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Technical report 06-003

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O. Gietelink, B. De Schutter, and M. Verhaegen, "Adaptive importance sampling for probabilistic validation of advanced driver assistance systems," *Proceedings of the 2006 American Control Conference*, Minneapolis, Minnesota, pp. 4002–4007, June 2006.

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* This report can also be downloaded via https://pub.bartdeschutter.org/abs/06_003

Adaptive Importance Sampling for Probabilistic Validation of Advanced Driver Assistance Systems

Olaf Gietelink, Bart De Schutter, and Michel Verhaegen

***Abstract*— We present an approach for validation of advanced driver assistance systems, based on randomized algorithms. The new method consists of an iterative randomized simulation using adaptive importance sampling. The randomized algorithm is more efficient than conventional simulation techniques. The importance sampling pdf is estimated by a kernel density estimate, based on the results from the previous iteration. The concept is illustrated with a simple adaptive cruise control problem.**

I. INTRODUCTION

A. Advanced Driver Assistance Systems

The increasing demand for safer passenger cars has stimulated the development of advanced driver assistance systems (ADASs). An ADAS is a control system that uses environment sensors to improve comfort and traffic safety by assisting the driver. An example is adaptive cruise control (ACC), which tries to maintain a pre-defined velocity set-point, unless a slower vehicle is detected ahead [14]. The ACC then controls the vehicle to follow the slower vehicle at a desired distance x_d (Fig. 1). Further defined are the position x, velocity v, and acceleration a of both vehicles, the relative velocity $v_\mathrm{r} = v_1 - v_2$, clearance $x_\mathrm{r} = x_1 - x_2$, and separation error $e_x = x_\mathrm{d} - x_\mathrm{r}$.

The demand for safety naturally increases with increasing automation of the driving task, since the driver must fully rely on a flawless operation of the ADAS. The ADAS should therefore be validated for a wide set of operating conditions.

An iterative process of *simulations* and *test drives* is often used for validation. Test drives give realistic results, but can never cover the entire set of operating conditions. Results are also difficult to analyze and not reproducible [3]. On the other hand, simulations have their limitations as well. For a realistic nonlinear model and multiple traffic disturbances, the validation problem will become difficult to solve, and eventually become *intractable* [15]. To make the simulation phase more efficient, a controller can be validated with a *grid search* over the operating range of all parameters [4]. However, an exhaustive grid search requires an intractably large number of experiments. Another possibility is a Monte Carlo strategy, where the system is simulated for a representative, but still very large, set of operating conditions, based on the probability that these conditions occur [12].

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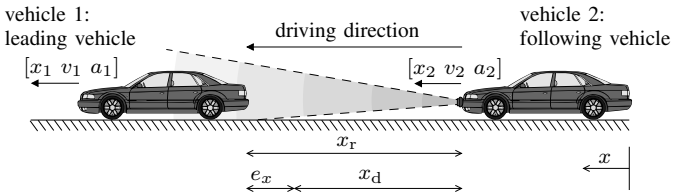

Fig. 1. Schematic representation of an ACC system.

In [6] we have introduced the use of *importance sampling* (IS) to make Monte Carlo simulation more efficient, i.e. using a much smaller number of tests. It was shown that IS considerably speeds up the simulation process, although the choice of probability density function (pdf) to sample from was not optimal. The choice of an optimal IS pdf has been the topic of much research, but has usually been constrained to simple parametric approaches [1], [5].

B. Objectives of this Paper

The objective of this paper is to present a new validation approach based on an iterative *randomized algorithm* (RA) using *adaptive importance sampling* (AIS), where the IS pdf is based on the results of the previous iteration. This provides an efficient test program in order to cover the entire set of operating conditions with a minimum number of samples.

Section II presents some background theory on RAs. Sections III and IV then present an iterative sampling algorithm and AIS to improve its efficiency, illustrated with the case study. Section V presents the conclusions.

II. RANDOMIZED ALGORITHMS

An alternative approach for solving a complex problem exactly, is to solve it *approximately* by using an RA. An RA is an algorithm that makes random choices during its execution [10]. The use of an RA can turn an intractable problem into a *tractable* one, but at the cost that the algorithm may fail to give a correct solution. The probability that the RA fails can be made arbitrarily close to zero, but never exactly equal to zero. This probability δ mainly depends on the *sample complexity* N, i.e. the number of simulations performed, but also on the specification of the problem to be solved.

A. Problem Specification

The performance of an ACC can be quantified in a number of measures ρ_i, e.g. tracking error, control effort, ride comfort, and string stability. In this paper the controller validation is restricted to the measure of *safety*, expressed as the probability p that an unsafe situation occurs. In [6] we have used a *discrete* safety measure $\rho_\mathrm{s} \in \{0, 1\}$, where

$\rho_s = 1$ means 'collision' and $\rho_s = 0$ means 'no collision'. However, a discrete safety measure cannot distinguish in severity between different situations for which $\rho_s = 1$. Therefore, a *continuous* safety measure ρ_t will be used, defined by the time-to-collision $t_{ttc} = \frac{x_r}{v_r}$. The time-to-collision has a lower value for unsafe situations, with a threshold value $\gamma_t = 6\,\text{s}$, since a traffic situation is subjectively regarded by a driver as 'dangerous' when $\rho_t < \gamma_t$ [3].

The value of ρ_t for a particular scenario depends on the perturbations imposed by that scenario, i.e. the motion of other vehicles. Apart from the acceleration of the lead vehicle a_1, also the initial conditions $x_r(0)$, $v_r(0)$ influence ρ_t. These scenario parameters, together with driver input, and other disturbances form the *parameter set* Q: an n-dimensional bounded set of all possible parameter combinations.

Since the system performance is defined by the scalar performance measure ρ_t, a cost function $\rho_t(q) : Q \to \mathbb{R}$ can be defined. The validation objective is then to identify the probability p that $\rho_t < \gamma_t$, by estimating it by $\hat{p}$ with a given *accuracy* ϵ and *confidence* level $1 - \delta$:

$$\Pr\{|\Pr\{\rho_t < \gamma_t\} - \hat{p}| \leq \epsilon\} \geq 1 - \delta, \text{ with } \delta, \epsilon \in (0, 1). \quad (1)$$

B. Monte Carlo Sampling

Consider an arbitrary process with only two possible outcomes, 'success' ($\rho = 0$) and 'failure' ($\rho = 1$), and suppose we wish to determine the probability p for a successful outcome[1]. A Monte Carlo sampling method can then be used [9], based on the 'law of large numbers', which states that

$$\lim_{N\to\infty} \Pr\{|\hat{p}_N - p| \geq \epsilon\} = 0, \quad (2)$$

where the *empirical probability* $\hat{p}_N$ of a failure of the process can be estimated as

$$\hat{p}_N = \frac{1}{N}\sum_{i=1}^{N} J(q_i), \quad (3)$$

which is known as the *simple sampling* estimator. The indicator function J represents the outcome of every i-th experiment for a parameter combination q_i, given by

$$J(q_i) = \begin{cases} 0, & \text{if } \rho = 0 \\ 1, & \text{if } \rho = 1. \end{cases} \quad (4)$$

The expected value of $\hat{p}_N$ is p, since $\hat{p}_N$ is an unbiased estimator. The variance of $\hat{p}_N$ is given by

$$\sigma_{ss}^2 = \text{var}\left\{\frac{1}{N}\sum_{i=1}^{N} J(q_i)\right\} = \frac{1}{N}\left[E\left\{J(q_i)^2\right\} - E\left\{J(q_i)\right\}^2\right]$$
$$= \frac{1}{N}\left(p - p^2\right) = \frac{p\,(1-p)}{N}. \quad (5)$$

The accuracy of the estimate $\hat{p}_N$ can then be expressed in the *relative rms error*

$$\sqrt{\frac{\text{var}\,\{\hat{p}_N\}}{p^2}} = \sqrt{\frac{1-p}{pN}}. \quad (6)$$

[1]Please note the difference in notation between the performance level ρ for one particular experiment and its probability p for all experiments.

From (6) the sample complexity could be calculated, given a desired relative error. However, this measure also requires *a priori* knowledge of p, which is exactly the parameter we wish to estimate. Furthermore, (6) does not give any information on the confidence interval for any particular estimate $\hat{p}_N$. In that respect, $\hat{p}_N$ is unlikely to be exactly equal to the real probability p, although it is reasonable to expect that $\hat{p}_N$ will approach p, as $N \to \infty$, and as long as the samples are chosen to be representative of the set Q.

The question thus arises how many samples N are necessary to give a reliable estimate $\hat{p}_N$, such that it differs from the real (unknown) value p by no more than $\epsilon > 0$, i.e.

$$p - \hat{p}_N \leq \epsilon. \quad (7)$$

Obviously, it is important to know whether the probability that the real performance p is *worse* than the estimated performance $\hat{p}_N$, i.e. the probability that $p - \hat{p}_N > \epsilon$, as in (7). Vice versa, the probability that the real p is better than expected by $\hat{p}_N$, i.e. $\hat{p}_N - p > \epsilon$, is not as important.

Since $\hat{p}_N$ is a random variable, the outcome of the inequality (7) is a random variable as well with a certain probability of realization. Therefore, we cannot always guarantee that $p - \hat{p}_N \leq \epsilon$, even for large N. By introducing a confidence level $1 - \delta$, the probability that $p - \hat{p}_{N_j} > \epsilon$ for any j-th set of N simulations (denoted by N_j) is then defined as

$$\Pr\{p - \hat{p}_N \leq \epsilon\} \geq 1 - \delta, \quad \text{with } \delta, \epsilon \in (0, 1). \quad (8)$$

It is then of interest to know the required N for (8) to hold.

C. Sample Complexity

Since the process is a binomial process, (8) can be calculated by

$$\Pr\{p - \hat{p}_N \leq \epsilon\} = 1 - \sum_{i=0}^{N^-} \binom{N^-}{i} p^i\,(1-p)^{(N^- - i)} > 1 - \delta, \quad (9)$$

where $N^- = (p - \epsilon)N$ is the minimum number of samples for which $J(q_i) = 1$. Unfortunately, (9) cannot be solved explicitly, such that the necessary sample complexity has to be approximated. From the central limit theorem follows that the distribution of $\hat{p}_N$ approaches a normal distribution as $N \to \infty$ [1]. In that case (8) can be approximated by

$$\Pr\{p - \hat{p}_N < k\sigma\} \geq 1 - \delta. \quad (10)$$

Here the variance σ can be approximated by the binomial parameter $\sqrt{Np(1-p)}$, and $k > 0$ can be derived from tables for a standard Gaussian distribution with a desired confidence $1 - \delta$. In this way, a fast approximation for N, denoted by N_{bin}, can be calculated. However, this measure again requires a priori knowledge of p.

To avoid this paradox we can use the Chernoff bound [2], which states that the probability $\delta > 0$ in (8) is no larger than $e^{-2N\epsilon^2}$. Therefore, to estimate the unknown quantity p to an accuracy ϵ and with a confidence $1 - \delta$, N should be chosen such that $2e^{-2N\epsilon^2} \leq \delta$. This can be rewritten as

$$N \geq \frac{1}{2\epsilon^2}\ln\frac{1}{\delta}, \quad (11)$$

which is known as the *one-sided additive Chernoff bound*. In order to compute $\hat{p}_N$ we can then use Algorithm 1.

Algorithm 1 (Probabilistic performance verification [13]) Given a desired $\epsilon, \delta \in (0,1)$, a threshold $\gamma \geq 0$, and the true pdf f_Q, this RA returns with a probability of at least $1-\delta$ an estimate $\hat{p}_N$ for p, such that $p - \hat{p}_N \leq \epsilon$.

1) Determine the necessary N with (11).
2) Draw N independent identically distributed (iid) samples $q_1, q_2, \ldots, q_N$ according to its pdf f_Q.
3) Return the empirical probability

$$\hat{p}_N = \frac{1}{N} \sum_{i=1}^{N} J\left(q_i\right),$$

where $J\left(q_i\right)$ is the indicator function

$$J(q_i) = \begin{cases} 0, & \text{if } \rho \geq \gamma \\ 1, & \text{if } \rho < \gamma. \end{cases} \tag{12}$$

Example 1 (Gaussian distributed disturbance) Consider a scenario where the lead vehicle brakes to a full stop. The initial conditions are $x_{\mathrm{r},0} = x_{\mathrm{d}} = 40\,\mathrm{m}$, and $v_1(0) = v_2(0) = 30\,\mathrm{m/s}$. For simplicity, assume that the deceleration of the lead vehicle a_1 is the only disturbance (note however that the approach can easily be extended to an n-dimensional parameter set). When the lead vehicle brakes hard, the ACC vehicle cannot always brake in time, since the ACC deceleration is limited to $-2.5\,\mathrm{m/s^2}$. Now, for fine-tuning the control parameters, we would like to know the percentage of situations (p) for which $\rho_{\mathrm{t}} < \gamma_{\mathrm{t}}$. The safety obviously decreases with a stronger deceleration a_1, such that the function $\rho_{\mathrm{t}}(a_1)$ is non-decreasing. For verification of this simple example the 'true' outcome can therefore be calculated numerically and is known exactly: p = 0.03630.

However, in practice it can be impossible to determine p in a deterministic way, when the dimension of Q increases and the function ρ_Q becomes non-convex. So instead of calculating p in deterministic sense, the function is randomized in such a way that it takes a random input q_i from its pdf $f(q)$. ACC field tests [3] suggest that the acceleration profile can be roughly described as a random signal with a Gaussian pdf f^{N}

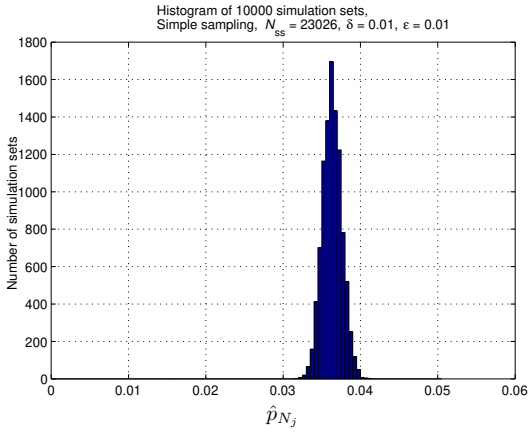

Fig. 2. Histogram of 10 000 estimates $\hat{p}_{N_j}$, with $N_j = 23026$ each, where the acceleration profile is sampled from a Gaussian pdf $\mathcal{N}(0, 1.5)$.

with mean $\mu = 0$ and standard deviation $\sigma = 1.5$, denoted as $\mathcal{N}(0, 1.5)$, truncated on the interval $[-10, 10]$ m/s^2.

In order to verify Algorithm 1, we execute it $M = 10\,000$ times. Suppose it is desired that $\epsilon = 0.01$ and $\delta = 0.01$, then (11) gives $N = 23026$. Every j-th simulation set gives an estimate $\hat{p}_{N_j}$ for $j = 1, \ldots, M$. The distribution of the estimate is shown in Fig. 2. The empirical mean $\hat{p}_M$ is 0.03633, as could be expected near the true probability p. The variance of the estimator $\hat{p}_{N_j}$ is $\hat{\sigma}^2_{\mathrm{ss}} = \frac{1}{M-1} \sum_{j=1}^{M} (\hat{p}_{N_j} - \hat{p}_M)^2 = 1.5373 \cdot 10^{-6}$. The accuracy for a single simulation set (each consisting of 23026 simulations), can be estimated from the results as $\hat{\epsilon} = 0.0028$ when $\hat{\delta} = 0.01$. □

These results suggest that the desired values for δ and ϵ can be achieved with a much lower N than given by (11).

III. ITERATIVE ESTIMATION USING THE BINOMIAL BOUND

As a solution to the conservatism of the Chernoff bound, we can make an initial estimate of p using (11), and subsequently use the binomial bound (10), after which p can be estimated with a lower bound on the sample complexity.

In order to obtain a lower N in total, the values for ϵ and δ must be modified. From (11) follows that N is proportional to $\log(1/\delta)$ and inversely proportional to $1/\epsilon^2$. Therefore, when δ is decreased and ϵ increased by a suitably chosen factor κ, a lower N can be obtained in a first iteration, that is $\delta_1 = \delta/\kappa$, and $\epsilon_1 = \kappa\epsilon$. To obtain the desired confidence $1-\delta$ for $\hat{p}_N$ in a second iteration, δ_2 should be chosen such that $(1-\delta_1)(1-\delta_2) \geq 1-\delta$. This is true when $\delta_2 = \delta - \delta_1$, since $(1-\delta/\kappa)(1-\delta+\delta/\kappa) \geq 1-\delta$, for all $\kappa \geq 1$. The accuracy in the second iteration is set to its desired value ϵ. This procedure is formalized in Algorithm 2.

Algorithm 2 (Iterative estimation using binomial bound) Given a desired $\epsilon, \delta \in (0,1)$ and a threshold $\gamma \geq 0$,

1) Draw $N_{\mathrm{ss},1} \geq \frac{1}{2\epsilon_1^2} \ln \frac{2}{\delta_1}$ iid samples, where $\delta_1 = \delta/\kappa$, and $\epsilon_1 = \kappa\epsilon$, and κ is a suitably chosen real number.
2) Return the empirical probability

$$\hat{p}_{N,1} = \frac{1}{N_{\mathrm{ss},1}} \sum_{i=1}^{N_{\mathrm{ss},1}} J\left(q_i\right).$$

The real p is at worst $p_{\mathrm{bin}} = \hat{p}_{N,1} + \epsilon_1$ with confidence $1-\delta_1$.
3) Determine $N_{\mathrm{bin},2}$ for p_{bin}, $\delta_2 = \delta - \delta_1$ and $\epsilon_2 = \epsilon$ using (10).
4) If $N_{\mathrm{bin},2} > N_{\mathrm{ss},1}$, draw $N_{\mathrm{ss},2} = N_{\mathrm{bin},2} - N_{\mathrm{ss},1}$ samples.
5) Return the empirical probability

$$\hat{p}_N = \frac{1}{N_{\mathrm{ss},1} + N_{\mathrm{ss},2}} \sum_{i=1}^{N_{\mathrm{ss},1}+N_{\mathrm{ss},2}} J\left(q_i\right)$$

with accuracy ϵ and confidence $(1-\delta_1)(1-\delta_2) \geq 1-\delta$.

In order to choose a suitable value for κ, we investigate the bounds on $N_{\mathrm{ss},1}$ and $N_{\mathrm{bin},2}$ for different values of κ, p, δ, and ϵ, see Fig. 3. This figure shows that Algorithm 2 is optimal when $N_{\mathrm{bin},2} = N_{\mathrm{ss},1}$. It is also clear that κ can better

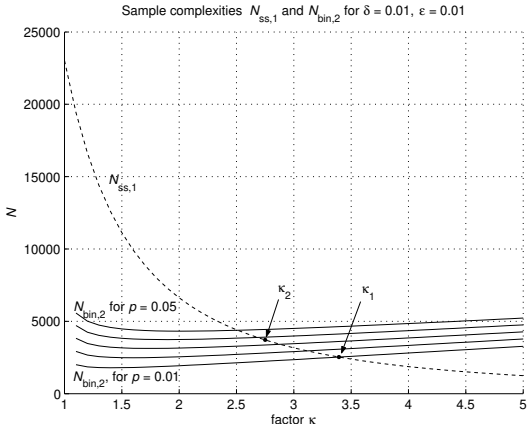

Fig. 3. Sample complexities $N_{\mathrm{bin},2}$ and $N_{\mathrm{ss},1}$ for varying κ, p, δ, and ϵ.

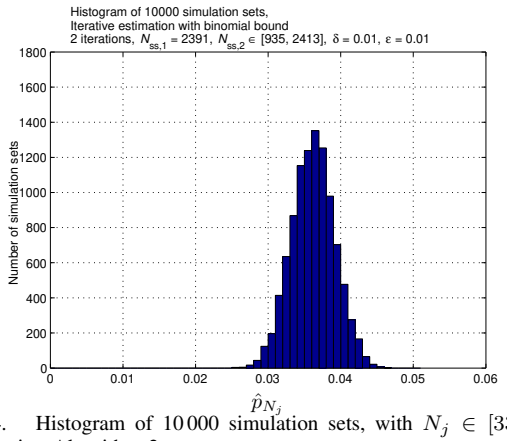

Fig. 4. Histogram of 10 000 simulation sets, with $N_j \in [3326, 4804]$ each, using Algorithm 2.

be selected too large instead of too small. In the former case, only a small number of extra samples $N_{\mathrm{ss},2} = N_{\mathrm{bin},2} - N_{\mathrm{ss},1}$ has to be drawn in the second iteration. In the latter case, the first iteration will draw too many samples $N_{\mathrm{ss},1}$, which is a waste of simulation time. The contours are similar for other values of δ, ϵ and p.

Therefore we choose κ at $N_{\mathrm{bin},2} = N_{\mathrm{ss},1}$, based on a reasonable value for p, preferably with p as low as possible. Since p should be greater than ϵ ($p < \epsilon$ implies the possibility of a negative probability), p is lower bounded by ϵ. Therefore κ should be chosen such that $N_{\mathrm{bin},2} = N_{\mathrm{ss},1}$ and $p = \epsilon$.

Example 2 (Iterative estimation using binomial bound) The problem in Example 1 is repeated using Algorithm 2, for the same values $\delta = 0.01$, $\epsilon = 0.01$, and the real p at 0.03630. However, since $\epsilon = 0.01$, p is assumed to be lower bounded at 0.01. We therefore choose $\kappa = 3.4$, using the procedure described above and illustrated in Fig. 3. Fig. 4 shows the results for 10 000 simulation sets, each with $N_j \in [3326, 4804]$, where $N_{\mathrm{ss},1,j} = 2391$ and $N_{\mathrm{ss},2,j} \in [935, 2413]$. The sample complexity $N_{\mathrm{ss},2}$ in the second iteration is quite large, caused by the fact that the factor κ is chosen larger than the optimal value (i.e. $\kappa_2 = 3.4$ at $p = 0.01$, instead of $\kappa_1 = 2.75$ at $p = 0.03630$, see Fig. 3). The results are $\hat{p}_N = 0.03619$ and the variance $\hat{\sigma}^2 = 8.8040 \cdot 10^{-6}$. When the empirical confidence is set $\hat{\delta} = 0.01$ the empirical accuracy is found to be $\hat{\epsilon} = 0.0070$, meaning that $N_j \in [935, 2413]$ is still slightly conservative, although better than in Example 1. Compared to the initial Chernoff bound $N = 23026$ for $\epsilon = 0.01$ and $\delta = 0.01$, this means an efficiency improvement of $\frac{N}{\max_j(N_{\mathrm{bin},2,j})} = 4.8$. □

IV. ADAPTIVE IMPORTANCE SAMPLING (AIS)

A. Principle of Importance Sampling (IS)

Example 2 shows that a significant reduction of N can be achieved, although the samples q_i themselves are not chosen more efficiently. One solution is to reduce the set Q by neglecting certain subsets that are impossible to occur [6]. Obviously, a low t_{ttc} is more likely with lower values for $x_{\mathrm{r},0}$, $v_{\mathrm{r},0}$, and a_1, such that the samples for which $J = 1$ are clustered in a specific subset Q_{bad}. This means that there is structure in the parameter set Q and in the function ρ_Q that can be used to reduce the sampling space by disregarding specific subsets of Q, of which the outcome is a priori known.

It therefore makes sense to give more attention to operating conditions that are more likely to become dangerous than others. Another possibility for using a priori knowledge on interesting samples is importance sampling, which is a technique to increase the number of occurrences of the event of which the probability p should be estimated [9].

Suppose that we want to estimate a probability p, given the parameter set Q. The goal is then to estimate

$$p = \int_Q J(q) f_Q(q) dq = E\left\{J(q)\right\}, \tag{13}$$

where we sample q from its joint pdf f_Q, denoted as $q \sim f_Q$.

In order to highlight the interesting subset Q_{bad} it thus makes sense not to sample from the original pdf f_Q, but instead use an artificial pdf, reflecting the 'importance' of the events, and then reweighing the observations to get an unbiased estimate.

We can now define an IS pdf φ that is strictly positive on Q. We can then write

$$p = \int_Q \frac{J(q) f_Q(q)}{\varphi(q)} \varphi(q) dq = E\left\{\frac{J(\Phi) f_Q(\Phi)}{\varphi(\Phi)}\right\}, \tag{14}$$

where $\Phi \sim \varphi$. The IS estimator based on φ is

$$\hat{p}_{N,\mathrm{is}} = \frac{1}{N} \sum\nolimits_{i=1}^{N} \frac{J(\Phi_i) f_Q(\Phi_i)}{\varphi(\Phi_i)}, \tag{15}$$

where $\Phi_1, \ldots, \Phi_N$ are iid with pdf φ. Every sample drawn from φ can be seen as a realization of $\frac{\varphi}{f}$ original samples, which must therefore be reweighed with the factor $\frac{f}{\varphi}$. The variance of the importance sampling estimator (15) is

$$\begin{aligned} \sigma^2_{\mathrm{is}} &= \mathrm{var}\left\{\hat{p}_{N,\mathrm{is}}\right\} = \mathrm{var}\left\{\tfrac{1}{N} \sum_{i=1}^{N} J\left(\Phi_i\right) \tfrac{f(\Phi_i)}{\varphi(\Phi_i)}\right\} = \\ &\tfrac{1}{N}\left[E\left\{J^2\left(\Phi_i\right) \tfrac{f^2(\Phi_i)}{\varphi^2(\Phi_i)}\right\} - E\left\{J\left(\Phi_i\right) \tfrac{f(\Phi_i)}{\varphi(\Phi_i)}\right\}^2\right]. \end{aligned} \tag{16}$$

An RA can then be formulated as follows.

Algorithm 3 (Importance sampling)
Given a desired $\epsilon, \delta \in (0,1)$ and a threshold $\gamma \geq 0$,

1) Determine a strictly positive IS pdf φ.
2) Select an initial number of samples N_{is}.
3) Draw N_{is} iid samples Φ_i according to φ.
4) Return the empirical probability

$$\hat{p}_{N,\text{is}} = \frac{1}{N_{\text{is}}} \sum\nolimits_{i=1}^{N_{\text{is}}} \frac{J(\Phi_i) f(\Phi_i)}{\varphi(\Phi_i)}.$$

In order to make this algorithm work, good choices for N_{is} and the IS pdf φ must be made in advance. We will discuss these two issues next.

B. Estimation of φ using Kernel Density Estimation

The performance of Algorithm 3 heavily depends on the reliability of the pdf φ to generate random variables, and of the models used in the simulation. An efficient estimator of $\hat{p}_{N,\text{is}}$ is obtained by choosing φ proportional to the importance of the individual samples, with importance defined as $|J(q) f_Q(q)|$. A rare but dangerous event can thus be equally important as a frequent but less critical event. Conventional IS methods consist of shifting the mean or variance of the original pdf f to form the IS pdf φ, a so-called parametric approach [1], [11]. However, the optimal IS pdf φ will most likely not be a standard type pdf (e.g. normal), but reflect an erratic multi-dimensional surface in Q. Parametric IS methods can therefore bias the results if not carefully chosen. Instead, a nonparametric approach is developed, where the *entire* pdf is estimated. Since importance is related to the samples q_i for which $J(q_i) = 1$, we apply a multivariate kernel density estimate on these samples.

Assume we have N_b independent observations $q_1, \ldots, q_{N_b}$ of this random variable with $J(q_i) = 1$ for $i = 1, \ldots, N_b$. The kernel density estimator $\widehat{f}_H(q)$ for the estimation of the density value $f(q)$ at point q is defined as

$$\widehat{f}_H(q) = \frac{1}{N_b} \sum_{i=1}^{N_b} K_H(q_i - q), \tag{17}$$

where $K_H(q) = |H|^{-1/2} K(H^{-1/2} q)$, $K(\cdot)$ is a multivariate kernel function, and H is a symmetric positive definite $n \times n$ matrix known as the bandwidth matrix. For more information on the use of suitable kernel density estimates, see [7].

C. Sample Complexity of IS

Instead of using a stopping criterion, we would like to know in advance the necessary sample complexity to achieve a specified δ and ϵ. Unfortunately, it is not possible to exactly calculate the minimum sample complexity beforehand [1]. However, here we will present a method that at least gives a reliable prediction of N after an initial limited iteration.

In order to provide an estimate of the reduction in sample complexity that can be achieved with IS, we would like to know the *importance sampling reduction factor*

$$\lambda_{\text{is}} = \frac{\sigma_{\text{is}}^2}{\sigma_{\text{ss}}^2}, \tag{18}$$

where σ_{ss}^2 and σ_{is}^2 are the variances of the simple sampling estimator and the IS estimator, respectively.

The accuracy of the resulting estimator $\hat{p}_{N,\text{is}}$ can then be expressed in its relative rms error

$$\sqrt{\frac{\text{var}\{\hat{p}_{N,\text{is}}\}}{p^2}} = \sqrt{\frac{\lambda_{\text{is}} \text{var}\{\hat{p}_N\}}{p^2}} = \sqrt{\frac{\lambda_{\text{is}}(1-p)}{pN}}. \tag{19}$$

The gain in efficiency can then be calculated by equating (6) and (19). From this follows that in order to get the same level of relative error, the reduction in samples for IS is

$$N_{\text{is}} \approx \lambda_{\text{is}} N_{\text{ch}}, \quad p \ll \lambda_{\text{is}}. \tag{20}$$

The simple sampling variance σ_{ss}^2 can be approximated from (5) using the first iteration. The IS variance σ_{is}^2 can be approximated by the empirical estimate of (16)

$$\hat{\sigma}_{\text{is}}^2 = \frac{\frac{1}{N_{\text{is}}} \sum\limits_{i=1}^{N_{\text{is}}} J^2(\Phi_i) \frac{f^2(\Phi_i)}{\varphi^2(\Phi_i)} - \left(\frac{1}{N_{\text{is}}} \sum\limits_{i=1}^{N_{\text{is}}} J(\Phi_i) \frac{f(\Phi_i)}{\varphi(\Phi_i)} \right)^2}{N_{\text{is}}}. \tag{21}$$

Unfortunately, (21) can only be calculated *a posteriori*, whereas we would like to know N before we start with importance sampling. We therefore seek to estimate $\hat{\sigma}_{\text{is}}^2$ a priori, thereby predicting the IS reduction factor in (18), which in turn is used to predict $N_{\text{is}} \approx \lambda_{\text{is}} N_{\text{ss}}$.

Suppose a limited set of samples has already been evaluated in a first iteration. Then consider the factor $\frac{f(q_i)}{\varphi(q_i)}$, which is known for every sampled value q_i. We then predict the number of 'hits' $(J = 1)$ that correspond to the IS pdf φ in a second iteration (but before these samples are actually drawn). We do this by using the assumption that every single sample obtained with simple sampling ($q_i \sim f$), corresponds to $\frac{\varphi(q_i)}{f(q_i)}$ samples using IS ($q_i \sim \varphi$), as shown in (15).

The expected IS variance $\hat{\sigma}_{\text{is}}^2$ can then be estimated using the first simple sampling iteration, where the first term in the numerator of (21) is multiplied with $\frac{\varphi(q_i)}{f(q_i)}$. The second term is equal to $\hat{p}_{N,\text{is}}$ and can be approximated by its simple sampling estimate. Rewriting the samples Φ_i to q_i we get

$$\tilde{\sigma}_{\text{is}}^2 = \frac{\frac{1}{N_{\text{ss}}} \sum\limits_{i=1}^{N_{\text{ss}}} J^2(q_i) \frac{f(q_i)}{\varphi(q_i)} - \left(\frac{1}{N_{\text{ss}}} \sum\limits_{i=1}^{N_{\text{ss}}} J(q_i) \right)^2}{N_{\text{ss}}}. \tag{22}$$

We can then estimate λ_{is} from (18). After the first iteration we can then predict N_{is}, as well as form the IS pdf φ. We therefore combine importance sampling with the iterative estimation, as discussed next.

D. Combination of AIS and the Binomial Bound

In the *first* iteration with sample complexity $N_{\text{ss},1}$ it is recommended to use a grid-based sampling strategy or Latin hypercube [8], in order to get a representative overview of the parameter set Q. The results of this first iteration can then be used to form an IS pdf φ, and estimate the IS reduction factor λ_{is}. The remaining number of samples to be taken in the *second* iteration is then $N_{\text{is},2} = \lambda_{\text{is}} (N_{\text{bin},2} - N_{\text{ss},1})$. If desired, these steps can be repeated to obtain a better IS pdf φ. We therefore combine Algorithms 2 and 3 as follows.

Algorithm 4 (Iterative estimation with AIS)
Given a desired $\epsilon, \delta \in (0,1)$ and a threshold $\gamma \geq 0$,

1) Draw $N_{\mathrm{ss},1} \geq \frac{1}{2\epsilon_1^2} \ln \frac{2}{\delta_1}$ iid samples, where $\delta_1 = \delta/\kappa$, and $\epsilon_1 = \kappa\epsilon$, and κ is a suitably chosen real number.
2) Return the empirical probability

$$\hat{p}_{N,1} = \frac{1}{N_{\mathrm{ss},1}} \sum\nolimits_{i=1}^{N_{\mathrm{ss},1}} J\left(q_i\right).$$

and the empirical variance $\hat{\sigma}^2_{\mathrm{ss}}$. The real p is at worst $p_{\mathrm{bin}} = \hat{p}_{N,1} + \epsilon_1$ with confidence $1 - \delta_1$.
3) Determine $N_{\mathrm{bin},2}$ for p_{bin}, $\delta_2 = \delta - \delta_1$ and $\epsilon_2 = \epsilon$ using (10).
4) Estimate the IS pdf φ, based on the kernel density estimate (17) of samples $q_{b,i}$, for which $\rho_j(q_i) \geq \gamma_j$.
5) Estimate the IS reduction factor $\hat{\lambda}_{\mathrm{is}} = \frac{\tilde{\sigma}^2_{\mathrm{is}}}{\hat{\sigma}^2_{\mathrm{ss}}}$, with $\tilde{\sigma}^2_{\mathrm{is}}$ from (22) and $\hat{\sigma}^2_{\mathrm{ss}}$ from step 3.
6) If $N_{\mathrm{bin},2} > N_{\mathrm{ss},1}$, then draw $N_{\mathrm{is},2} = \hat{\lambda}_{\mathrm{is}}\left(N_{\mathrm{bin},2} - N_{\mathrm{ss},1}\right)$ samples.
7) If $N_{\mathrm{bin},2} > N_{\mathrm{ss},1}$, then do the remaining samples.
8) Return the empirical probability

$$\hat{p}_N = \frac{1}{N_{\mathrm{ss},1} + N_{\mathrm{is},2}} \left[\sum_{i=1}^{N_{\mathrm{ss},1}} J\left(q_i\right) + \sum_{i=1}^{N_{\mathrm{is},2}} \frac{J(\Phi_i) f(\Phi_i)}{\varphi(\Phi_i)} \right]$$

with accuracy ϵ and confidence $(1-\delta_1)(1-\delta_2) \geq 1-\delta$.

Example 3 (Iterative estimation using AIS)
To illustrate the efficiency of Algorithm 4, Example 1 is again repeated, but now using Algorithm 4. Fig. 5 shows the results for 10 000 simulation sets, each with $N_j \in [2519, 2810]$, where $N_{\mathrm{ss},1,j} = 2391$ and $N_{\mathrm{is},2,j} \in [128,\ 419]$. The sample complexity $N_{\mathrm{is},2}$ in the second iteration is considerably small, since the use of IS reduces $N_{\mathrm{is},2}$ in the second iteration. The results are $\hat{p}_N$ = 0.03628 and the variance $\hat{\sigma}^2_{\mathrm{is}} = 1.2140 \cdot 10^{-5}$. When the empirical confidence is set $\hat{\delta} = 0.01$ the empirical accuracy is found to be $\hat{\epsilon} = 0.0081$, meaning that $N_j \in [2519, 2810]$ is still slightly conservative, although much better than in Example 1. The fact that the bound $N_{\mathrm{bin},2,j}$ is not strict (such that $\hat{\delta} = 0.01$) is caused by the fact that the factor κ is not chosen optimal, as discussed in Section III. Nevertheless, there is an efficiency improvement of $\frac{N_{\mathrm{ss}}}{\max_j(N_{\mathrm{bin},2,j})} = 8.2$. This means an improvement with respect to Algorithm 2, where the improvement was a factor of 4.8. Obviously the resulting variance is slightly larger than in Example 2, but the accuracy and confidence are still within the desired values. In other words, the bound on the sample complexity is more strict. □

V. CONCLUSIONS

We have presented an approach for probabilistic performance validation of advanced driver assistance systems (ADASs), and applied a randomized algorithm, based on adaptive importance sampling, to a simple adaptive cruise control problem. For this problem the new approach leads to roughly a tenfold increase in efficiency.

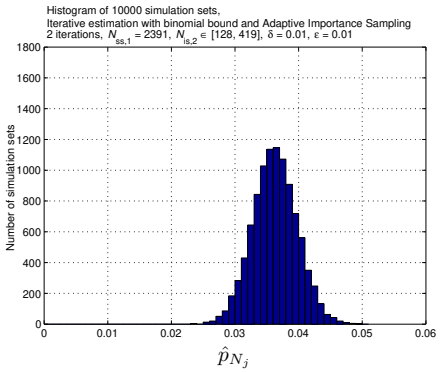

Fig. 5. Histogram of 10 000 simulation sets, with $N_j \in [2519, 2810]$ each, using Algorithm 4.

VI. ACKNOWLEDGMENTS

Research supported by TNO, TRAIL, Transport Research Centre Delft, and the European Network of Excellence HYCON, contract number FP6-IST-511368.

REFERENCES

[1] J. Bucklew. *Introduction to Rare Event Simulation*. Springer-Verlag, New York, NY, USA, 2004.
[2] H. Chernoff. A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. *Annals of Mathematical Statistics*, 23:493–507, 1952.
[3] P. Fancher, R. Ervin, J. Sayer, M. Hagan, S. Bogard, Z. Bareket, M. Mefford, and J. Haugen. Intelligent cruise control field operational test. Final report DOT HS 808 849, National Highway Traffic Safety Administration, Washington, DC, USA, May 1998.
[4] C. Fielding, A. Vargas, S. Bennani, and M. Selier, editors. *Advanced Techniques for Clearance of Flight Control Laws*. Springer, Berlin, Germany, 2002.
[5] K. Gerlach. New results in importance sampling. *IEEE Transactions on Aerospace and Electronic Systems*, 35(3):917–925, July 1999.
[6] O. Gietelink, B. De Schutter, and M. Verhaegen. Probabilistic validation of advanced driver assistance systems. In *Proc. of the 16th IFAC World Congress*, Prague, Czech Republic, July 3–8, 2005. paper nr. 4254.
[7] W. Härdle, M. Müller, S. Sperlich, and A. Werwatz. *Nonparametric and Semiparametric Models*. Springer Verlag, Berlin, Germany, May 2004.
[8] R. Iman, J. Campbell, and J. Helton. An approach to sensitivity analysis of computer models. I - Introduction, input, variable selection and preliminary variable assessment. *Journal of Quality Technology*, 13:174–183, July 1981.
[9] N. Madras. *Lectures on Monte Carlo Methods*. American Mathematical Society, Providence, RI, USA, 2002.
[10] R. Motwani. *Randomized Algorithms*. Cambridge University Press, New York, 1995.
[11] J. Stadler. Adaptive importance sampling. *IEEE Journal on Selected Areas in Communications*, 11(3):309–316, Apr. 1993.
[12] R. Stengel and L. Ray. Stochastic robustness of linear time-invariant control systems. *IEEE Trans. Automatic Control*, 36(1):82–87, Jan. 1991.
[13] R. Tempo, G. Calafiore, and F. Dabbene. *Randomized Algorithms for Analysis and Control of Uncertain Systems*. Springer-Verlag, Berlin, Germany, 2004.
[14] A. Vahidi and A. Eskandarian. Research advances in intelligent collision avoidance and adaptive cruise control. *IEEE Trans. on Intelligent Transportation Systems*, 4(3):143–153, Sept. 2003.
[15] M. Vidyasagar. Statistical learning theory and randomized algorithms for control. *IEEE Control Systems Magazine*, 18(6):69–85, Dec. 1998.