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# CONTROL OF PWA SYSTEMS USING A STABLE RECEDING HORIZON METHOD

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Abstract: In this paper we derive stabilization conditions for the class of piecewise affine (PWA) systems using the linear matrix inequality (LMI) framework. We take into account the piecewise structure of the system and therefore the matrix inequalities that we solve are less conservative. Using the upper bound of the infinite-horizon quadratic cost as a terminal cost and constructing also a convex terminal set we show that the receding horizon control stabilizes the PWA system. We derive also an algorithm for enlarging the terminal set based on a backward procedure; therefore, the prediction horizon can be chosen shorter, removing some computations off-line.

Keywords: piecewise affine system, linear matrix inequalities, model predictive control.

## 1. INTRODUCTION

Hybrid systems model the interaction between continuous and logic components. Currently, general analysis and control design methods for hybrid systems are not yet available. For this reason, several authors have studied special subclasses of hybrid systems for which control techniques are currently being developed such as manufacturing systems (Cassandras *et al.*, 2001) and piecewise affine (PWA) systems (Bemporad and Morari, 1999; Rantzer and Johansson, 2000).

Model Predictive Control (MPC) is the most successful advanced control technology implemented in industry due to its ability to handle complex systems with hard input-output constraints. Recently, the research has focused on developing stabilizing controllers for hybrid systems and in particular for PWA systems. PWA systems are defined by partitioning the state space of the system in a finite number of polytopes and associating to each polytope a different affine dynamic. Several results about stability of PWA systems and MPC schemes for such systems can be found in the literature, e.g. (Bemporad and Morari, 1999; Rantzer and Johansson, 2000; Mignone *et al.*, 2000; Lazar *et al.*, 2004; Mayne and Rakovic, 2003; Goebel *et al.*, 2004).

One of the first results about the stability of MPC for PWA systems is obtained in (Bemporad and Morari, 1999), where a terminal equality constraint approach is presented. This type of constraint is rather restrictive. Therefore, in order to guarantee feasibility of the MPC problem we need a long prediction horizon, which results in computationally demanding optimization problems. A stabilizing terminal set and cost MPC scheme for PWA systems has been developed in (Mayne and Rakovic, 2003; Lazar *et al.*, 2004). Stability has been guaranteed in (Lazar *et al.*, 2004) using the LMI framework and by developing an algorithm for constructing a polyhedral positively invariant set for the PWL dynamics.

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In this paper we derive a stable MPC scheme using the LMI framework. For the piecewise linear (PWL) dynamics we derive LMI conditions that provide a piecewise linear feedback controller that stabilizes those dynamics. We take into account also the piecewise structure of the system; conservativeness is reduced by implementing the S-procedure. We derive a stable MPC scheme with a convex terminal set and the upper bound of the infinite-horizon quadratic cost is used as a terminal cost. We present an algorithm for enlarging this set based on a backward procedure. By enlarging the terminal set the prediction horizon can be chosen shorter. Therefore the computational complexity decreases, removing some computations off-line.

We define a PWA system as:

$$\begin{aligned} x(k+1) &= A_i x(k) + B_i u(k) + a_i, \text{ if } x(k) \in \mathcal{P}_i \\ y(k) &= C_i x(k) + c_i, \end{aligned} \tag{1}$$

where $\{\mathcal{P}_i\}_{i\in\mathcal{I}}$ is a partition of $\mathbb{R}^n$ into a number of polyhedral cells ($n, n_u, n_y$ denotes the number of states, inputs and outputs). Let $\mathcal{I}_0 \subseteq \mathcal{I}$ ($\mathcal{I}_1 \subseteq \mathcal{I}$) be the set of indexes for the cells that contain (do not contain) the origin in their closure. So $a_i = 0$ for any $i \in \mathcal{I}_0$. We assume that the closure of $\mathcal{P}_i$ can be written as: $\mathrm{cl}(\mathcal{P}_i) = \{x \in \mathbb{R}^n : \tilde{E}_i x \geq e_i\}$ with $e_i = 0$ for $i \in \mathcal{I}_0$. We define the infinite-horizon quadratic cost:

$$J_\infty(x_0, \mathbf{u}) = \sum_{k=0}^{\infty} x^T(k) Q x(k) + u^T(k) R u(k)$$

with $Q = Q^T \geq 0$, $R = R^T > 0$ and $\mathbf{u} = (u(0), u(1), ...)$ We consider the constraints:

$$U_c = \{u \in \mathbb{R}^{n_u} : |u_j| \leq u_{j,\max}, j = 1, ..., n_u\} \tag{2}$$

$$X_c = \{x \in \mathbb{R}^n : |y_j| \leq y_{j,\max}, j = 1, ..., n_y\}, \tag{3}$$

with $u_{j,\max}, y_{j,\max} > 0$.We define now the problem that we would like to solve:

*Definition 1.1.* **Problem P** Design a feedback controller $u = F(x)$ for system (1) that: (i) limits the infinite-horizon quadratic cost in a *positively invariant set* $\mathcal{E}$, i.e. $x(0) = x_0 \in \mathcal{E} \Rightarrow x(k) \in \mathcal{E}, \text{ for all } k \geq 0$. (ii) makes the closed-loop asymptotically stable, i.e. $x(\infty) = 0$. (iii) satisfies the constraints $u(k) \in U_c,\ x(k) \in X_c$ for all $k \geq 0$ . ♢

## 2. DETERMINATION OF A CONVEX INVARIANT SET AND THE CONTROLLER

First we want to derive a local controller that stabilizes the PWL dynamics of the system (i.e. for all $i \in \mathcal{I}_0$ for which $a_i = 0$) and using this controller we construct also a convex positive invariant set corresponding to these dynamics. We define the local PWL feedback controller: $u(k) = F_i x(k)$ if $x(k) \in \mathcal{P}_i$, and the piecewise quadratic function:

$$V(k) = x^T(k) P(k) x(k), \quad P(k) = P_i \text{ if } x(k) \in \mathcal{P}_i$$

We want to find $F_i, P_i \in \mathbb{R}^{n\times n},\ i \in \mathcal{I}_0$, with $P_i > 0$, such that the following conditions are fulfilled:

$$\begin{aligned} & x(\infty) = 0, \\ & V(k+1) - V(k) \leq -l(x(k), F(k)x(k)),\ \forall\, k \geq 0, \end{aligned}$$

where $l(x, u) := x^T Q x + u^T R u$. The second condition can be written as:

$$\begin{aligned} & x^T(A_i + B_i F_i)^T P_j (A_i + B_i F_i)x - x^T P_i x + x^T Q x \\ & + x^T F_i^T R F_i x \leq 0, \text{ for all } x \in \mathcal{P}_i,\ i, j \in \mathcal{I}_0. \end{aligned} \tag{4}$$

The matrix inequalities (4) must be valid only for $x \in \mathcal{P}_i$. Using the *S-procedure* (see e.g. (Jonsson, 2001)) we can rewrite them to be valid on the entire space $\mathbb{R}^n$. One method (Rantzer and Johansson, 2000) to relax the matrix inequalities (4) is: find $F_i, P_i, U_{ij},\ i, j \in \mathcal{I}_0$, with $P_i > 0$ and $U_{ij}$ having all entries non-negative that satisfy the following matrix inequalities:

$$\begin{aligned} & (A_i + B_i F_i)^T P_j (A_i + B_i F_i) - P_i + Q \\ & + F_i^T R F_i + E_i^T U_{ij} E_i \leq 0, \text{ for all } i, j \in \mathcal{I}_0. \end{aligned} \tag{5}$$

We define $X_0 = \cup_{i\in\mathcal{I}_0} \mathcal{P}_i$ and we consider an inner approximation with an ellipsoid of this set:

$$\{x \in \mathbb{R}^n : x^T H x \leq 1\} \subseteq X_0.$$

In the sequel the symbol * is used to induce a symmetric structure in an LMI. We have:

*Proposition 2.1.* (i) If the following LMIs:

$$\begin{bmatrix} P_i - Q - E_i^T U_{ij} E_i & * & * \\ A_i + B_i F_i & S_j & 0 \\ F_i & 0 & R^{-1} \end{bmatrix} > 0 \tag{6}$$

and the following bilinear matrix inequalities (BMIs)

$$S_i P_i + P_i S_i \leq 2I \tag{7}$$

have solution $P_i, S_i, F_i, U_{ij}, i, j \in \mathcal{I}_0$, with $P_i > 0$ and all entries of $U_{ij}$ non-negative, then this is also a solution of (5).

(ii) We define the set $\mathcal{E}(\rho) = \{x \in \mathbb{R}^n : x^T P_j x \leq \rho,\ j \in \mathcal{I}_0\}$, with $\rho > 0$. Note that this set is convex and contains the origin in the interior. If the following LMIs are satisfied:

$$\begin{bmatrix} \tau_j H - P_j & 0 \\ 0 & -\tau_j + \rho \end{bmatrix} \leq 0 \tag{8}$$

for all $j \in \mathcal{I}_0$ and with $\tau_j > 0$, then $\mathcal{E}(\rho)$ is a positive invariant set for the closed-loop system [2] .

(iii) If we require $u(k + l) \in U_c$ for all $l \geq 0$, once $x(k) \in \mathcal{E}(\rho)$ then an additional LMI must be satisfied:

$$\begin{bmatrix} \Lambda - E_i^T W_i E_i & F_i \\ * & P_i \end{bmatrix} \geq 0 \text{ with } \Lambda_{jj} \leq u_{j,\max}^2/\rho \tag{9}$$

for all $i \in \mathcal{I}_0$ and $j = 1, ..., n_u$, where the matrices $W_i$ have all entries non-negative.

(iv) If we require $y(k+1+l) \in X_c$ for all $l \geq 0$, once $x(k) \in \mathcal{E}(\rho)$ then the following additional LMI must be satisfied:

$$\begin{bmatrix} \Gamma - E_i^T V_i E_i & C_i(A_i + B_i F_i) \\ * & P_i \end{bmatrix} \geq 0, \Gamma_{jj} \leq \frac{y_{j,\max}^2}{\rho}$$

[2] If we assume that $X_0$ is a polytope $X_0 = \{x : c_l^T x \leq 1,\ l = 1, ..., s\}$, then the LMI (8) can be replaced with: $c_l^T \rho P_j^{-1} c_l \leq 1$, which can be written as LMI (see also (Necoara *et al.*, 2004)).

for all $i \in \mathcal{I}_0$ and $j = 1, ..., n_y$, where $V_i$ have all entries non-negative. Note that by taking $\gamma = 1/\rho$ all formulas (6)–(2) become LMIs except (7).

Proof: (i) The BMIs (7) imply that

$$0 < S_i \le P_i^{-1} \text{ if and only if } 0 < P_i \le S_i^{-1}$$

(see also (Slupphaug and Foss, 1999)). Applying now the Schur complement to (6), this leads to:

$$\begin{aligned} 0 <& P_i - Q - E_i^T U_{ij} E_i - (A_i + B_i F_i)^T S_j^{-1} (*) \\ &- F_i^T R F_i \le P_i - Q - E_i^T U_{ij} E_i \\ &- (A_i + B_i F_i)^T P_j (*) - F_i^T R F_i \end{aligned}$$

i.e. formula (5).

(ii) It is well-known (Vandenberghe and Boyd, 1996) that inclusion of ellipsoids

$$\{x : x^T P_j x \le \rho\} \subseteq \{x \in \mathbb{R}^n : x^T H x \le 1\} \subseteq X_0$$

can be expressed as an LMI (8). Then if $x(k) \in \mathcal{E}(\rho) \cap \mathcal{P}_{i_0}$, for some $i_o \in \mathcal{I}_0$ then $x^T(k) P_{i_0} x(k) \le \rho$ and $x(k+1) = (A_{i_0} + B_{i_0} F_{i_0}) x(k)$. Therefore, for any $j \in \mathcal{I}_0$ according to (i) we have:

$$x^T(k+1) P_j x(k+1) \le x^T(k) P_{i_0} x(k) \le \rho.$$

So $x(k+1) \in \mathcal{E}(\rho)$. By induction we can see that if $x(k) \in \mathcal{E}(\rho)$ then $x(k+l) \in \mathcal{E}(\rho)$ for any $l \ge 0$. So $\mathcal{E}(\rho)$ is an invariant set. $\mathcal{E}(\rho)$ is convex, because it is the intersection of ellipsoids ($P_i > 0$) and it contains the origin because each ellipsoid contains the origin.

(iii) The constraint on the input (2) is equivalent with $u_j^2(k) \le u_{j,\max}^2$. We have $\mathcal{E}(\rho) \subseteq \{x : x^T P_i x \le \rho\}$ and thus if $x(k) \in \mathcal{E}(\rho) \cap \mathcal{P}_i$ then

$$\begin{aligned} u_j^2(k) &\le \max_{x(k) \in \mathcal{E}(\rho)} (F_i x(k))_j^2 \le \max_{x^T P_i x \le \rho} (F_i x)_j^2 \\ &\le \max_{x^T \frac{P_i}{\rho} x \le 1} (F_i x)_j^2 \le \|\sqrt{\rho} (F_i P_i^{-1/2})_j\|_2^2 \\ &= \rho (F_i P_i^{-1} F_i^T)_{jj} = \rho (F_i P_i^{-1} F_i^T)_{jj} \\ &\le \rho \Lambda_{jj} \le u_{j,\max}^2. \end{aligned}$$

Taking $W_i$ with all entries non-negative and applying the S-procedure, the last inequality translates into:

$$\Lambda - F_i P_i^{-1} F_i^T - E_i^T W_i E_i \ge 0$$

and $\Lambda_{jj} \le \frac{u_{j,\max}^2}{\rho}$, which is the LMI (9) using $\gamma = \frac{1}{\rho}$.

(iv) The LMI (2) is derived in the same way. ♢

**Remark 2.2** The matrix inequalities (5) can be rewritten as BMIs by introducing dummy variables. We use the BMI formulation from Proposition 2.1 because there are algorithms in the literature (see (Fares *et al.*, 2001)) for solving BMIs in the form (6)–(7). ♢

**Remark 2.3** Using Finsler's lemma, we can provide the general solution of the LMIs (5). For a more detailed discussion about the general solution of the LMIs (5), the reader is referred to (Necoara *et al.*, 2004).

♢

If we do not apply the S-procedure for (4), i.e. we replace the condition "$x \in \mathcal{P}_i$", with $x \in \mathbb{R}^n$, then (4) becomes:

$$\begin{aligned} (A_i + B_i F_i)^T P_j (A_i + B_i F_i)& \\ - P_i + Q + F_i^T R F_i \le 0& \end{aligned} \quad (10)$$

for all $i, j \in \mathcal{I}_0$. The matrix inequalities (10) were solved in (Kothare *et al.*, 1996) for linear systems with polytopic uncertainty, making a so-called linearizing change of variables by introducing: $S_i = P_i^{-1}, F_i = Y_i S_i$. This linearization is also employed in (Lazar *et al.*, 2004) for the particular case of PWL systems. We use here another linearization of (4), namely $P_i = S_i^{-1}, F_i = Y_i G^{-1}$. Using this change of variables we see that the determination of the control law does not depend explicitly on the Lyapunov matrices $P_i$. The extra degree of freedom introduced by the matrices $G$ which is not considered symmetric, is incorporated in the control variable (see (Daafouz and Bernussou, 2001) for more details about this type of linearization).

*Proposition 2.4.* (i) If the following LMIs in $G, Y_i, S_i$

$$\begin{bmatrix} G + G^T - S_i & * & * & * \\ A_i G + B_i Y_i & S_j & * & * \\ \bar{Q}^{1/2} G & 0 & I & * \\ R^{1/2} Y_i & 0 & 0 & I \end{bmatrix} > 0 \quad (11)$$

for all $i, j \in \mathcal{I}_0$ have a solution then $F_i = Y_i G^{-1}, P_i = S_i^{-1}$ are a solution of (10).

(ii) Let $\mathcal{E}(\rho)$ be defined as in Proposition 2.1. If the following LMIs are satisfied:

$$\begin{bmatrix} \tau_j H^{-1} - S_j & 0 \\ 0 & -\tau_j + 1/\rho \end{bmatrix} \ge 0 \quad (12)$$

for all $j \in \mathcal{I}_0$ and with $\tau_j > 0$, then $\mathcal{E}(\rho)$ is a positive invariant set for the closed-loop system.

(iii) If we require $u(k+l) \in U_c$ for all $l \ge 0$, once $x(k) \in \mathcal{E}(\rho)$ then an additional LMI must be satisfied:

$$\begin{bmatrix} \Lambda & Y_i \\ * & G + G^T - S_i \end{bmatrix} \ge 0 \text{ with } \Lambda_{jj} \le u_{j,\max}^2 / \rho \quad (13)$$

for all $i \in \mathcal{I}_0$ and $j = 1, ..., n_u$.

(iv) If we require $y(k+1+l) \in X_c$ for all $l \ge 0$, once $x(k) \in \mathcal{E}(\rho)$ then the additional LMIs must be satisfied:

$$\begin{bmatrix} \Gamma & C_i(A_i G + B_i Y_i) \\ * & G + G^T - S_i \end{bmatrix} \ge 0 \text{ with } \Gamma_{jj} \le y_{j,\max}^2 / \rho. \quad (14)$$

for all $i \in \mathcal{I}_0$ and $j = 1, ..., n_y$. Note that taking $\gamma = 1/\rho$ all previous formulas become LMIs.

Proof: Basically the proof for (i) uses some matrix manipulations and the Schur complement (see also (Daafouz and Bernussou, 2001; Necoara *et al.*, 2004)). The points (ii)–(iv) can be proved using similar arguments as in Proposition 2.1 (ii)–(iv). ♢

Now we assume that by applying one of the approaches proposed before (Proposition 2.1 or Proposition 2.4) we obtained $F_i, P_i,$ for all $i \in \mathcal{I}_0$. Then we have:

*Corollary 2.5.* (i) If we consider only the PWL dynamics of the system (1), then the PWL feedback controller $u(k) = F_i x(k),\ x(k) \in \mathcal{P}_i, i \in \mathcal{I}_0$ asymptotically stabilizes these dynamics with a region of attraction $X_0$ and the infinite-horizon quadratic cost is bounded: $J_\infty(x_0) \le x_0^T P_i x_0,$ for any $x_0 \in \mathcal{P}_i$, for all $i \in \mathcal{I}_0$.

(ii) The PWL feedback controller $u(k) = F_i x(k), x(k) \in \mathcal{P}_i$ makes the origin locally asymptotically stable, with the input and output satisfying the constraints (2)–(3), and it has a region of attraction $\mathcal{E} = \cup_{i\in\mathcal{I}_0}(\{x : x^T P_i x \le \rho\} \cap \mathcal{P}_i)$, i.e. the feedback controller $u(k) = F_i x(k), x(k) \in \mathcal{P}_i$ solves locally the Problem **P**, and moreover $J_\infty(x_0) \le \rho,$ for any $x_0 \in \mathcal{E} \cap \mathcal{P}_i$.

Proof: It can be easily seen that $V(x) = x^T P_i x$, $x \in \mathcal{P}_i$ is a piecewise quadratic Lyapunov function for the closed-loop system: $x(k+1) = (A_i + B_i F_i)x(k)$, $x(k) \in \mathcal{P}_i, i \in \mathcal{I}_0$. The rest of the proof follows immediately. ♢

## 3. MODEL PREDICTIVE CONTROL LAW

### 3.1 Stable MPC

In the previous section we have found a PWL feedback controller $u(k) = F(k)x(k)$ that solves Problem **P** with a positive invariant set $\mathcal{E}$. In general this set is small in comparison with $\mathcal{E}_{\max}$, defined as the largest domain of attraction achievable by a control law solving problem **P**. In this section we show the benefits of MPC applied to solve Problem **P**.

We consider a prediction horizon $N$, we assume that at sample time $k$ the state $x(k)$ is available (i.e. can be measured or estimated), and we split the infinite-horizon cost into two parts:

$$J_\infty(x(k), \mathbf{u}) = J_N(x(k)) + J_\infty(x(k+N)).$$

From Section 2 we have available $K_i, P_i,$ for all $i \in \mathcal{I}_0$ and moreover we have obtained an upper bound for $J_\infty$: $x(k+N)^T P_i x(k+N) \ge J_\infty(x(k+N))$, if $x(k+N) \in \mathcal{P}_i$. The quasi-infinite methods replaces the second infinite term with its upper bound (Chen and Allgower, 2000; Kothare *et al.*, 1996). Then at each sample step $k$ we propose to solve the following optimization problem which will be called Problem **QI**($N$):

$$J^*(k) = \min_{\mathbf{u}_k} \sum_{j=k}^{k+N-1} x^T(j)Qx(j) + u^T(j)Ru(j) + x(k+N)^T P(k+N) x(k+N)$$

$$\text{subject to} \begin{cases} \mathbf{u}_k = (u(k), ..., u(k+N-1)) \in U_c^N \\ \text{equation } (1) \\ (y(k+1), ..., y(k+N)) \in X_c^N \\ \text{hard constraint: } x(k+N) \in \mathcal{E}(\rho), \end{cases}$$

where $P(k+N) = P_i$ if $x(k+N) \in \mathcal{P}_i$.

In the above formulation we detect the standard ingredients for a stable MPC scheme: a terminal cost and constraint set (see (Mayne *et al.*, 2000)). According to (Mayne *et al.*, 2000), ideally, the terminal cost should be the infinite-horizon cost, but in contrast to the linear case this cannot be computed explicitly due to the nonlinearity of the system. Therefore, we replace it with the upper bound that we derived in Section 2.

According to the receding horizon principle, at each step $k$ we apply to the system only the first sample:

$$u(k) = F^{\mathrm{RH},N}(x(k)) := \mathbf{u}_k^*(1).$$

Let $\mathcal{F}(N, x_0)$ be the set of all feasible inputs corresponding to **QI**($N$) and let $\mathcal{E}^{\mathrm{RH}}(N)$ be the set of initial states $x_0$ such that $\mathcal{F}(N, x_0) \ne \emptyset$. Consider the closed-loop system given by the receding horizon control:

$$\Sigma^{\mathrm{RH}} \begin{cases} x(k+1) = A_i x(k) + B_i F^{\mathrm{RH},N}(x(k)) + a_i \\ y(k) = C_i x(k) + c_i, \text{ if } x(k) \in \mathcal{P}_i. \end{cases}$$

*Proposition 3.1.* We assume that we obtained $F_i$, $P_i$, $\mathcal{E}(\rho)$ using Section 2. Then we have:

(i) $\mathcal{E}^{\mathrm{RH}}(N)$ is a positive invariant set for $\Sigma^{\mathrm{RH}}$ and

$$\mathcal{E}(\rho) \subseteq \mathcal{E}^{\mathrm{RH}}(N), \text{for all } N > 0 \qquad (15)$$

(ii) the MPC scheme corresponding to Problem **QI**($N$) asymptotically stabilizes the system (1) with $u(k) = \mathbf{u}_k^*(1)$. Therefore, this quasi-infinite receding horizon control solves Problem **P**.

(iii) $\mathcal{E}^{\mathrm{RH}}(N) \subset \mathcal{E}^{\mathrm{RH}}(N+1)$ and $\lim_{N\to\infty} \mathcal{E}^{\mathrm{RH}}(N) = \cup_{N=1}^{\infty} \mathcal{E}^{\mathrm{RH}}(N) = \mathcal{E}_{\max}$.Moreover, if there exists an $N^*$ such that $\mathcal{E}^{\mathrm{RH}}(N^*) = \mathcal{E}^{\mathrm{RH}}(N^*+1)$ then $\mathcal{E}_{\max} = \mathcal{E}^{\mathrm{RH}}(N^*)$.

Proof: (i) Let $x_0 \in \mathcal{E}^{\mathrm{RH}}(N) \cap \mathcal{P}_i$. Then the optimization problem **QI**($N$) has an optimal solution $\mathbf{u}_0^* = (u(0)^*, ..., u(N-1)^*) \in U_c^N,\ (y(1)^*, ..., y(N)^*) \in X_c^N$. At the next sample step if $x(N)^* \in \mathcal{E}(\rho) \cap \mathcal{P}_j$ with $j \in \mathcal{I}_0$, we have a feasible input: $\mathbf{u}_1 = (u(1)^*, ..., u(N-1)^*, F_j x(N)^*) \in \mathcal{F}(N, A_i x_0 + B_i F^{\mathrm{RH},N}(x_0) + a_i)$. In conclusion $x_1 = A_i x_0 + B_i F^{\mathrm{RH},N}(x_0) + a_i \in \mathcal{E}^{\mathrm{RH}}(N)$. Therefore, (applying induction) we can prove that $\mathcal{E}^{\mathrm{RH}}(N)$ is a positively invariant set for $\Sigma^{\mathrm{RH}}$. Moreover, for any $x_0 \in \mathcal{E}(\rho)$ there exists a feasible input sequence for Problem **QI**($N$), namely $(F(0)x_0, ..., F(N-1)x(N-1))$, where $F(\cdot) \in \{F_i,\ i \in \mathcal{I}_0\}$ and thus $x_0 \in \mathcal{E}^{\mathrm{RH}}(N)$, so that $\mathcal{E}(\rho) \subseteq \mathcal{E}^{\mathrm{RH}}(N),$ for all $N > 0$.

(ii) It can be proved easily using inequalities (4) that:

$$J^*(k+1) - J^*(k) \le -\|x(k)^*\|_Q^2$$

i.e. the optimal quasi-infinite cost $J^*(k)$ is a Lyapunov function for the closed-loop system, and due to the previous inequality we have asymptotic stability. Therefore, in this way we can solve Problem **P** with the feedback controller $u(k) = F^{\mathrm{RH},N}(x(k))$ and the positive invariant set $\mathcal{E}^{\mathrm{RH}}(N)$ .

(iii) Let $x_0 \in \mathcal{E}^{\mathrm{RH}}(N)$. Then $(u(0), ..., u(N-1), F(N)x^*(N)) \in \mathcal{F}(N+1, x_0)$, so that $x_0 \in$

$\mathcal{E}^{\rm RH}(N+1)$. Therefore $\mathcal{E}^{\rm RH}(N) \subseteq \mathcal{E}^{\rm RH}(N+1)$. As $N \to \infty$ the Problem **QI**($N$) becomes an infinite-horizon model predictive control problem implying that $\lim_{N\to\infty} \mathcal{E}^{\rm RH}(N) = \mathcal{E}_{\max}$.

Moreover, from the equality $\mathcal{E}^{\rm RH}(N^*) = \mathcal{E}^{\rm RH}(N^*+1)$ it follows that there does not exist a state $x_0 \notin \mathcal{E}^{\rm RH}(N^*)$ such that with a feasible input $u$ the state $x_1 \in \mathcal{E}^{\rm RH}(N^*)$. Hence, $\mathcal{E}_{\max} = \mathcal{E}^{\rm RH}(N)$. $\diamondsuit$

*3.2 Enlargement of the terminal set using backward procedure*

The optimization problem **QI**($N$) that we have to solve on-line at each sample step $k$ is nonlinear and non-convex (except in case $N = 1$ when it is convex), and the computational time increases with the prediction horizon $N$. If the terminal set is small, then we need a long prediction horizon in order to have feasibility for Problem **QI**($N$). Therefore, the optimization problem will be computationally intensive. A larger terminal set is $\mathcal{E} = \cup_{i\in\mathcal{I}_0}(\{x : x^T P_i x \le \rho\} \cap \mathcal{P}_i)$, but this is not a convex set (it is a union of convex sets). In the sequel we develop a method to enlarge the terminal set based on a *backward procedure* that can be done *off-line*, and thus we can efficiently implement the stable MPC scheme derived before using a shorter prediction horizon. So, we move some computations off-line, resulting in a more efficient on-line implementation. We consider again only the PWL dynamics of the system (1). The approach consists of 3 steps:

Step 1 Solve the following *convex* optimization:

$$\min_{G,Y_i,S_i} -\sum_{i\in\mathcal{I}} \log\det S_i$$

$$\text{subject to LMIs}: \ (11),(13),(14), \text{ for all } i,j \in \mathcal{I}_0$$

and define: $F_{i,1} = Y_i G^{-1}, P_{i,1} = S_i^{-1}$,

$$\mathcal{E}_1 = \{x \in \mathbb{R}^n : x^T P_{i,1} x \le 1,\ i \in \mathcal{I}_0\}.$$

By Proposition 2.4 for any $x \in \mathcal{E}_1$, the controller $u = F_{i,1}x$, if $x \in \mathcal{P}_i$ satisfies the input and output constraints and maintains the trajectory of the closed-loop system inside $\mathcal{E}_1$ converging to the origin.

Step 2 Using the previous terminal set $\mathcal{E}_{\rm prev} = \{x \in \mathbb{R}^n : x^T P_{i,\rm prev} x \le 1,\ i \in \mathcal{I}_0\}$, we construct a new larger terminal set $\mathcal{E}_{\rm new}$ based on a controller $F_{i,\rm new}$, that steers the system from $\mathcal{E}_{\rm new}$ but not within $\mathcal{E}_{\rm prev}$ to the last terminal set $\mathcal{E}_{\rm prev}$, by solving the *convex* optimization problem:

$$\min_{G,Y_i,S_i} -\sum_{i\in\mathcal{I}} \log\det S_i$$

$$\text{subject to} \begin{cases} \begin{bmatrix} G + G^T - S_i & * \\ A_i G + B_i Y_i & P_{j,\rm prev}^{-1} \end{bmatrix} > 0 \\ S_i \ge \tau_i P_{i,\rm prev}^{-1},\ \tau_i \ge 1 \\ \text{LMIs}: (13),(14) \text{ for all } i,j \in \mathcal{I}_0 \end{cases}$$

Proof: We denote with $P_{i,\rm new} = S_i^{-1}, F_{i,\rm new} = Y_i G^{-1}$. Applying the Schur complement to the first LMI from the previous optimization problem we have:

$$P_{i,\rm new} = S_i^{-1} \ge (A_i + B_i F_{i,\rm new}) P_{j,\rm prev} (*)^T$$

i.e. if $x_0 \in (\mathcal{E}_{\rm new} \cap \mathcal{P}_i) \setminus \mathcal{E}_{\rm prev}$ and applying the feedback controller $u_0 = F_{i,\rm new}x_0$ then $x_1 = (A_i + B_i F_{i,\rm new})x_0 \in \mathcal{E}_{\rm prev}$. The second LMI is equivalent with: $\mathcal{E}_{\rm prev} \subseteq \mathcal{E}_{\rm new} = \{x \in \mathbb{R}^n : x^T P_{i,\rm new} x \le 1,\ i \in \mathcal{I}_0\}$. The LMIs (13)–(14) guarantee that the controller $u(x) = F_{i,\rm new}x$, if $x \in \mathcal{P}_i$ satisfies the input and output constraints. Step 2 is an iterative procedure, i.e. we repeat it as long as we want, let us say $L$ times (and we stop when there is no more increase in the volume of the set $\mathcal{E}_{\rm new}$). Therefore we have available a sequence of controllers $u = F_{i,l}x$, if $x \in (\mathcal{E}_l \setminus \mathcal{E}_{l-1}) \cap \mathcal{P}_i,\ i \in \mathcal{I}_0,\ l \in \{1, \cdots, L\}$ where by definition $\mathcal{E}_0$ is the empty set.

Step 3 We want to find a piecewise quadratic terminal cost $P(x) = x^T P_i x$ if $x \in \mathcal{P}_i$ such that stability is guaranteed when we apply the MPC scheme based on Problem **QI**($N$) with the terminal set $\mathcal{E}_L$. The sequence $\{P_i\}_{i\in\mathcal{I}_0}$ is determined solving the following LMIs, with $U_{i,j}$ having all entries non-negative:

$$(A_i + B_i F_{i,l})^T P_j (A_i + B_i F_{i,l}) - P_i + Q + F_{i,l}^T R F_{i,l} + E_i^T U_{i,j} E_i \le 0 \qquad (16)$$

for all $i, j \in \mathcal{I}_0$, $l \in \{1, \cdots, L\}$ (see the proof of (ii) of Proposition 3.1 where the condition $J^*(k+1) - J^*(k) \le -l(x(k), u(k))$ is implied by the LMIs (16)).

*Corollary 3.2.* (i) The controller $u(x) = F_{i,l}x$, if $x \in (\mathcal{E}_l \setminus \mathcal{E}_{l-1}) \cap \mathcal{P}_i, l \in \{1, ..., L\}$ solves Problem **P** .

(ii) $\mathcal{E}_L$ is positive invariant for the closed-loop system.

(iii) Using $\mathcal{E}_L$ as a terminal set and the terminal cost $P(x) = x^T P_i x$ if $x \in \mathcal{P}_i$, with $P_i$ given by (16) in Problem **QI**($N$), then Proposition 3.1 still holds.

Proof: It is obvious that this controller stabilizes the system, because for any $x_0 \in \mathcal{E}_L$ in at most $L$ steps $x(L) \in \mathcal{E}_1$, and then according to Proposition 2.4 $x(L)$ will converge asymptotically towards zero. Moreover, this controller fulfills the input and output constraints. For the last part, we observe that if $x_0 \in \mathcal{E}_l \subseteq \mathcal{E}_L$, then applying this feedback controller we have $(A_i + B_i F_{i,l})x_0 \in \mathcal{E}_{l-1} \subseteq \mathcal{E}_L$. Therefore, $\mathcal{E}_L$ is a positive invariant set for the closed-loop system, and the LMIs (22) guarantee stability for the MPC scheme corresponding to Problem **QI**($N$). $\diamondsuit$

**Remark 3.3** We can use also polyhedral or union of polyhedral sets: $\cup_{i\in\mathcal{I}_0}\mathcal{E}(i)$ with $\mathcal{E}(i) = \{x \in \mathbb{R}^n : H_i x \le h_i\} \subseteq \mathcal{P}_i$ as a positive invariant terminal set. In this case Problem **QI**($N$) becomes a mixed-integer quadratic programming problem. One way of obtaining such a union of polyhedral sets is:

$$\{x : x^T P_{i,L-1} x \le 1\} \cap \mathcal{P}_i \subseteq \mathcal{E}(i) \subseteq \{x : x^T P_{i,L} x \le 1\} \cap \mathcal{P}_i$$

and then use $\cup_{i\in\mathcal{I}_0}\mathcal{E}(i)$ as a terminal set, and as terminal cost $P(x) = x^T P_{i,L} x$ if $x \in \mathcal{P}_i$, where $P_{i,L}$ are given by the LMIs (16). Finding such a set $\mathcal{E}(i)$ is an LMI problem (see (Necoara *et al.*, 2004)).

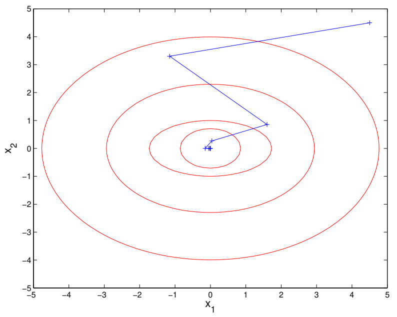


Fig. 1. Enlargement of ellipsoidal terminal set and the trajectory corresponding to MPC scheme QI(1).

**Example:** We consider the following system taken from (Bemporad *et al.*, 2000):

$$A_1 = \begin{bmatrix} 0.35 & -0.6062 \\ 0.6062 & 0.35 \end{bmatrix}, A_2 = \begin{bmatrix} 0.35 & 0.6062 \\ -0.6062 & 0.35 \end{bmatrix}$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ |x_1| \leq 5, \ |x_2| \leq 5, \ |u| \leq 1,$$

$$E_1 = [1 \ \ 0], \ E_2 = [-1 \ \ 0], \ Q = I, \ R = 0.1.$$

Iterating Step 2 for $L = 3$ we obtain the following terminal set (positive invariant set):

$$\mathcal{E}_3 = \{x \in \mathbb{R}^2 : x^T \begin{bmatrix} 0.0441 & 0 \\ 0 & 0.0627 \end{bmatrix} x \leq 1\}$$

and applying then Step 3 we obtain the terminal cost:

$$P(x) = x^T \begin{bmatrix} 6.7534 & 0 \\ 0 & 9.2863 \end{bmatrix} x.$$

For $N = 1$ the optimization problem is feasible for any $x \in [-5 \ \ 5] \times [-5 \ \ 5]$ (see also Fig. 1). Therefore, at each step we solve a convex optimization problem.

## 4. CONCLUSIONS

We have derived stabilization conditions for the class of PWA systems using the LMI framework. The LMIs are derived using the piecewise structure of the system; therefore, less conservatism is introduced in comparison with other approaches. Using the upper bound of the infinite-horizon quadratic cost as a terminal cost and constructing also a convex terminal set (ellipsoidal or polyhedral) we have shown that the quasi-infinite receding horizon control stabilizes the PWA system. We have proposed an algorithm based on a backward procedure to enlarge the terminal set in order to reduce the on-line computational complexity by off-line computations.

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