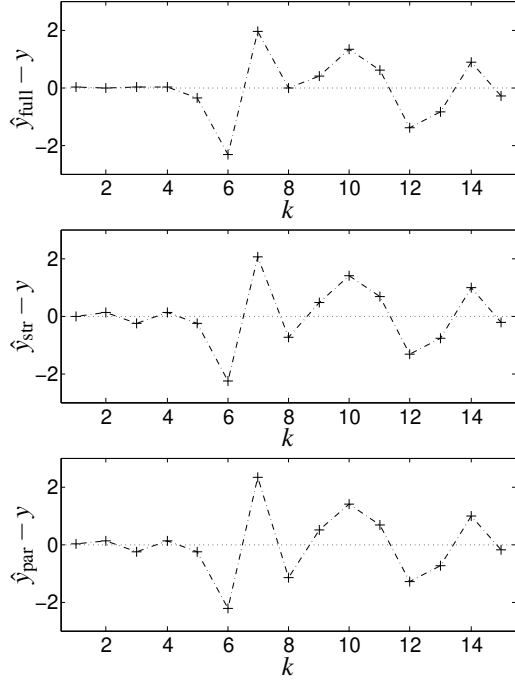
Figure 3: The difference between the estimated and measured output signals for structured identification, parameter identification, and fully parameterized identification.

$$\hat{A}_{\text{str}} = \begin{bmatrix} 10.00 & \varepsilon & \varepsilon \\ \varepsilon & 6.38 & \varepsilon \\ 16.98 & 19.93 & 1.55 \end{bmatrix}, \quad \hat{B}_{\text{str}} = \begin{bmatrix} 3.02 \\ 0.00 \\ 11.00 \end{bmatrix}, \quad (20)$$

$$\hat{C}_{\text{str}} = \begin{bmatrix} \varepsilon & \varepsilon & 6.40 \end{bmatrix}, \quad \hat{x}_{\text{str}}(0) = \begin{bmatrix} 1.13 \\ 15.85 \\ 38.69 \end{bmatrix}, \quad (21)$$

with an error $\|\hat{y}_{\text{tot}} - y_{\text{tot}}\|_2$ of 3.98. Note that the entries of the system matrices given by (20)–(21) are not consistent with the model (18)–(19) since, e.g., $a_{32} \neq 2a_{22}$ and $a_{33} \neq c_{13}$. Using the identification method of this paper to directly estimate the parameters of the model (18)–(19) yields

$$\hat{d}_1 = 9.28, \ \hat{d}_2 = 6.38, \ \hat{d}_3 = 1.80, \ \hat{t}_1 = 3.51, \ \hat{t}_2 = 2.83,$$

and $\hat{x}_{\text{par}}(0) = \begin{bmatrix} -5.14 & 27.62 & 43.04 \end{bmatrix}^T$ with an error of 4.23. Note that the parameters values are not estimated accurately. This is probably due to the fact that the parameters of the model cannot be identified uniquely from the input-output behavior only, and that we did not specify extra constraints on the (approximate) range or the relative order of the parameters. This issue will be a topic for further research.

The fully parameterized state space identification results in

$$\hat{A}_{\text{full}} = \begin{bmatrix} -12.21 & -18.85 & -19.55 \\ 13.89 & 5.98 & -13.38 \\ 18.91 & 16.03 & 1.79 \end{bmatrix}, \quad \hat{B}_{\text{full}} = \begin{bmatrix} 4.29 \\ -0.89 \\ 14.39 \end{bmatrix},$$

$$\hat{C}_{\text{full}} = \begin{bmatrix} -12.21 & 0.32 & 2.94 \end{bmatrix}, \quad \hat{x}_{\text{full}}(0) = \begin{bmatrix} -1.07 \\ 7.55 \\ 41.91 \end{bmatrix},$$

with an error of 3.89. Note that this realization has no physical meaning or any clear relation with the structure of the system anymore.

4 Conclusions and further research

We have presented a method to identify max-plus-linear state space models of discrete event systems from input-output data. This method does not require measurements of the state — although (partial) state measurements can easily be taken into account, — and it works for both structured and fully parameterized state space models. In order to solve optimization problem that corresponds to the max-plus-linear identification problem we have transformed it into a mixed integer quadratic programming problem.

Topics for future research include: development of more efficient algorithms for max-plus-linear state space identification, further investigation and comparison of the computational requirements of the ELCPs with and without state estimates, and development of methods to obtain good estimates for the system order based on input-output data.

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