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Stable Receding Horizon Control for Max-Plus-Linear Systems

I. Necoara, B. De Schutter, T.J.J. van den Boom and J. Hellendoorn

Abstract— **We develop a stabilizing receding horizon control (RHC) scheme for the class of discrete-event systems called max-pus-linear (MPL) systems. MPL systems can be described by models that are "linear" in the max-plus algebra, which has maximization and addition as basic operations. In this paper we extend the concept of positively invariant set from classical system theory to discrete-event MPL systems. We define stability for the class of MPL systems in the sense of Lyapunov. For a particular convex piecewise affine cost function and linear input-state constraints the RHC optimization problem can be recast as a linear program. Using a dual-mode approach we are able to prove exponential stability of the RHC scheme. We derive also a constrained time-optimal controller by solving a sequence of parametric linear programs.**

I. Introduction

In the last decades Receding Horizon Control (RHC) or Model Predictive Control (MPC) [1], [2] has gained wide acceptance in the process industry. An important advantage of RHC is that the use of a finite horizon allows the inclusion of constraints on the inputs and states. Recently, the RHC approach was extended to a class of discrete-event systems (DES) called max-plus-linear (MPL) systems [3]. MPL systems are linear in the max-plus algebra [4] and they usually arise in the context of manufacturing systems, telecommunication networks, railway networks, parallel computing etc. Several authors have already developed methods to compute optimal controllers for MPL systems [3], [5]–[8]. The main advantage of the RHC scheme presented in this paper is that it allows to include linear constraints on inputs and states and the RHC controller guarantees a priori stability of the closed-loop system.

We start the paper with an introduction of the main concepts from max-plus algebra. We introduce stability in the sense of Lyapunov for the class of MPL systems, using similar concepts as in [9]. In Section II we take into account constraints on input and states. We define the concept of positively invariant (PI) set for the class of MPL systems. We prove that, under some mild conditions, the PI set is a polyhedron. For a particular convex piecewise affine cost function, we prove that the MPL-RHC optimization problem can be recast as a linear program (LP). Using a dual-mode approach [1] we prove that the RHC controller stabilizes in the sense of Lyapunov the MPL system. In Section III we derive a time-optimal controller using parametric linear programming. We conclude with an example.

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A. Max-Plus Algebra

Define $\varepsilon := -\infty$ and $\mathbb{R}_\varepsilon := \mathbb{R} \cup \{\varepsilon\}$. The max-plus-algebraic (MPA) addition ($\oplus$) and multiplication ($\otimes$) are defined as [4]: $x \oplus y := \max\{x,y\}$, $x \otimes y := x+y$, for $x,y \in \mathbb{R}_\varepsilon$. For matrices $A,B \in \mathbb{R}_\varepsilon^{m\times n}$ and $C \in \mathbb{R}_\varepsilon^{n\times p}$ one can extend the definition as follows: $(A\oplus B)_{ij} := A_{ij} \oplus B_{ij}, (A\otimes C)_{ij} := \bigoplus_{k=1}^{n} A_{ik} \otimes C_{kj}$, $\forall\ i,\ j$. The matrix $\mathcal{E}$ denotes the MPA zero matrix of appropriate dimension: $\mathcal{E}_{ij} := \varepsilon,\ \forall\ i,\ j$ and E_n is the $n\times n$ MPA identity matrix: $(E_n)_{ii} := 0, \forall\ i$ and $(E_n)_{ij} := \varepsilon$, $\forall\ i,j$ with $i \neq j$. For any matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$, let $A^{\otimes^k} := A\otimes A\otimes \cdots \otimes A$ (i.e. the kth MPA power of A) and define A^*, whenever it exists, by $A^* := E_n \oplus A \oplus \cdots \oplus A^{\otimes^k} \oplus \cdots$ Given a vector $x \in \mathbb{R}_\varepsilon^n$ we denote with $\|x\|_\oplus := \max\{x_1 \cdots x_n\}$. For a positive integer n, we denote with $\underline{n} := \{1,2,\cdots,n\}$. A matrix $\Gamma \in \mathbb{R}_\varepsilon^{n\times m}$ is *row-finite* if for any row $i \in \underline{n}$, $\max_{j\in \underline{m}} \Gamma_{ij} > \varepsilon$; *column-finite* is similarly defined.

We denote with $x \oplus' y := \min\{x,y\}$ and $x \otimes' y := x+y$ (the operations $\otimes$ and $\otimes'$ differ only in that $(-\infty)\otimes(+\infty) := -\infty$, while $(-\infty)\otimes'(+\infty) := +\infty$). The matrix multiplication and addition for $(\oplus',\otimes')$ are defined similarly as for $(\oplus,\otimes)$. It can be shown that for any matrices A,B and any vectors x,y of appropriate dimensions over $\mathbb{R}_\varepsilon$ we have [10]:

$$A\otimes'(B\otimes x) \geq (A\otimes' B)\otimes x,\ ((-A^T)\otimes' A)\otimes x \geq x, \quad \text{(1a)}$$

$$x \leq y \Rightarrow\ A\otimes x \leq A\otimes y \text{ and } A\otimes' x \leq A\otimes' y. \quad \text{(1b)}$$

Lemma 1.1: [4] (i) The inequality $A\otimes x \leq b$ has the largest solution given by $x_{\text{opt}} = (-A^T)\otimes' b = -(A^T\otimes(-b))$ (by the largest solution we mean that for all x satisfying $A\otimes x \leq b$ we have $x \leq x_{\text{opt}}$).

(ii) The equation $x = A\otimes x \oplus b$ has a solution $x = A^*\otimes b$. If $A_{ij} < 0$ for all i,j, then the solution is unique.

B. Max-Plus-Linear Systems

DES with only synchronization and no concurrency can be modeled by an MPA model of the following form [4]:

$$\begin{cases} x_{\text{sys}}(k) = A_{\text{sys}} \otimes x_{\text{sys}}(k-1) \oplus B_{\text{sys}} \otimes u_{\text{sys}}(k), \\ y_{\text{sys}}(k) = C_{\text{sys}} \otimes x_{\text{sys}}(k) \end{cases} \quad (2)$$

where $x_{\text{sys}}(k) \in \mathbb{R}_\varepsilon^n$ represents the state, $u_{\text{sys}}(k) \in \mathbb{R}_\varepsilon^m$ is the input, $y_{\text{sys}}(k) \in \mathbb{R}_\varepsilon^p$ is the output and where $A_{\text{sys}} \in \mathbb{R}_\varepsilon^{n\times n}$, $B_{\text{sys}} \in \mathbb{R}_\varepsilon^{n\times m}$, $C_{\text{sys}} \in \mathbb{R}_\varepsilon^{p\times n}$ are the system matrices[1]. Since the states and the inputs represent times, typical constraints

[1]We may assume without loss of generality that B_{sys} is column-finite and C_{sys} is row-finite.

for MPL systems are (see [8] for more details):

$$\begin{cases} u_{\rm sys}(k+1)-u_{\rm sys}(k)\geq 0, \\ H_{\rm sys}x_{\rm sys}(k)+G_{\rm sys}u_{\rm sys}(k)\leq h_{\rm sys}(k) \end{cases} \tag{3}$$

Let λ^* be the largest MPA eigenvalue of $A_{\rm sys}$ (see [4] for an appropriate definition). We consider a reference signal that the output should track of the form:

$$r_{\rm sys}(k) = y_{\rm sys,t} + k\rho. \tag{4}$$

Since through the term $B_{\rm sys}\otimes u_{\rm sys}$ it is only possible to create delays in the starting times of activities, we should choose the growth rate of the due dates such that is larger than the growth rate of the system, i.e. $\rho \geq \lambda^*$. If $\lambda^* > \varepsilon$ (in practical applications we even have $\lambda^*\geq 0$) then there exists an MPA invertible matrix $P\in\mathbb{R}_\varepsilon^{n\times n}$ such that the matrix $\bar{A} = P^{\otimes^{-1}}\otimes A_{\rm sys}\otimes P$ satisfies $\bar{A}_{ij}\leq\lambda^*$, for all $i,j\in\underline{n}$ [11] ($P^{\otimes^{-1}}$ denotes the MPA inverse of P). We make the following change of coordinates: $\bar{x}(k) = P^{\otimes^{-1}}\otimes x_{\rm sys}(k)$. We denote with $\bar{B} = P^{\otimes^{-1}}\otimes B_{\rm sys}$, $\bar{C} = C_{\rm sys}\otimes P$ and $\bar{y}(k) = y_{\rm sys}(k)$, $\bar{u}(k) = u_{\rm sys}(k)$. In the new coordinates the system (2) becomes:

$$\bar{x}(k) = \bar{A}\otimes\bar{x}(k-1)\oplus\bar{B}\otimes\bar{u}(k),\ \bar{y}(k) = \bar{C}\otimes\bar{x}(k)$$

We now consider the *normalized* system: $x(k) = \bar{x}(k) - \rho k$, $u(k) = \bar{u}(k)-\rho k$, $y(k) = \bar{y}(k)-\rho k$, $A = \bar{A}-\rho$ (i.e. by subtracting in the conventional algebra all entries of $\bar{x},\bar{u},\bar{y}$ and of $\bar{A}$ by ρk and ρ, respectively) and $B = \bar{B}$, $C = \bar{C}$. The normalized system can be written as:

$$x(k) = A\otimes x(k-1)\oplus B\otimes u(k) \tag{5a}$$

$$y(k) = C\otimes x(k). \tag{5b}$$

We assume that in the new coordinates, the constraint (3) becomes:

$$u(k+1)-u(k)\geq -\rho,\ \ Hx(k)+Gu(k)\leq h. \tag{6}$$

The following assumption will be used throughout the paper:

Assumption **A**: We consider that $\rho > \lambda^* \geq 0$, the system is controllable and observable[2] and $H\geq 0$ in (6).

The conditions from Assumption **A** are quite weak and are usually met in applications. Note that ρ can be chosen arbitrarily close to λ^*. From Assumption **A** it follows that $A_{ij} < 0$, for all $i,j\in\underline{n}$. In the new coordinates the output should be regulated to the desired target $y_{\rm t} := y_{\rm sys,t}$.

Since $A_{ij}<0$ for all $i,j\in\underline{n}$, $A^* = E_n\oplus A\oplus\cdots\oplus A^{\otimes^{n-1}}$ (see [4]). Note that for any finite vector u there exists a state equilibrium x (i.e. $x = A\otimes x\oplus B\otimes u$), given by $x = A^*\otimes B\otimes u$. Note that x is unique (according to Lemma 1.1 (ii)) and finite (due to controllability assumption). We associate to $y_{\rm t}$ the largest[3] equilibrium pair $(x_{\rm e},u_{\rm e})$ satisfying $C\otimes x_{\rm e}\leq y_{\rm t}$. From the previous discussion and taking into account that the system is observable it follows that $(x_{\rm e},u_{\rm e})$ is unique, finite and given by (see also [8]):

$$u_{\rm e} = (-(C\otimes A^*\otimes B))^T\otimes' y_{\rm t},\ x_{\rm e} = A^*\otimes B\otimes u_{\rm e} \tag{7}$$

[2]See [4] for appropriate definitions for observability and controllability.

[3]By the largest we mean that any other feasible equilibrium pair (x,u) satisfies $x\leq x_{\rm e}, u\leq u_{\rm e}$.

Throughout the paper $\|\cdot\|_\infty$ denotes the ∞-norm ($\|x\|_\infty := \max_{i\in\underline{n}}|x_i|$).

C. Lyapunov stability for MPL systems

In this section we adopt the formulation developed in [9] to the study of stability of MPL systems. Let d_∞ denote the metric on $\mathbb{R}^n$ induced by ∞-norm. Given a set $\mathscr{O}\subset\mathbb{R}^n$ then $d_\infty(x_0,\mathscr{O}) = \min_{x\in\mathscr{O}}\|x_0-x\|_\infty$ denotes the distance from a point x_0 to the set $\mathscr{O}$. An *r-neighborhood* of a set $\mathscr{O}$ is defined as the set $\mathscr{N}(\mathscr{O},r) = \{x : 0 < d_\infty(x,\mathscr{O}) < r\}$, where $r>0$. Given an MPL system (2) in closed-loop with a feedback law $\mu(x)$, we study the stability properties of the closed-loop system:

$$x(k) = A\otimes x(k-1)\oplus B\otimes\mu(x(k-1)) \tag{8}$$

The set $\mathscr{O}$ is called *positive invariant* for the system (8) if for all $x\in\mathscr{O}$ it follows that $A\otimes x\oplus B\otimes\mu(x)\in\mathscr{O}$.

Definition 1.2: A closed invariant set $\mathscr{O}$ is called *stable in the sense of Lyapunov* for the system (8) if for any $\theta>0$, there exists a $\delta>0$ such that for all $x(0)$ satisfying $d_\infty(x(0),\mathscr{O})<\delta$ we have $d_\infty(x(k),\mathscr{O})<\theta$, for all $k\geq 0$. If, furthermore, $d_\infty(x(k),\mathscr{O})\to 0$ as $k\to\infty$, then $\mathscr{O}$ is *asymptotically stable* for (8). In the case when $d_\infty(x(k),\mathscr{O})\leq c\gamma^{-\alpha k}d_\infty(x(0),\mathscr{O})$ for some $c,\alpha>0$ and $0<\gamma<1$, then the set $\mathscr{O}$ is *exponentially stable*. ◊

The following theorem gives sufficient conditions for exponential stability.

Theorem 1.3: [9] The closed invariant set $\mathscr{O}$ is exponentially stable, if in a sufficient small neighborhood $\mathscr{N}(\mathscr{O},r)$ of the set $\mathscr{O}$ there exists a functional V with the following properties:

(i) $c_1 d_\infty(x,\mathscr{O})\leq V(x)\leq c_2 d_\infty(x,\mathscr{O})$, for all $x\in\mathscr{N}(\mathscr{O},r)$

(ii) $V(x(k+1))-V(x(k))\leq -c_3 d_\infty(x(k),\mathscr{O})$ for $x(0)\in\mathscr{N}(\mathscr{O},r)$, for all $k\geq 0$ provided that $x(k)\in\mathscr{N}(\mathscr{O},r)$, where c_1,c_2 and c_3 are positive constants and $0<\frac{c_3}{c_2}<1$. ◊

II. Stabilizing RHC: constrained case

The main advantage of RHC is that it can accommodate constraints on states and inputs. In this section we derive a stabilizing RHC scheme for MPL systems (5a)–(5b) where we consider constraints of the type (6), using a dual-mode approach as in [1].

A. Maximal invariant set $\mathscr{O}_\infty$

We consider the normalized MPL system (5a)–(5b) together with the constraints (6). We may assume that the equilibrium pair $(x_{\rm e},u_{\rm e})$ defined in (7) satisfies the constraints (6) (otherwise $(x_{\rm e},u_{\rm e})$ is determined as the optimal solution of the following linear programming problem: $\max_u\sum_i u_i$, s.t. $x = A^*\otimes B\otimes u, C\otimes x\leq y_{\rm t}, Hx+Gu\leq h$).

We consider the following closed-loop system:

$$x(k) = A\otimes x(k-1)\oplus B\otimes u_{\rm e}. \tag{9}$$

In [8] it is proved that $\mathscr{O} = \{x_{\rm e}\}$ is asymptotically stable for the closed-loop system (9). We define the state constraint set associated to the closed-loop system (9)

$$\mathscr{O}_0 = \{x\in\mathbb{R}^n : Hx+Gu_{\rm e}\leq h\} \tag{10}$$

We define recursively for all $k \geq 1$ the sets

$$\mathcal{O}_k = \{x \in \mathcal{O}_0 : A \otimes x \oplus B \otimes u_e \in \mathcal{O}_{k-1}\} \tag{11}$$

It is trivial to see that $\mathcal{O}_k \subseteq \mathcal{O}_{k-1} \subseteq \cdots \subseteq \mathcal{O}_1 \subseteq O_0$. Therefore, the limit of $\mathcal{O}_k$ exists and we have

$$\mathcal{O}_\infty = \bigcap_{k \geq 0} \mathcal{O}_k = \lim_{k \to \infty} \mathcal{O}_k \tag{12}$$

By induction we can prove that $x_e \in \mathcal{O}_k$, for all $k \geq 0$ and therefore $x_e \in \mathcal{O}_\infty$ i.e. $\mathcal{O}_\infty$ is non-empty.

Lemma 2.1: If Assumption **A** is satisfied then $\mathcal{O}_k$ is a polyhedral set having the form

$$\mathcal{O}_k = \{x \in \mathbb{R}^n : H_k x \leq h_k\} \tag{13}$$

with the matrix $H_k \geq 0$.

Proof: For $k = 0$ the statement is obvious (see Assumption **A**). Let us assume that $\mathcal{O}_{k-1} = \{x \in \mathbb{R}^n : H_{k-1}x \leq h_{k-1}\}$, with $H_{k-1} \geq 0$ and we prove that $\mathcal{O}_k$ has a similar form. Since $A \otimes x \oplus B \otimes u_e$ is a "max" expression of the form $[\max_j\{a_{ij} + x_j, c_i\}]_i$ for some $a_{ij} \in \mathbb{R}_\varepsilon$ and a constant vector c, it is straightforward to show that the inequality $H_{k-1}(A \otimes x \oplus B \otimes u_e) \leq h_{k-1}$ can be rewritten in the form $\bar{H}_k x \leq \bar{h}_k$, with $\bar{H}_k \geq 0$. Then, $H_k = [H_{k-1}^T \; \bar{H}_k^T]^T \geq 0$ and $h_k = [h_{k-1}^T \; \bar{h}_k^T]^T$. ∎

From the previous lemma it is clear that the set $\mathcal{O}_\infty$ is convex (it is a countable intersection of polyhedral sets). We derive now conditions when $\mathcal{O}_\infty$ is a polyhedron.

Theorem 2.2: (i) If there exists a t^* such that $\mathcal{O}_{t^*} = \mathcal{O}_{t^*+1}$ then $\mathcal{O}_\infty = \mathcal{O}_{t^*}$ (i.e. $\mathcal{O}_\infty$ is *finitely determined* and it is a polyhedral set).

(ii) The set $\mathcal{O}_\infty$ is the *maximal* positively invariant set for (9) contained in $\mathcal{O}_0$.

Proof: (i) Let us assume that there exists a t^* such that $\mathcal{O}_{t^*} = \mathcal{O}_{t^*+1}$. It is obvious that $\mathcal{O}_{t^*+2} \subseteq \mathcal{O}_{t^*+1}$. Moreover, for any $x \in \mathcal{O}_{t^*+1}$ it follows that $A \otimes x \oplus B \otimes u_e \in \mathcal{O}_{t^*} = \mathcal{O}_{t^*+1}$, i.e. $x \in \mathcal{O}_{t^*+2}$. In conclusion, $\mathcal{O}_{t^*+1} \subseteq \mathcal{O}_{t^*+2}$ and thus $\mathcal{O}_{t^*+2} = \mathcal{O}_{t^*+1} = \mathcal{O}_{t^*}$. Iterating this procedure and using (12) we conclude that $\mathcal{O}_\infty = \mathcal{O}_{t^*}$.

(ii) Let $T \subseteq \mathcal{O}_0 = \{x : H_0 x \leq h_0\}$ be a positive invariant set for (9) and let $x \in T$. Then from the definition of a positively invariant set we have $H_0(A \otimes x \oplus B \otimes u_e) \leq h_0$. This implies that $x \in \mathcal{O}_1$ (according to the recursion (11)). Therefore, $T \subseteq \mathcal{O}_1$. By iterating this procedure we obtain that $T \subseteq \mathcal{O}_k$ for all $k \geq 0$. In conclusion, for any positive invariant set T it follows that $T \subseteq \mathcal{O}_\infty$ and thus $\mathcal{O}_\infty$ is maximal. ∎

From Theorem 2.2 we have obtained that if $\mathcal{O}_\infty$ is finitely determined then $\mathcal{O}_\infty$ is a polyhedron of the form $\mathcal{O}_\infty = \{x \in \mathbb{R}^n : H_\infty x \leq h_\infty\}$, where $H_\infty \geq 0$. Now, we give sufficient conditions under which the set $\mathcal{O}_\infty$ is finitely determined. Note that the recursive relation (11) can be written equivalently as

$$\mathcal{O}_k = \{x \in \mathcal{O}_{k-1} : H(A^{\otimes^k} \otimes x \oplus A^{\otimes^{k-1}} \otimes B \otimes u_e \oplus \cdots \oplus B \otimes u_e) + G u_e \leq h\}. \tag{14}$$

Theorem 2.3: Suppose that there exists a positive integer t_0 and $a \in \mathbb{R}^n$ such that $\mathcal{O}_{t_0} \subseteq \{x \in \mathbb{R}^n : x \leq a\}$. Then, $\mathcal{O}_\infty$ is finitely determined (i.e. $\exists\, t^*$ such that $\mathcal{O}_\infty = \mathcal{O}_{t^*}$).

Proof: Since $A_{ij} < 0, \forall i, j$ it follows that for all $x \in \mathbb{R}^n$: $A^{\otimes^k} \otimes x \to \varepsilon$ as $k \to \infty$. Moreover, for any $b \in \mathbb{R}^n$ we have: $b \oplus A \otimes b \oplus \cdots \oplus A^{\otimes^{k+n}} \otimes b = A^* \otimes b$, for all $k \geq 0$. Since $x_e = A^* \otimes B \otimes u_e$ is finite, there exists $t^* \geq \max\{n, t_0\}$ such that $A^{\otimes^k} \otimes a \leq x_e$, for all $k \geq t^*$. We show that $\mathcal{O}_{t^*} = \mathcal{O}_{t^*+1}$. Since $\mathcal{O}_{t^*+1} \subseteq \mathcal{O}_{t^*}$, to complete the proof we now show that the other inclusion is also valid, i.e. $\mathcal{O}_{t^*} \subseteq \mathcal{O}_{t^*+1}$.

Let $x \in \mathcal{O}_{t^*} \subseteq \mathcal{O}_{t_0} \subseteq \{x \in \mathbb{R}^n : x \leq a\}$. Then, $A^{\otimes^{t^*+1}} \otimes x \leq A^{\otimes^{t^*+1}} \otimes a \leq x_e$. It follows that: $H(A^{\otimes^{t^*+1}} \otimes x \oplus A^{\otimes^{t^*}} \otimes B \otimes u_e \oplus \cdots \oplus B \otimes u_e) = H(A^{\otimes^{t^*+1}} \otimes x \oplus A^* \otimes B \otimes u_e) = H x_e \leq h - G u_e$, i.e. $x \in \mathcal{O}_{t^*+1}$. ∎

It is often the case that the set $\mathcal{O}_0$ can be written as $\mathcal{O}_0 = \{x \in \mathbb{R}^n_\varepsilon : x_i \leq a_i^0, \text{ for } i = 1, \cdots, n\}$, where a_i^0 is either a finite number or $+\infty$ (when there are no restrictions on x_i). Then, we can prove that all the sets $\mathcal{O}_k$ can be written in a similar form $\mathcal{O}_k = \{x \in \mathbb{R}^n_\varepsilon : x_i \leq a_i^k, \text{ for } i = 1, \cdots, n\}$, where a_i^k is either a finite number or $+\infty$ (i.e. every $\mathcal{O}_k$ is described by at most n inequalities). We prove this by induction. For $k = 0$ this statement is true. Let us assume that $\mathcal{O}_k = \{x \in \mathbb{R}^n_\varepsilon : x_i \leq a_i^k, \text{ for } i = 1, \cdots, n\}$ and we prove that $\mathcal{O}_{k+1}$ has a similar form. We denote with $a^k = [a_1^k \cdots a_n^k]^T$. From the recursive relation (11) we have:

$$\mathcal{O}_{k+1} = \{x \in \mathbb{R}^n_\varepsilon : x \leq a^k, A \otimes x \leq a^k\} = \{x \in \mathbb{R}^n_\varepsilon : x \leq a^k, x \leq (-A^T) \otimes' a^k\} = \{x \in \mathbb{R}^n_\varepsilon : x \leq a^{k+1}\}$$

where $a^{k+1} = \min\{a^k, (-A^T) \otimes' a^k\}$. We conclude that $\mathcal{O}_\infty$ is described by at most n inequalities and in fact $\mathcal{O}_\infty = \{x \in \mathbb{R}^n_\varepsilon : x \leq a^\infty\}$ where a_i^∞ is either in $\mathbb{R}$ or equal to $+\infty$ for any $i = 1, \cdots, n$.

Note that the results obtained in this section concerning the maximal positively invariant set $\mathcal{O}_\infty$ for the MPL system (9) are similar to the one obtained in [12] for the linear case.

B. Stable constrained RHC

In this section it is assumed that the maximal positively invariant set $\mathcal{O}_\infty = \{x \in \mathbb{R}^n : H_\infty x \leq h_\infty\}$ is available , where $H_\infty \geq 0$. We give now a lemma that will be used in the sequel:

Lemma 2.4: (i) Let $X_f = \{x \in \mathbb{R}^n : Px \leq q\}$, where $P \geq 0$. Then,

$$d_\infty(x_0, X_f) = \min_{x \in X_f} \max\{\|x_0 - x\|_\oplus, \; 0\}$$

(ii) In particular if $X_f(\alpha) := \{x \in \mathbb{R}^n : x \leq \alpha\}$ then

$$d_\infty(x_0, X_f(\alpha)) = \max\{\|x_0 - \alpha\|_\oplus, 0\}$$

Proof: (i) It is straightforward to see that the statement is true when $x_0 \in X_f$. Therefore, we consider the case when $x_0 \notin X_f$, i.e. $d_\infty(x_0, X_f) > 0$. We prove this case by contradiction. Let $x^* \in X_f$ be the optimal solution, i.e. $0 < d_\infty(x_0, X_f) = \|x_0 - x^*\|_\infty$. We define the set $\mathcal{I} \subseteq \underline{n}$ as follows: if $i \in \mathcal{I}$ then $\|x_0 - x^*\|_\infty = x_i^* - (x_0)_i > 0$ and for any $j \in \underline{n} \setminus \mathcal{I} : \|x_0 - x^*\|_\infty > (x_0)_j - x_j^*$; otherwise, if such $\mathcal{I}$ does not exists, then define $\mathcal{I} = \emptyset$.

Assume that $\mathcal{I} \neq \emptyset$. Then, we define x_{feas} as: $(x_{\text{feas}})_i = (x_0)_i$, if $i \in \mathcal{I}$ and $(x_{\text{feas}})_i = x_i^*$, if $i \notin \mathcal{I}$. Since $P \geq 0$ and $x_{\text{feas}} \leq x^*, x_{\text{feas}} \neq x^*$ it follows that $x_{\text{feas}} \in X_f$. Moreover, $0 < d_\infty(x_0, X_f) = \|x_0 - x^*\|_\infty = \max_{i \in \underline{n}}\{x_i^* - (x_0)_i, (x_0)_i -$

$x_i^*\} \leq \|x_{\text{feas}} - x_0\|_\infty = \max_{i \notin \mathscr{I}}\{x_i^* - (x_0)_i, (x_0)_i - x_i^*, 0\} < \max_{i \in \underline{n}}\{x_i^* - (x_0)_i, (x_0)_i - x_i^*\} = \|x_0 - x^*\|_\infty$ i.e. a contradiction. Therefore, $\mathscr{I} = \emptyset$ and then $\|x_0 - x^*\|_\infty = \|x_0 - x^*\|_\oplus$.

(ii) If $x_0 \notin X_{\text{f}}(\alpha)$ and $x \leq \alpha$, the following inequality is valid: $\max_{i \in \underline{n}}\{(x_0)_i - x_i\} \geq \max_{i \in \underline{n}}\{(x_0)_i - \alpha_i\}$. We conclude that $\min_{x \in X_{\text{f}}(\alpha)} \max_{i \in \underline{n}}\{(x_0)_i - x_i\} \geq \max_{i \in \underline{n}}\{(x_0)_i - \alpha_i\}$. From (i) it follows that $d_\infty(x_0, X_{\text{f}}(\alpha)) \geq \max_{i \in \underline{n}}\{(x_0)_i - \alpha_i\} = \|x_0 - \alpha\|_\oplus$ (according to the first part of this lemma). But $d_\infty(x_0, X_{\text{f}}(\alpha)) \leq \|x_0 - \alpha\|_\oplus$ since $\alpha \in X_{\text{f}}(\alpha)$. It follows that $d_\infty(x_0, X_{\text{f}}(\alpha)) = \|x_0 - \alpha\|_\oplus$. ■

For initial conditions $x(0), u(0)$ and a future input sequence $\tilde{u} = (u(1) \cdots u(N))$, the following cost function is introduced:

$$J(x(0), \tilde{u}) = \sum_{j=0}^{N} d_\infty(x(j), \mathscr{O}_\infty) + \beta \|u(j) - u_{\text{e}}\|_\infty,$$

where $\beta > 0$ and N is the prediction horizon. Usually, it is the case that $\mathscr{O}_\infty = \{x : x \leq a_\infty\}$. Then, from Lemma 2.4 we have that $J(x(0), \tilde{u}) = \sum_{j=0}^{N} \max\{\|x(j) - a_\infty\|_\oplus, 0\} + \beta \|u(j) - u_{\text{e}}\|_\infty$. In the context of manufacturing systems the first term expresses the tardiness with respect to a_∞, while the second term penalizes the delay with respect to u_{e}.

Since we want to feed raw material as late as possible, we impose the constraint $u(k) \geq u_{\text{e}}$ for all $k \geq 1$. For simplicity, we assume that $B \otimes u_{\text{e}}$ is a finite vector. We have that $x(k) \geq B \otimes u_{\text{e}}$ for all $k \geq 1$. In conclusion, $\mathscr{O}_\infty \cap \{x : x \geq B \otimes u_{\text{e}}\}$, which is bounded, is in fact an invariant set for (9). Given $x(k-1)$ and $u(k-1)$, the RHC optimization problem at stage $k-1$ is defined as follows:

$$J^*(x(k-1)) = \min_{\tilde{u}(k)} J(x(k-1), \tilde{u}(k)) \tag{15}$$

$$\text{s.t.} \begin{cases} u(k+j|k-1) - u(k+j-1|k-1) \geq -\rho \\ u(k+j|k-1) \geq u_{\text{e}} \\ Hx(k+j|k-1) + Gu(k+j|k-1) \leq h, \forall j \in \{0, \cdots, N-1\} \\ u_{\text{e}} - u(k+N-1|k-1) \geq -\rho \\ x(k+N-1|k-1) \in \mathscr{O}_\infty \end{cases}$$

where $x(k+j|k-1)$ is the system state at $k+j$ as predicted at $k-1$, based on (5a)–(5b), $x(k-1|k-1) = x(k-1), u(k-1|k-1) = u(k-1)$ and the future input sequence $\tilde{u}(k) := (u(k|k-1) \cdots u(k+N-1|k-1))$. By including extra variables and using Lemma 2.4, the entire optimization problem can be written as a linear program. We apply the optimal controller in a receding horizon fashion: at event k we apply $u^{\text{RHC}}(k) := u^*(k|k-1)$ to the system (5a)–(5b), where $\tilde{u}^*(k)$ is the optimal solution of (15). Recall that the set $\mathscr{O}_\infty \cap \{x : x \geq B \otimes u_{\text{e}}\}$ is bounded. We can derive the following lemma:

Lemma 2.5: There exist $r > 0$ and $c_2 > 1$ such that for all $x \in \mathscr{N}(\mathscr{O}_\infty, r)$ we have

$$d_\infty(x, \mathscr{O}_\infty) \leq J^*(x) \leq c_2 d_\infty(x, \mathscr{O}_\infty)$$

Proof: Let us take $r > 0$. The following facts are easy to prove.

Fact 1: For any finite vectors x, u, y, v and matrices A, B satisfying Assumption 1 we have:

$$\|A \otimes x \oplus B \otimes u - A \otimes y \oplus B \otimes v\|_\infty \leq \|x - y\|_\infty \oplus \|u - v\|_\infty$$

Fact 2: It is well-known (see [13], [14]) that the optimal RHC solution of (15) is a piecewise affine function of the current state $x(k-1)$: $u^*(k+j|k-1) = \mu(x(k-1))$, for all $j \in \{0, \cdots, N-1\}$, where $\mu(\cdot)$ is a piecewise affine function.

Fact 3: Given a polytope $\mathscr{P} \subseteq \mathscr{N}(\mathscr{O}_\infty, r) \cap \{x : x \geq B \otimes u_{\text{e}}\}$ then, exists a $c > 0$ such that $\|Fx + g\|_\infty \leq c d_\infty(x, \mathscr{O}_\infty)$, $\forall x \in \mathscr{P}$ for any matrix F and vector g. (Indeed, the functions $x \longmapsto \|Fx + g\|_\infty$ and $x \longmapsto d_\infty(x, \mathscr{O}_\infty)$ are continuous on the compact set $\mathscr{P}$. Moreover, $d_\infty(x, \mathscr{O}_\infty) > 0$ for all $x \in \mathscr{P}$. Then, the function $x \longmapsto \frac{\|Fx+g\|_\infty}{d_\infty(x, \mathscr{O}_\infty)}$ is continuous on the compact set $\mathscr{P}$. From the Weierstrass theorem, this function is bounded. Therefore, there exists a $c > 0$ such that $\frac{\|Fx+g\|_\infty}{d_\infty(x, \mathscr{O}_\infty)} \leq c$ for all $x \in \mathscr{P}$.)

Using Fact 2 and Fact 3 we conclude that $\|u^*(k+j|k-1) - u_{\text{e}}\|_\infty \leq \bar{c}_j d_\infty(x(k-1), \mathscr{O}_\infty)$ for all $x(k-1) \in \mathscr{N}(\mathscr{O}_\infty, r)$, where $\bar{c}_j > 0$.

From Fact 1 and Fact 3 we have

$$\begin{aligned} &d_\infty(x(k+j|k-1), \mathscr{O}_\infty) = \min_{x \in \mathscr{O}_\infty} \|A^{\otimes^{j+1}} \otimes x(k-1) \oplus \\ &A^{\otimes^j} \otimes B \otimes u^*(k|k-1) \oplus \cdots \oplus B \otimes u^*(k+j|k-1) - x\|_\infty \leq \\ &\min_{x \in \mathscr{O}_\infty} \|A^{\otimes^{j+1}} \otimes x(k-1) \oplus A^{\otimes^j} \otimes B \otimes u^*(k|k-1) \oplus \cdots \oplus \\ &B \otimes u^*(k+j|k-1) - A^{\otimes^{j+1}} \otimes x \oplus A^{\otimes^j} \otimes B \otimes u_{\text{e}} \oplus \cdots \oplus \\ &B \otimes u_{\text{e}}\|_\infty \leq d_\infty(x(k-1), \mathscr{O}_\infty) \oplus \bar{c}_j d_\infty(x(k-1), \mathscr{O}_\infty) \oplus \cdots \oplus \\ &\bar{c}_1 d_\infty(x(k-1), \mathscr{O}_\infty) \leq \tilde{c}_j d_\infty(x(k-1), \mathscr{O}_\infty), \forall j \in \{0, \cdots, N-1\} \end{aligned}$$

where $\tilde{c}_j > 0$. In conclusion, there exists a $c_2 > 1$ such that $J^*(x(k-1)) \leq c_2 d_\infty(x(k-1), \mathscr{O}_\infty)$ for all $x(k-1) \in \mathscr{N}(\mathscr{O}_\infty, r)$. It is obvious that $J^*(x(k-1)) \geq d_\infty(x(k-1), \mathscr{O}_\infty)$ for all $x(k-1) \in N(\mathscr{O}_\infty, r)$. ■

We define the feasible set:

$$X_N^{\text{RHC}} = \{x \in \mathbb{R}_\varepsilon^n : (15) \text{ is feasible for } x(k-1) = x\}$$

Theorem 2.6: If $x(0) \in X_N^{\text{RHC}}$ then all subsequent stages of the optimization problem (15) will be feasible. Moreover, the bounded set $\mathscr{O}_\infty \cap \{x : x \geq B \otimes u_{\text{e}}\}$ is exponentially stable for the system (5a)–(5b) in closed-loop with the RHC controller $u^{\text{RHC}}(k) = u^*(k|k-1)$.

Proof: The proof is done by induction. If (15) has an optimal solution at step $k-1 : \tilde{u}^*(k) = (u^*(k|k-1) \cdots u^*(k+N-1|k-1))$, then at step k a feasible solution is $u_{\text{feas}} = (u^*(k+1|k-1) \cdots u^*(k+N-1|k-1)\ u_{\text{e}})$ since $\mathscr{O}_\infty$ is a positively invariant set, $x(k+N-1|k-1) \in \mathscr{O}_\infty$ and then u_{e} keeps the state in $\mathscr{O}_\infty$. Using the receding horizon principle the state at k becomes $x(k) = A \otimes x(k-1) \oplus B \otimes u^{\text{RHC}}(k) = x(k|k-1)$.

Since $x(k+N-1|k-1) \in \mathscr{O}_\infty$, we have

$$\begin{aligned} J^*(x(k)) - J^*(x(k-1)) \leq &J(x(k), u_{\text{feas}}) - J^*(x(k-1)) \leq \\ &-d_\infty(x(k-1), \mathscr{O}_\infty) \end{aligned}$$

Therefore, $\{J^*(x(k))\}_{k \geq 0}$ is a non-increasing sequence and from Lemma 2.5 we have

$$d_\infty(x(k-1), \mathscr{O}_\infty) \leq J^*(x(k-1)) \leq c_2 d_\infty(x(k-1), \mathscr{O}_\infty)$$

Using Theorem 1.3 and $x(k) \geq B \otimes u_{\text{e}}$ for all $k \geq 1$, we conclude that the compact set $\mathscr{O}_\infty \cap \{x : x \geq B \otimes u_{\text{e}}\}$ is exponentially stable for the system (5a) in closed-loop with the RHC controller provided by (15). ■

Assuming $x(k) \in \mathscr{O}_\infty$ we switch then to the feasible controller u_e for all the subsequent motion and we need finite number of steps to attain x_e. Indeed, $x(k+j) = A^{\otimes^j} \otimes x(k) \oplus (\bigoplus_{i=1}^{j} A^{\otimes^{j-i}} \otimes B \otimes u_e) \in \mathscr{O}_\infty$ for any $j \geq 1$. Since $A^{\otimes^j} \rightarrow \boldsymbol{\mathcal{E}}$ the first term $A^{\otimes^j} \otimes x(k) \rightarrow \boldsymbol{\mathcal{E}}$ while the second is equal to x_e for $j \geq n$.

Interpretation of Lyapunov stability: In the context of discrete-event systems, the Lyapunov stability of the compact set $\mathscr{O}_\infty \cap \{x : x \geq B \otimes u_e\}$ implies boundedness of the buffer levels.

III. TIME-OPTIMAL CONTROL

Given a maximum horizon length $N_{\max}$ we now consider the problem of ensuring that the completion times after N events, where $N \in \{1, 2, \cdots, N_{\max}\}$ are less than or equal to a specified target time α ($x(N) \leq \alpha$ with the initial conditions $x(0)$ and $u(0)$), using the largest controller that satisfies the state-input constraints (6). Note that such a problem, but without considering input and state constraints, was considered also in [4] in terms of lattice theory.

We define an equivalent system for (5a)– (5b) such that we do not need to impose the constraint $u(k+1) - u(k) \geq -\rho$, this constraint being satisfied automatically. We introduce a new state vector $x_{\text{new}}(k) = [x^T(k) \;\; u^T_{\text{new}}(k)]^T$ with the dynamics:

$$\begin{cases} x_{\text{new}}(k) = \begin{bmatrix} A & B - \rho \\ \boldsymbol{\mathcal{E}} & E_m - \rho \end{bmatrix} \otimes x_{\text{new}}(k-1) \oplus \begin{bmatrix} B \\ E_m \end{bmatrix} \otimes u(k) \\ y_{\text{new}}(k) = [C \;\; \boldsymbol{\mathcal{E}}] \otimes x_{\text{new}}(k) \end{cases} \tag{16}$$

and the extra constraint:

$$u_{\text{new}}(k) \leq u(k) \tag{17}$$

We denote with $A_{\text{new}} = \begin{bmatrix} A & B - \rho \\ \boldsymbol{\mathcal{E}} & E_m - \rho \end{bmatrix}$, $B_{\text{new}} = \begin{bmatrix} B \\ E_m \end{bmatrix}$ and $C_{\text{new}} = [C \;\; \boldsymbol{\mathcal{E}}]$. Given the initial conditions $x(0)$ and $u(0)$ for the system (5a)–(5b) with constraints (6) and the initial conditions $x_{\text{new}}(0) = [x(0)^T \; u(0)^T]^T$ and $u(0)$ for the new system (16) with the extra constraint (17) then by applying the same input $u(k)$ for both systems we obtain that the first n components of $x_{\text{new}}(k)$ coincide with $x(k)$ and the last m components of $x_{\text{new}}(k)$ coincide with $u(k)$. Note that the constraints (6) for the normalized system (5a)– (5b) can be written for the new system (16) as $[H \;\; \mathbf{0}]x_{\text{new}}(k) + Gu(k) \leq h$, $[\mathbf{0} \;\; I_m]x_{\text{new}}(k) - I_m u(k) \leq 0$, i.e.

$$H_{\text{new}} x_{\text{new}}(k) + G_{\text{new}} u(k) \leq h_{\text{new}} \tag{18}$$

where[4] $H_{\text{new}} \geq 0$. Moreover, for the new system (16) the target time is $\alpha_{\text{new}} = [\alpha^T \;\; ((-B^T) \otimes' \alpha)^T]^T$. The time-optimal control problem can be posed in terms of an optimization problem: given $x_{\text{new}} := x_{\text{new}}(0)$, find

$$\begin{cases} N^0(x_{\text{new}}) = \max_{(N, u(1), \cdots, u(N))} N \\ \text{s.t. } x_{\text{new}}(j) = A_{\text{new}} \otimes x_{\text{new}}(j-1) \oplus B_{\text{new}} \otimes u(j) \\ \quad H_{\text{new}} x_{\text{new}}(j) + G_{\text{new}} u(j) \leq h_{\text{new}}, \;\; \forall j \in \underline{N} \\ \quad x_{\text{new}}(N) \leq \alpha_{\text{new}} \end{cases}$$

making $u(1), \cdots, u(N)$ as big as possible. We denote with $\tilde{X}_N$ the set of initial states such that after N steps the trajectory is below α applying the largest controller:

$$\tilde{X}_N = \{x_{\text{new}} : x_{\text{new}}(N) \leq \alpha_{\text{new}}\} \tag{19}$$

We give first a lemma that will be useful in the sequel:

Lemma 3.1: [15] Suppose $Z = \{(x, u) : \bar{H}x + \bar{G}u \leq \bar{h}\}$, with $\bar{H} \geq 0$. Let X be defined as $X = \{x \in \mathbb{R}^n : \exists\, u \text{ s.t. } (x, u) \in Z\}$. Then, $X = \{x : \tilde{H}x \leq \tilde{h}\}$, where $\tilde{H} \geq 0$.

We determine the expression of $\tilde{X}_N$ using dynamic programming. We initialize with $\tilde{X}_0 = \{x_{\text{new}} : x_{\text{new}} \leq \alpha_{\text{new}}\}$. The set $\tilde{X}_1$ is defined as follows:

$$\begin{aligned} \tilde{X}_1 = \{x_{\text{new}} : \exists\; u \text{ s.t. } & A_{\text{new}} \otimes x_{\text{new}} \oplus B_{\text{new}} \otimes u \in \tilde{X}_0, \\ & H_{\text{new}}(A_{\text{new}} \otimes x_{\text{new}} \oplus B_{\text{new}} \otimes u) + G_{\text{new}} u \leq h_{\text{new}}\} \\ = \{x_{\text{new}} : \exists\; u \text{ s.t. } & \bar{H}_1 x_{\text{new}} + \bar{G}_1 u \leq \bar{h}_1\} \end{aligned}$$

for some matrices $\bar{H}_1, \bar{G}_1$ and $\bar{h}_1$, with $\bar{H}_1 \geq 0$. Using Lemma 3.1 we conclude: $\tilde{X}_1 = \{x_{\text{new}} : \tilde{H}_1 x_{\text{new}} \leq \tilde{h}_1\}$, with $\tilde{H}_1 \geq 0$. Moreover, we search for the largest controller u. In order to find a Pareto optimal u we use the following criterion:

$$\max_u \sum_{i=1}^{m} u_i \;\; \text{s.t. } \bar{H}_1 x_{\text{new}} + \bar{G}_1 u \leq \bar{h}_1$$

Solving this optimization problem as a parametric linear program with the parameter x_{new} we find the time-optimal controller $u^{\text{t}}_1(\cdot) : \tilde{X}_1 \rightarrow \mathbb{R}^m$ which is a continuous piecewise affine function of the state x_{new}.

Iterating this procedure backwards, we can compute

$$\begin{aligned} \tilde{X}_N = \{x_{\text{new}} : \exists\; u \text{ s.t. } & A_{\text{new}} \otimes x_{\text{new}} \oplus B_{\text{new}} \otimes u \in \tilde{X}_{N-1}, \\ & H_{\text{new}}(A_{\text{new}} \otimes x_{\text{new}} \oplus B_{\text{new}} \otimes u) + G_{\text{new}} u \leq h_{\text{new}}\} \\ = \{x_{\text{new}} : \exists\; u \text{ s.t. } & \bar{H}_N x_{\text{new}} + \bar{G}_N u \leq \bar{h}_N\} \end{aligned}$$

with matrix $\bar{H}_N$ having all entries non-negative. Using Lemma (3.1) we obtain that $\tilde{X}_N = \{x_{\text{new}} : \tilde{H}_N x_{\text{new}} \leq \tilde{h}_N\}$, with $\tilde{H}_N \geq 0$.

Similarly, we obtain the piecewise affine time-optimal controller $u^{\text{t}}_N(\cdot) : \tilde{X}_N \rightarrow \mathbb{R}^m$ by solving a parametric linear program in x_{new}:

$$\max_u \sum_{j=1}^{m} u_j \;\; \text{s.t. } \bar{H}_N x_{\text{new}} + \bar{G}_N u \leq \bar{h}_N.$$

It follows that

$$N^0(x_{\text{new}}) = \max\{N \in \underline{N_{\max}} : x_{\text{new}} \in \tilde{X}_N\}.$$

The time-optimal controller is implemented as follows:

1) For each $N \in \underline{N_{\max}}$, find $\tilde{X}_N$. Define $N := N^0(x_{\text{new}}(0))$.

2) Apply the control policy $u(k) = u^{\text{t}}_{N-k+1}(x_{\text{new}}(k-1))$ for $k = 1, 2, \cdots, N^0(x_{\text{new}}(0))$.

[4]Here $H_{\text{new}} = \begin{bmatrix} H & \mathbf{0} \\ \mathbf{0} & I_m \end{bmatrix}$, $G_{\text{new}} = \begin{bmatrix} G \\ -I_m \end{bmatrix}$ and $h_{\text{new}} = \begin{bmatrix} h \\ 0 \end{bmatrix}$.

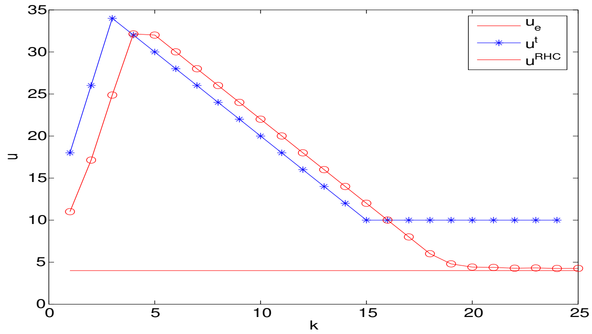

Fig. 1. The RHC controller and the time-optimal controller.

IV. Example

We consider the following system:

$$x_{sys}(k) = \begin{bmatrix} \varepsilon & 1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 2 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 3 \\ 4 & \varepsilon & \varepsilon & \varepsilon \end{bmatrix} \otimes x_{sys}(k-1) \oplus \begin{bmatrix} 2 \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix} \otimes u_{sys}(k)$$

For this example the MPA eigenvalue is $\lambda = 2.5$. We consider the reference signal $r_{sys}(k) = [17\ \ 15\ \ 1\ \ 10]^T + 4.5k$. We take the following constraints:

$$\begin{cases} u_{sys}(k) - u_{sys}(k+1) \leq 0 \\ x_{sys,1}(k) - u_{sys}(k) \leq 4 \\ x_{sys,2}(k) - x_{sys,1}(k) \leq 9 \end{cases} \tag{20}$$

Note that the constraint $x_{sys,2}(k) - x_{sys,1}(k) \leq 9$ is implied by the more conservative constraint $x_{sys,2}(k) - u_{sys}(k) \leq 11$.

We obtain for the normalized system the invariant set

$$\mathcal{O}_\infty = \mathcal{O}_3 = \{x \in \mathbb{R}^n_\varepsilon : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x \leq \begin{bmatrix} 5 \\ 7 \\ 9 \\ 11 \end{bmatrix} \} \tag{21}$$

The initial conditions are $x(0) = [20\ \ 30\ \ 40\ \ 50]^T, u(0) = 20$. The plots correspond to the normalized system. Fig. 1 displays both: the RHC controller and the time optimal controller while the constraints for the RHC controller are depicted in Fig. 2. Note that the RHC controller keeps the system behavior as close as possible to the constraints.

V. Conclusions

In this paper we have discussed the problem of stabilization of an MPL system using an RHC approach. We have considered state-input constraints and using a dual-mode RHC scheme we have proved that the system is exponentially stable in the sense of Lyapunov in the closed-loop with the RHC controller. Moreover the optimization problem that is solved at each step is a linear program for which efficient algorithms exist. We have also derived a time-optimal controller that satisfies the constraints using parametric linear programming.

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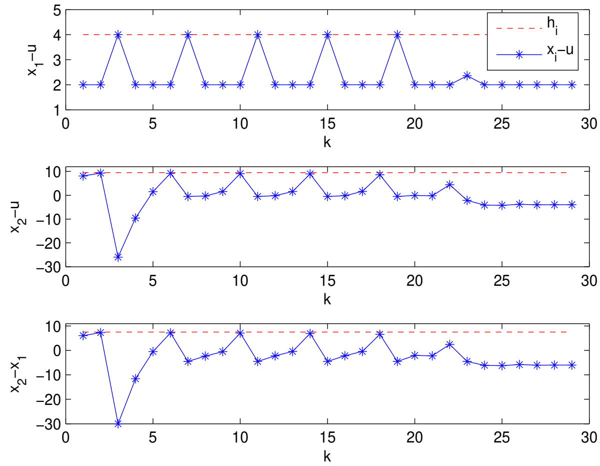

Fig. 2. Constraints. Sometimes the constraints are active.

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