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Worst-case optimal control of uncertain max-plus-linear systems

Ion Necoara, Eric C. Kerrigan, Bart De Schutter and Ton J.J. van den Boom

Abstract— **In this paper the finite-horizon min-max optimal control problem for uncertain max-plus-linear (MPL) discrete-event systems is considered. We assume that the uncertain parameters lie in a given convex and compact set and it is required that the input and state sequence satisfy a given set of linear inequality constraints. The optimal control policy is computed via dynamic programming using tools from polyhedral algebra and multi-parametric linear programming. Although the controlled system is nonlinear, we provide sufficient conditions, which are usually satisfied in practice, such that the value function is guaranteed to be convex, continuous and piecewise affine, and the optimal control policy is continuous and piecewise affine on a polyhedral domain.**

I. INTRODUCTION

We extend the classical formulation of worst-case optimal control for linear and nonlinear systems to a particular class of discrete-event systems (DES), called max-plus-linear (MPL) systems. DES are event-driven dynamical systems (i.e. the state transitions are initiated by events, rather than a clock). This is in contrast to conventional time-driven systems, where the state changes as time progresses. In the last couple of decades there has been an increase in the amount of research on DES that can be modeled as max-plus-linear (MPL) systems. MPL systems are *nonlinear* dynamic systems that are "linear" in the max-plus-algebra [1], i.e. the algebra having maximization and addition as basic operations. MPL systems model DES with synchronization and no concurrency and they often arise in the context of manufacturing systems, telecommunication networks, railway networks, parallel computing, etc.

Although there are many papers on optimal control for MPL systems (see [5], [11] and the references therein), there are only few papers on the robust control for this class of systems. In [15] the authors propose an open-loop min-max model predictive control scheme while in [8], [9] optimal controllers based on residuation theory are derived.

In this paper we consider the MPL version of the finite-horizon min-max control [3] for uncertain dynamic systems using a dynamic programming (DP) approach as in [2], [6]. The main advantage of this paper compared to the papers mentioned previously on robust control of MPL systems is the fact that we optimize over feedback policies, rather than open-loop input sequences, and that we incorporate state and input constraints directly into the problem formulation. This results in increased feasibility and better performance. One of the key contributions of this paper is to provide sufficient conditions, which are not restrictive in practice, such that we can employ results from convex analysis to compute robust optimal controllers for MPL systems. It is important to note that we require the stage cost to have a particular representation in which the coefficients corresponding to the state vector are non-negative *and* that the matrix associated with the state constraints is also non-negative. Note that in practice these conditions are often satisfied.

The paper is organized as follows. We continue this section by introducing some notation and defining the finite-horizon min-max problem of interest. In Section II we present the DP solution to this problem. We conclude with an example in Section III, where we compare our approach with the open-loop controller of [15].

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A. Preliminaries

Define $\varepsilon := -\infty$ and $\mathbb{R}_\varepsilon := \mathbb{R} \cup \{\varepsilon\}$. The max-plus-algebraic (MPA) addition ($\oplus$) and multiplication ($\otimes$) are defined as [1]: $x \oplus y := \max\{x, y\}$, $x \otimes y := x + y$, for $x, y \in \mathbb{R}_\varepsilon$. For matrices $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ and $C \in \mathbb{R}_\varepsilon^{n\times p}$ we can extend the definition as follows: $(A \oplus B)_{ij} := A_{ij} \oplus B_{ij}$, $(A \otimes C)_{ij} := \bigoplus_{k=1}^{n} A_{ik} \otimes C_{kj}$ for all i,j. Define the matrix $\mathcal{E}_{m\times n}$ as the $m \times n$ MPA zero matrix: $(\mathcal{E}_{m\times n})_{ij} := \varepsilon$ for all i, j. The matrix E_n is the $n \times n$ MPA identity matrix: $(E_n)_{ii} := 0$ for all i and $(E_n)_{ij} := \varepsilon$ for all i, j with $i \neq j$. For a positive integer l, we denote with $\underline{l} := \{1, 2, \ldots, l\}$. Given a matrix $H = (H_{ij})$, by $H \geq 0$ we mean that $H_{ij} \geq 0$ for all i, j; $H \leq 0$ is similarly defined. Given a set $\mathcal{Z} \subseteq \mathbb{R}^n \times \mathbb{R}^m$, the operator $\mathrm{Proj}_n(\cdot)$ denotes the projection on $\mathbb{R}^n$, defined by $\mathrm{Proj}_n\mathcal{Z} := \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^m$ such that $(x, y) \in \mathcal{Z}\}$. A polyhedron is the intersection of a finite number of closed half-spaces.

A function $J : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty, -\infty\}$ is called *proper* if $\{x \in \mathbb{R}^n : -\infty < J(x) < +\infty\} \neq \emptyset$ [12]. The epigraph of a function $J : X \to \mathbb{R}$ with $X \subseteq \mathbb{R}^n$ is defined as $\mathrm{epi}\, J := \{(x, t) \in X \times \mathbb{R} : J(x) \leq t\}$. A function $J(\cdot)$ is piecewise affine (PWA) if its epigraph is a finite union of polyhedra [12]. Let $\mathcal{F}_{\mathrm{mps}}$ denote the set of *max-plus-scaling* functions, i.e. functions $J : X \to \mathbb{R}$ such that $J(x) = \max_{j\in\underline{l}}\{\alpha_j^T x + \beta_j\}$ for all $x \in X$, with X a polyhedron in $\mathbb{R}^q$, $\alpha_j \in \mathbb{R}^q$ and $\beta_j \in \mathbb{R}$. Let $\mathcal{F}^+_{\mathrm{mps}}$ denote the set of *max-plus-non-negative-scaling* functions, i.e. functions defined by $J(x) = \max_{j\in\underline{l}}\{\alpha_j^T x + \beta_j\}$ with $\alpha_j \geq 0$ for all $j \in \underline{l}$.

B. MPL systems

The system matrices of a discrete event MPL system usually consist of sums or maximization of process times and transportation times. Therefore, we consider the following uncertain MPL system (see also [15]):

$$x(k) = A(w(k)) \otimes x(k-1) \oplus B(w(k)) \otimes u(k) \tag{1a}$$
$$y(k) = C(w(k)) \otimes x(k), \tag{1b}$$

where $A(\cdot) \in \mathcal{F}_{\text{mps}}^{n\times n}, B(\cdot) \in \mathcal{F}_{\text{mps}}^{n\times m}$ and $C(\cdot) \in \mathcal{F}_{\text{mps}}^{p\times n}$ (it is important to note that these matrix functions are nonlinear). We frequently use the short-hand notation

$$f(x,u,w) := A(w) \otimes x \oplus B(w) \otimes u. \tag{2}$$

We gather in the vector w all the uncertainty caused by disturbances and errors in the estimation of physical variables. At each step, the value of the disturbance $w(k)$ is unknown, but takes on values from a compact and convex set W. We also consider a reference signal $\{r(k) \in \mathbb{R}^p\}_{k\geq 0}$ which the system (1) may be required to track.

In the context of DES, k is an event counter while $x(k)$, $y(k)$ and $u(k)$ represent dates, i.e. processing times, finishing times and feeding times, respectively. Since the states $x(k)$ represent times we assume they can always be measured. Note that time is not scalable, therefore typical constraints for DES are:

$$y(k) \leq r(k) + h^{\text{yr}}(k),\ u_i(k) - u_j(k) \leq h^{\text{u}}_{ij}(k) \tag{3a}$$
$$x_i(k) - u_j(k) \leq h^{\text{xu}}_{ij}(k),\ u(k+1) - u(k) \geq 0 \tag{3b}$$

The constraints in (3) can equivalently be written as:

$$\begin{aligned} Hx(k) + Gu(k) + Fr(k) + Ew(k) &\leq h(k), \\ u(k+1) - u(k) &\geq 0, \end{aligned} \tag{4}$$

where $H \geq 0$. The first constraint might express safety or performance requirements, while the constraint $u(k+1) - u(k) \geq 0$ appears in the context of DES where the input represents times, so the signal $u(\cdot)$ should be non-decreasing. Since it is more difficult to cope with this constraint when we use the DP framework, we would like to eliminate it. By remodeling the system, this constraint can be removed. Indeed, introducing a new state vector $\tilde{x} = [x^T, \tilde{u}^T]^T$ and the dynamics:

$$\tilde{x}(k) = \begin{bmatrix} A(w(k)) & B(w(k)) \\ \mathcal{E}_{m\times n} & E_m \end{bmatrix} \otimes \tilde{x}(k-1) \oplus \begin{bmatrix} B(w(k)) \\ E_m \end{bmatrix} \otimes u(k) \tag{5a}$$

$$y(k) = [C(w(k)) \ \ \mathcal{E}_{p\times m}] \otimes \tilde{x}(k) \tag{5b}$$

with the constraint:

$$\tilde{u}(k) \leq u(k) \tag{6}$$

then it is clear that both systems have the same behavior. Note that the constraints (4) and (6) can be written equivalently as $[H \ 0]\tilde{x}(k) + Gu(k) + Fr(k) + Ew(k) \leq h(k)$, $[0 \ I_m]\tilde{x}(k) - I_m u(k) \leq 0$. So we get

$$\tilde{H}\tilde{x}(k) + \tilde{G}u(k) + \tilde{F}r(k) + \tilde{E}w(k) \leq \tilde{h}(k).$$

Note that $\tilde{H} \geq 0$, i.e. by remodeling the system, the properties of H are preserved.

C. Min-max control for MPL systems: problem formulation

We consider the uncertain MPL system (1) subject to general control and state constraints over a finite horizon N:

$$H_k x(k) + G_k u(k) + F_k r(k) + E_k w(k) \leq h_k, k \in \underline{N}, \tag{7}$$

where (7) may vary with k, i.e. $H_k \in \mathbb{R}^{n_{H_k}\times n}, G_k \in \mathbb{R}^{n_{H_k}\times m}, F_k \in \mathbb{R}^{n_{H_k}\times p}, E_k \in \mathbb{R}^{n_{H_k}\times q}$ and $h_k \in \mathbb{R}^{n_{H_k}}$.

We now formulate the problem of finite-horizon min-max control of this class of systems. Effective control in the presence of disturbance requires one to optimize over *feedback policies* [3], [10], rather than open-loop input sequences. Therefore, we will define the decision variable in the optimal control problem, for a given initial condition x and the reference signal $\mathbf{r} := [r_1^T, r_2^T, \ldots, r_N^T]^T$, as a control policy $\pi := (\mu_1(\cdot), \mu_2(\cdot), \ldots, \mu_N(\cdot))$, where each $\mu_i : \mathbb{R}^n \times \mathbb{R}^{Np} \to \mathbb{R}^m$ is a state feedback control. Let $\mathbf{w} := [w_1^T, w_2^T, \ldots, w_N^T]^T$ denote a realization of the disturbance over the horizon $k = 1, 2, \ldots, N$. Also, let $\phi(i; x, \pi, \mathbf{w})$ denote the solution of (1) at step i when the initial state is x at step 0, the control is determined by the policy π, i.e. $u(i) = \mu_i(\phi(i-1; x, \pi, \mathbf{w}), \mathbf{r})$, and the disturbance sequence is $\mathbf{w}$. By definition, $\phi(0; x, \pi, \mathbf{w}) := x$.

The cost $V_N(x, \pi, \mathbf{r}, \mathbf{w})$, for the initial condition x, the due dates $\mathbf{r}$, the control policy π and the disturbance realization $\mathbf{w}$, is:

$$V_N(x, \pi, \mathbf{r}, \mathbf{w}) := \sum_{i=1}^{N} \ell_i(x_i, u_i, r_i, w_i), \tag{8}$$

where $x_i := \phi(i; x, \pi, \mathbf{w})$, $u_i := \mu_i(x_{i-1}, \mathbf{r})$, and ℓ_i is a variable stage cost (see e.g. (12)), for all $i \in \underline{N}$. Note that we do not consider a terminal cost on the state, since u_N directly influences the final state x_N.

For each initial condition x and due dates $\mathbf{r}$ we define the set of feasible policies π:

$$\begin{aligned} \Pi_N(x,\mathbf{r}) := \{\pi : & H_i\phi(i;x,\pi,\mathbf{w}) + G_i\mu_i(\phi(i-1;x,\pi,\mathbf{w}),\mathbf{r}) + \\ & F_i r_i + E_i w_i \leq h_i, \forall \mathbf{w} \in \mathcal{W}, i \in \underline{N}\} \end{aligned} \tag{9}$$

where $\mathcal{W} := W^N$. Also, let X_N denote the set of initial states *and* reference signals for which a feasible policy exists, i.e.

$$X_N := \{(x, \mathbf{r}) : \Pi_N(x, \mathbf{r}) \neq \emptyset\}. \tag{10}$$

The finite-horizon min-max problem considered is:

$$\mathbb{P}_N(x,\mathbf{r}) : V_N^0(x,\mathbf{r}) := \inf_{\pi\in\Pi_N(x,\mathbf{r})} \max_{\mathbf{w}\in\mathcal{W}} V_N(x,\pi,\mathbf{r},\mathbf{w}). \tag{11}$$

Let $\pi_N^0(x,\mathbf{r}) =: (\mu_1^0(x,\mathbf{r}), \mu_2^0(\cdot,\mathbf{r}), \ldots, \mu_N^0(\cdot,\mathbf{r}))$ denote a minimizer of $\mathbb{P}_N(x,\mathbf{r})$, i.e. $\pi_N^0(x,\mathbf{r}) \in \arg\min_{\pi\in\Pi_N(x,\mathbf{r})} \max_{\mathbf{w}\in\mathcal{W}} V_N(x,\pi,\mathbf{r},\mathbf{w})$, whenever the infimum is attained.

Standard optimal control implements the policy $u(k) = \mu_k^0(x(k-1), [r^T(1), \ldots, r^T(N)]^T)$ for $k = 1, 2, \ldots, N$, while receding horizon control [10] involves an iterative approach in which at each step k the first sample of the policy is applied to the system, i.e. $u(k) = \mu_1^0(x(k-1), [r^T(k), \ldots, r^T(k+N-1)]^T)$ for $k = 1, 2, \ldots$

The following key assumptions will be used:

A1: The matrices H_i in (7), (9) are non-negative $\forall i \in \underline{N}$.

A2: The stage cost $\ell_i(\cdot)$ satisfies $\ell_i(\cdot, u, r, w) \in \mathcal{F}^+_{\text{mps}}$, $\forall (u, r, w)$ and $\ell_i(x, \cdot) \in \mathcal{F}_{\text{mps}}$, $\forall x$.

Assumptions **A1** and **A2** are key components of the proofs of Proposition 2.4 and Theorem 2.8 respectively. See the discussion from Section I-B for a justification of why **A1** is not restrictive. A typical example of a stage cost that satisfies **A2** is the following [15]:

$$\ell_i(x_i, u_i, r_i, w_i) = \sum_{j=1}^{p} \max\{(C(w_i) \otimes x_i - r_i)_j, 0\} - \gamma \sum_{j=1}^{m} (u_i)_j, \quad (12)$$

where $(v_i)_j$ denotes the j^{th} component of a vector v_i and $0 \leq \gamma$. In the context of manufacturing systems, this stage cost has the interpretation that the first term penalizes the delay of the finishing times with respect to the due dates, while the second term tries to maximize the feeding times.

We will proceed to show how for the above assumptions, in conjunction with the convexity and monotonicity of the system dynamics (1), an explicit expression of the solution to problem $\mathbb{P}_N(x, \mathbf{r})$ can be computed using results from polyhedral algebra and multi-parametric linear programming. As will be seen below, the results developed in this paper are also valid for a larger class of systems that the MPL systems, namely for convex PWA systems (i.e. $f(\cdot) \in \mathcal{F}^n_{\text{mps}}$) that satisfy $f(\cdot, u, w) \in (\mathcal{F}^+_{\text{mps}})^n$ for any fixed (u, w).

II. Dynamic programming solution

Dynamic programming (DP) [3], [10] is a well-known method for solving sequential, or multi-stage, decision problems. More specifically, we compute sequentially the partial return functions $\{V_i^0(\cdot)\}_{i=1}^N$, the associated set-valued optimal control laws $\{\kappa_i(\cdot)\}_{i=1}^N$ (such that $\mu^0_{N-i+1}(x, \mathbf{r}) \in \kappa_i(x, \mathbf{r})$ for any $(x, \mathbf{r}) \in X_i$) and their domains $\{X_i\}_{i=1}^N$; here $i \in \underline{N}$ denotes "time-to-go". If we define

$$J_i(x, \mathbf{r}, u) := \max_{w \in W} \{\ell_{N-i+1}(f(x, u, w), u, r_{N-i+1}, w) + V^0_{i-1}(f(x, u, w), \mathbf{r})\}, \forall (x, \mathbf{r}, u) \in Z_i, \quad (13a)$$

where

$$Z_i := \{(x, \mathbf{r}, u) : H_{N-i+1} f(x, u, w) + G_{N-i+1} u + F_{N-i+1} r_{N-i+1} + E_{N-i+1} w \leq h_{N-i+1}, (f(x, u, w), \mathbf{r}) \in X_{i-1}, \forall w \in W\}, \quad (13b)$$

then we can compute $\{V_i^0(\cdot), \kappa_i(\cdot), X_i\}_{i=1}^N$ recursively, as follows [3], [10]:

$$V_i^0(x, \mathbf{r}) = \min_u \{J_i(x, \mathbf{r}, u) : (x, \mathbf{r}, u) \in Z_i\}, \ \forall (x, \mathbf{r}) \in X_i, \quad (13c)$$

$$\kappa_i(x, \mathbf{r}) = \arg\min_u \{J_i(x, \mathbf{r}, u) : (x, \mathbf{r}, u) \in Z_i\}, \forall (x, \mathbf{r}) \in X_i, \quad (13d)$$

$$X_i = \text{Proj}_{n+pN} Z_i, \quad (13e)$$

with the boundary conditions

$$X_0 = \mathbb{R}^n \times \mathbb{R}^{pN}, \ V_0^0(x, \mathbf{r}) = 0, \ \forall (x, \mathbf{r}) \in \mathbb{R}^n \times \mathbb{R}^{pN}. \quad (13f)$$

Alternatively, we could also impose a terminal set for the state as in the constraint formulation (7), in which case the first part of (13f) would become

$$X_0 = X_{\text{f}} = \{(x, \mathbf{r}) : \Gamma x + \Phi \mathbf{r} \leq \gamma\} \quad (14)$$

for appropriately defined Γ, Φ, γ with $\Gamma \geqslant 0$.

To simplify notation in the rest of the paper, we define two prototype problems and we study their properties. The maximization problem $\mathbb{P}_{\max}(x, \mathbf{r}, u)$ is defined as:

$$\mathbb{P}_{\max}(x, \mathbf{r}, u): \ J(x, \mathbf{r}, u) := \max_{w \in W} \{\ell(f(x, u, w), u, r, w) + V(f(x, u, w), \mathbf{r})\}, \ \forall (x, \mathbf{r}, u) \in Z, \quad (15)$$

where $\ell : \mathbb{R}^{n+m+p+q} \to \mathbb{R}$, $V : \Omega \to \mathbb{R}$, $\mathbf{r}$ has the form $\mathbf{r} = [\ldots, r^T, \ldots]^T$ (i.e., $\exists k : r_k = r$) and

$$Z := \{(x, \mathbf{r}, u) : H f(x, u, w) + G u + F r + E w \leq h, (f(x, u, w), \mathbf{r}) \in \Omega, \forall w \in W\}, \quad (16a)$$

$$X := \text{Proj}_{n+pN} Z. \quad (16b)$$

The minimization problem $\mathbb{P}_{\min}(x, \mathbf{r})$ is defined as:

$$\mathbb{P}_{\min}(x, \mathbf{r}): \quad V^0(x, \mathbf{r}) := \min_u \{J(x, \mathbf{r}, u) : (x, \mathbf{r}, u) \in Z\}, \ \forall (x, \mathbf{r}) \in X \quad (17a)$$

$$\kappa(x, \mathbf{r}) := \arg\min_u \{J(x, \mathbf{r}, u) : (x, \mathbf{r}, u) \in Z\}, \ \forall (x, \mathbf{r}) \in X. \quad (17b)$$

In terms of these prototype problems, it is easy to identify the DP recursion (13) by setting $r \leftarrow r_{N-i+1}$, $\ell(\cdot) \leftarrow \ell_{N-i+1}(\cdot)$, $V(\cdot) \leftarrow V^0_{i-1}(\cdot)$, $V^0(\cdot) \leftarrow V_i^0(\cdot)$, $X \leftarrow X_i$, $Z \leftarrow Z_i$ and $\Omega \leftarrow X_{i-1}$. Moreover, H, G, F, E, h are identified with $H_{N-i+1}, G_{N-i+1}, F_{N-i+1}, E_{N-i+1}, h_{N-i+1}$, respectively.

Clearly, we can now proceed to show, via induction, that a certain set of properties is possessed by each element in the sequence $\{V_i^0(\cdot), \kappa_i(\cdot), X_i\}_{i=1}^N$ by showing that if $\{V(\cdot), \Omega\}$ has a given set of properties, then $\{V^0(\cdot), X\}$ also has these properties, with the properties of $\kappa(\cdot)$ being the same as those of each of the elements in the sequence $\{\kappa_i(\cdot)\}_{i=1}^N$. In the sequel, constructive proofs of the main results are presented, so that the reader can develop a prototype algorithm for computing the sequence $\{V_i^0(\cdot), \kappa_i(\cdot), X_i\}_{i=1}^N$.

A. Invariance properties of X

The first result states that some basic properties of max-plus-scaling functions are preserved under addition, composition and multiplication with a non-negative scalar.

Lemma 2.1: Suppose the functions g_1, g_2 and $g_3 = [g_{31}, \ldots, g_{3n}]^T$ with g_1, g_2, g_{3j} of the form $g : W \times Z \to \mathbb{R} : (w, z) \mapsto g(w, z)$ have the property that for each $w \in W$, $g_i(\cdot, w)$, $g_{3j}(\cdot, w) \in \mathcal{F}^+_{\text{mps}}$ and for each $z \in Z$, $g_i(z, \cdot)$, $g_{3j}(z, \cdot) \in \mathcal{F}_{\text{mps}}$, for all i, j. Then, for any scalar $\lambda \geq 0$, $(\lambda g_1)(\cdot, w)$, $(g_1 + g_2)(\cdot, w)$, $g_1(g_3(\cdot, w), w) \in \mathcal{F}^+_{\text{mps}}$ for any fixed $w \in W$, and $(\lambda g_1)(z, \cdot)$, $(g_1 + g_2)(z, \cdot)$, $g_1(g_3(z, \cdot), \cdot) \in \mathcal{F}_{\text{mps}}$ for any fixed $z \in Z$.

Proof: The proof is straightforward and uses the fact that $\lambda \max\{a,b\} = \max\{\lambda a, \lambda b\}$ if $\lambda \geq 0$, $\max\{a,b\} + \max\{c,d\} = \max\{a+c, a+d, b+c, b+d\}$ and $\max\{\max\{a,b\}, c\} = \max\{a,b,c\}$. ∎

Lemma 2.2: The set $\mathcal{Z} = \{(x,u) : \bar{H} f(x,u,w) + \bar{G}u + \bar{E}w \leq \bar{h}, \forall w \in W\}$ with $\bar{H} \geq 0$, can be written equivalently as $\mathcal{Z} = \{(x,u) : \tilde{H}x + \tilde{G}u \leq \tilde{h}\}$ with $\tilde{H} \geq 0$.

Proof: Since $f(\cdot) \in \mathcal{F}^n_{\text{mps}}$ and $f(\cdot,w) \in (\mathcal{F}^+_{\text{mps}})^n$ for each w, it follows from Lemma 2.1 that the function $x \mapsto \bar{H} f(x,u,w)$ is in $(\mathcal{F}^+_{\text{mps}})^n$ for any (u,w). Recall that, given any finite set of scalar-valued functions $\{\varphi_j(\cdot)\}_{j \in \underline{l}}$ we have that $\{z : \max_{j \in \underline{l}}\{\varphi_j(z)\} \leq \alpha\} = \{z : \varphi_j(z) \leq \alpha, \forall j \in \underline{l}\}$. Hence, it is easy to verify that the set $\mathcal{Z}$ has the equivalent representation $\mathcal{Z} = \{(x,u) : \tilde{H}x + \tilde{G}u + \tilde{F}w \leq h, \forall w \in W\}$, where $\tilde{H} \geq 0$ and $\tilde{G}$, $\tilde{F}$ and h are suitably defined. If we define $f^*_j := \max_{w \in W}\{\tilde{F}_j w\}$, where $\tilde{F}_j$ denotes the j^{th} row of $\tilde{F}$, then the result follows by letting $\tilde{h} := h - f^*$, where $f^* := (f^*_1, f^*_2, \ldots)$. Note that f^* can be computed by solving a set of convex optimization problems (Recall that W is a compact, convex set.) ∎

The next result shows that some useful properties of a class of polyhedra are inherited by its projection.

Lemma 2.3: Let $\mathcal{Z} = \{(x,r,t,u) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m : \bar{H}x + \bar{F}r + \bar{K}t + \bar{G}u \leq \bar{h}\}$ be given, where $\bar{H} \geq 0$ and $\bar{K} \leq 0$. The set $\mathcal{X} := \{(x,r,t) : \exists u \text{ s.t. } (x,r,t,u) \in \mathcal{Z}\}$ is a polyhedral set of the form $\mathcal{X} = \{(x,r,t) : \tilde{H}x + \tilde{F}r + \tilde{K}t \leq \tilde{h}\}$, where $\tilde{H} \geq 0$ and $\tilde{K} \leq 0$.

Proof: Since $\mathcal{X} = \text{Proj}_{n+p+q}\mathcal{Z}$, it is clear that $\mathcal{X}$ is a polyhedron. We begin by considering the case $m = 1$. The proof for this case will lead to a solution for the case $m > 1$. Let $\bar{H}_i, \bar{F}_i, \bar{K}_i$ and $\bar{G}_i$ be the i^{th} row of respectively the matrices $\bar{H}$, $\bar{F}$, $\bar{K}$ and the vector $\bar{G}$. Define $I_+ := \{i \in \underline{q} : \bar{G}_i > 0\}$, $I_- := \{i \in \underline{q} : \bar{G}_i < 0\}$ and $I_0 := \{i \in \underline{q} : \bar{G}_i = 0\}$. We have the following cases:

1) $i \in I_0 \Rightarrow \bar{H}_i x + \bar{F}_i r + \bar{K}_i t \leq h_i$ and $\bar{H}_i \geq 0$, $\bar{K}_i \leq 0$.
2) $j \in I_+ \Rightarrow u \leq -\frac{1}{\bar{G}_j}\bar{H}_j x - \frac{1}{\bar{G}_j}\bar{F}_j r - \frac{1}{\bar{G}_j}\bar{K}_j t + \frac{\bar{h}_j}{\bar{G}_j}$ and $\frac{1}{\bar{G}_j}\bar{H}_j \geq 0$, $\frac{1}{\bar{G}_j}\bar{K}_j \leq 0$.
3) $l \in I_- \Rightarrow u \geq -\frac{1}{\bar{G}_l}\bar{H}_l x - \frac{1}{\bar{G}_l}\bar{F}_l r - \frac{1}{\bar{G}_l}\bar{K}_l t + \frac{\bar{h}_l}{\bar{G}_l}$ and $-\frac{1}{\bar{G}_l}\bar{H}_l \geq 0$, $-\frac{1}{\bar{G}_l}\bar{K}_l \leq 0$.

It is then straightforward to combine the above and show that the set $\mathcal{X}$ is described by the following inequalities:

$$
\begin{aligned}
&\bar{H}_i x + \bar{F}_i r + \bar{K}_i t \leq \bar{h}_i, \ \forall i \in I_0 \\
&(-\frac{1}{\bar{G}_l}\bar{H}_l + \frac{1}{\bar{G}_j}\bar{H}_j)x + (-\frac{1}{\bar{G}_l}\bar{F}_l + \frac{1}{\bar{G}_j}\bar{F}_j)r + \\
&(-\frac{1}{\bar{G}_l}\bar{K}_l + \frac{1}{\bar{G}_j}\bar{K}_j)t \leq -\frac{\bar{h}_l}{\bar{G}_l} + \frac{\bar{h}_j}{\bar{G}_j}, \ \forall j \in I_+, l \in I_-
\end{aligned}
$$

The result follows, since the rows of $\tilde{H}$ are composed of the vectors $\bar{H}_i \geq 0$ and $-\frac{1}{\bar{G}_l}\bar{H}_l + \frac{1}{\bar{G}_j}\bar{H}_j \geq 0$ for all $i \in I_0$, $j \in I_+$, $l \in I_-$, while the rows of $\tilde{K}$ are composed of the vectors $\bar{K}_i \leq 0$ and $-\frac{1}{\bar{G}_l}\bar{K}_l + \frac{1}{\bar{G}_j}\bar{K}_j \leq 0$ for all $i \in I_0$, $j \in I_+$, $l \in I_-$.

When $m > 1$, the previous reasoning for the case $m = 1$ can be repeated m times, eliminating one component of the vector u at a time. ∎

We are now in a position to show that X has the same structural properties as Ω.

Proposition 2.4: Suppose Ω is a polyhedral set given by $\Omega = \{(x,\mathbf{r}) : \Gamma x + \Phi \mathbf{r} \leq \gamma\}$ with $\Gamma \geq 0$, and assume that H in (16a) satisfies $H \geq 0$. Then, the set X defined in (16b) is a polyhedron given by $X = \{(x,\mathbf{r}) : \hat{H}x + \hat{F}\mathbf{r} \leq \hat{h}\}$, where $\hat{H} \geq 0$.

Proof: The set Z is described as follows:

$$
Z = \{(x,\mathbf{r},u) : \begin{bmatrix} H \\ \Gamma \end{bmatrix} f(x,u,w) + \begin{bmatrix} G \\ 0 \end{bmatrix} u + \begin{bmatrix} Fr \\ \Phi\mathbf{r} \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} w \leq \begin{bmatrix} h \\ \gamma \end{bmatrix}, \forall w \in W\}, \tag{18}
$$

with $H, \Gamma \geq 0$. From Lemma 2.2 it follows that Z can be written as $Z = \{(x,\mathbf{r},u) : \tilde{H}x + \tilde{G}u + \tilde{F}\mathbf{r} \leq \tilde{h}\}$ where $\tilde{H} \geq 0$. The result follows by applying a particular case of Lemma 2.3. ∎

Note that the set X_0 in (13f) and (14) is of the form given in Proposition 2.4.

B. Invariance properties of $\mathbb{P}_{\max}(x,\boldsymbol{r},u)$

This section derives an invariance property of the prototype maximization problem $\mathbb{P}_{\max}$.

Proposition 2.5: If $\ell(\cdot,u,r,w)$, $V(\cdot,\mathbf{r}) \in \mathcal{F}^+_{\text{mps}}$ for any fixed $(u,\mathbf{r},w)$ and $\ell(x,\cdot)$, $V(x,\cdot) \in \mathcal{F}_{\text{mps}}$ for any fixed x, then $J(\cdot)$ possesses the same properties, i.e. $J(\cdot,\mathbf{r},u) \in \mathcal{F}^+_{\text{mps}}$ for any fixed $(\mathbf{r},u)$ and $J(x,\cdot) \in \mathcal{F}_{\text{mps}}$ for fixed x.

Proof: It follows from Lemma 2.1 that we can write $\ell(f(x,u,w),u,r,w) + V(f(x,u,w),\mathbf{r}) = \max_{j \in \underline{l}}\{\alpha_j^T x + \beta_j^T w + \gamma_j^T u + \delta_j^T \mathbf{r} + \tilde{\theta}_j\}$, where $\alpha_j \geq 0 \ \ \forall j \in \underline{l}$, so that

$$
\begin{aligned}
J(x,\mathbf{r},u) &= \max_{w \in W}\{\max_{j \in \underline{l}}\{\alpha_j^T x + \beta_j^T w + \gamma_j^T u + \delta_j^T \mathbf{r} + \tilde{\theta}_j\}\} \\
&= \max_{j \in \underline{l}}\{\max_{w \in W}\{\alpha_j^T x + \beta_j^T w + \gamma_j^T u + \delta_j^T \mathbf{r} + \tilde{\theta}_j\}\} \\
&= \max_{j \in \underline{l}}\{\alpha_j^T x + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j\},
\end{aligned}
$$

where $\theta_j := \tilde{\theta}_j + \max_{w \in W}\{\beta_j^T w\}$ for all $j \in \underline{l}$. Note that $\{\theta_j\}_{j \in \underline{l}}$ can be computed by solving a sequence of convex optimization problems. ∎

Note that $\ell(\cdot)$ and $V_0^0(\cdot)$ given in (12) and (13f) satisfy the conditions of Proposition 2.5.

C. Invariance properties of $\mathbb{P}_{\min}(x,\boldsymbol{r})$

This section derives the main properties of $V^0(\cdot)$ and $\kappa(\cdot)$. Before proceeding, we show that if $V^0(\cdot)$ is proper, then $V^0(\cdot)$ is finite everywhere on X. Note that since we always have that $u(0)$ should be larger than the current time instant, i.e. the time instant at which we start performing the computations, $u(0)$ is bounded from below and $V^0(\cdot)$ will always be proper.

Lemma 2.6: Suppose Ω is a polyhedral set given by $\Omega = \{(x,\mathbf{r}) : \Gamma x + \Phi\mathbf{r} \leq \gamma\}$ with $\Gamma \geq 0$, and assume that H in (16a) satisfies $H \geq 0$. Suppose also that $Z \neq \emptyset$ and $J(\cdot) \in \mathcal{F}_{\text{mps}}$. There exists a $(\bar{x},\bar{\mathbf{r}}) \in X$ such that $V^0(\bar{x},\bar{\mathbf{r}})$ is finite if and only if $V^0(x,\mathbf{r})$ is finite for all $(x,\mathbf{r}) \in X$.

Proof: From the proof of Proposition 2.4 it follows that Z is a non-empty polyhedron: $Z = \{(x,\mathbf{r},u) : \tilde{H}x + \tilde{G}u +$

$\tilde{F}\mathbf{r} \le \tilde{h}\}$, with $\tilde{H} \ge 0$. Since $J(\cdot) \in \mathcal{F}_{\rm mps}$, we can write $J(x, \mathbf{r}, u) = \max_{j\in \underline{l}}\{\alpha_j^T x + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j\}$. The prototype minimization problem $\mathbb{P}_{\min}(x, \mathbf{r})$ becomes:

$$\begin{aligned} &V^0(x, \mathbf{r}) = \\ &\min_u\{\max_{j\in \underline{l}}\{\alpha_j^T x + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j\} : (x, \mathbf{r}, u) \in Z\} \\ =&\min_{\mu,u}\{\mu : \alpha_j^T x + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j \le \mu, \forall j \in \underline{l}, \\ &\quad \tilde{H}x + \tilde{G}u + \tilde{F}\mathbf{r} \le \tilde{h}\}, \end{aligned} \tag{19}$$

i.e. we have obtained a feasible linear program for any fixed $(x, \mathbf{r}) \in X$.

Note that the feasible set of the dual of (19) does not depend on x or $\mathbf{r}$. Assume that $V^0(\bar{x}, \mathbf{r})$ is finite. From strong duality for linear programs [12], [13] it follows that the dual problem of (19) is feasible, independent of x and $\mathbf{r}$. Using again strong duality for linear programs, we conclude that $V^0(x, \mathbf{r})$ is finite if $(x, \mathbf{r}) \in X$ and $V^0(x, \mathbf{r}) = +\infty$ if $(x, \mathbf{r}) \notin X$. The reverse implication is obvious. ∎

The following proposition gives a characterization of the solution and of the optimal value of the prototype minimization problem $\mathbb{P}_{\min}$.

Proposition 2.7: Suppose Ω is a polyhedral set given by $\Omega = \{(x, \mathbf{r}) : \Gamma x + \Phi \mathbf{r} \le \gamma\}$ with $\Gamma \ge 0$, and assume that H in (16a) satisfies $H \ge 0$. Suppose also that $Z \ne \emptyset$, $J(\cdot) \in \mathcal{F}_{\rm mps}$ and $V^0(\cdot)$ is proper. Then, the value function $V^0(\cdot)$ is in $\mathcal{F}_{\rm mps}$ and has domain X, where X is a polyhedral set. The (set-valued) control law $\kappa(x, \mathbf{r})$ is a polyhedron for a given $(x, \mathbf{r}) \in X$. Moreover, it is always possible to select a continuous and PWA control law $\mu(\cdot)$ such that $\mu(x, \mathbf{r}) \in \kappa(x, \mathbf{r})$ for all $(x, \mathbf{r}) \in X$.

Proof: It follows from (19) that $\mathbb{P}_{\min}(x, \mathbf{r})$ is a multi-parametric linear program of the type $\min_z\{c^T z : \bar{H}\phi + \bar{G}z \le \bar{h}\}$, where the vector of parameters is ϕ and the optimization variable is z (in our case, from (19) we conclude that $\phi = [x^T \; \mathbf{r}^T]^T$ and $z = [\mu \; u^T]^T$). The properties stated above then follow from the properties of the multi-parametric linear program (see [4], [7], [14]). ∎

Now we can state the following key result that, together with Propositions 2.4–2.7, allow us to deduce, via induction, some important properties of the sequence $\{V_i^0(\cdot), \kappa_i(\cdot), X_i\}_{i=1}^N$:

Theorem 2.8: Suppose that the same assumptions as in Proposition 2.7 hold. If, in addition, $J(\cdot, \mathbf{r}, u) \in \mathcal{F}^+_{\rm mps}$ for any $(\mathbf{r}, u)$, then the value function $V^0(\cdot, \mathbf{r}) \in \mathcal{F}^+_{\rm mps}$ for any $\mathbf{r} \in \mathbb{R}^{Np}$.

Proof: Using Proposition 2.4 it follows that $Z = \{(x, \mathbf{r}, u) : \tilde{H}x + \tilde{G}u + \tilde{F}\mathbf{r} \le \tilde{h}\}$, with $\tilde{H} \ge 0$. The function $J(\cdot)$ can be written as: $J(x, \mathbf{r}, u) = \max_{j\in\underline{l}}\{\alpha_j^T x + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j\}$, where $\alpha_j \ge 0$ for all j. From Proposition 2.7 and the fact that $V^0(\cdot)$ is proper, it follows that $V^0(\cdot) \in \mathcal{F}_{\rm mps}$ and its domain is X. The epigraph of $V^0(\cdot)$ is given by:

$$\begin{aligned} \operatorname{epi} V^0 :=& \{(x, \mathbf{r}, t) : V^0(x, \mathbf{r}) \le t, x \in X\} = \\ &\{(x, \mathbf{r}, t) : \exists u \text{ s.t. } (x, \mathbf{r}, u) \in Z, J(x, \mathbf{r}, u) \le t\} \\ =& \{(x, \mathbf{r}, t) : \exists u \text{ s.t. } \tilde{H}x + \tilde{G}u + \tilde{F}\mathbf{r} \le \tilde{h}, \\ & \alpha_j^T x + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j \le t, \forall j \in \underline{l}\} \\ =& \{(x, \mathbf{r}, t) : \exists u \text{ s.t. } \bar{H}x + \bar{F}\mathbf{r} + \bar{K}t + \bar{G}u \le \bar{h}\}, \end{aligned}$$

where $\bar{H} = [\tilde{H}^T \; \alpha_1^T \; \cdots \; \alpha_l^T]^T \ge 0$ and $\bar{K} = [0, -1, \ldots, -1]^T \le 0$. From Lemma 2.3 we obtain that the epigraph of $V^0(\cdot)$ is a polyhedral set given by epi $V^0 = \{(x, \mathbf{r}, t) : \hat{H}x + \hat{F}\mathbf{r} + \hat{K}t \le \hat{h}\}$, where $\hat{H} \ge 0, \hat{K} \le 0$. Let l be the number of inequalities describing $\operatorname{epi} V^0$. We arrange the indices[1] $j \in \underline{l}$ such that $\hat{K}_j < 0$ for $j = 1, 2, \ldots, v$ but $\hat{K}_j = 0$ for $j = v+1, v+2, \ldots, l$ (possibly $v = 0$, i.e. $\hat{K}_j = 0$ for all j). Taking $a_j = -\hat{H}_j/\hat{K}_j, b_j = -\hat{F}_j/\hat{K}_j$ and $c_j = -\hat{h}_j/\hat{K}_j$ for $j = 1, 2, \ldots, v$, we get that the epigraph of $V^0(\cdot)$ is expressed as:

$$\begin{aligned} \operatorname{epi} V^0 =& \{(x, \mathbf{r}, t) : a_j x + b_j \mathbf{r} - c_j \le t, \forall j \in \underline{v}; \\ & \hat{H}_j x + \hat{F}_j \mathbf{r} \le \hat{h}_j \text{ for } j = v+1, \ldots, l\} \end{aligned} \tag{20}$$

But $V^0(\cdot)$ is proper, therefore $v > 0$. Since $V^0(\cdot) \in \mathcal{F}_{\rm mps}$, (20) gives us a representation of $V^0(\cdot)$ as $V^0(x, \mathbf{r}) = \max_{j\in\underline{v}}\{a_j x + b_j \mathbf{r} - c_j\}$, where $a_j = -\hat{H}_j/\hat{K}_j \ge 0$, $\forall j \in \underline{v}$, i.e. $V^0(\cdot, \mathbf{r}) \in \mathcal{F}^+_{\rm mps}$ for any fixed $\mathbf{r} \in \mathbb{R}^p$. Moreover, the domain of $V^0(\cdot)$ is $\{(x, \mathbf{r}) : \hat{H}_j x + \hat{F}_j \mathbf{r} \le \hat{h}_j$ for $j = v+1, \ldots, l\}$ and coincides with X (Proposition 2.7). ∎

We now summarize the main results. Based on the invariance properties of the two prototype problems $\mathbb{P}_{\max}$ and $\mathbb{P}_{\min}$, we can derive the properties of $V_i^0(\cdot)$, $\kappa_i(\cdot)$ and X_i for all $i \in \underline{N}$. The next result follows by applying Propositions 2.4–2.7 and Theorem 2.8 to the DP equations (13):

Theorem 2.9: Suppose that **A1** and **A2** hold, Z_i is non-empty and $V_i^0(\cdot)$ is proper for all $i \in \underline{N}$. The following holds for each $i \in \underline{N}$:

(i) X_i is a non-empty polyhedron,
(ii) $V_i^0(\cdot)$ is a convex, continuous PWA function with domain X_i,
(iii) $V_i^0(\cdot, \mathbf{r}) \in \mathcal{F}^+_{\rm mps}$ for any fixed $\mathbf{r}$.
(iv) There exists a continuous PWA function $\mu^0_{N-i+1}(\cdot)$ such that $\mu^0_{N-i+1}(x, \mathbf{r}) \in \kappa_i(x, \mathbf{r})$ for all $(x, \mathbf{r}) \in X_i$.

Since the proofs of all the above results are constructive, the sequences $\{V_i^0(\cdot), \kappa_i(\cdot), X_i\}_{i=1}^N$ and $\{\mu_i^0(\cdot)\}_{i=1}^N$ can be computed iteratively, without gridding, by noting the following:

- Given X_{i-1}, we can compute X_i by first computing Z_i, as in the proof of Proposition 2.4, followed with a projection operation,
- Given $V^0_{i-1}(\cdot)$, a max-plus-scaling expressions of $J_i(\cdot)$ can be computed by referring to the proof of Proposition 2.5,
- Given $J_i(\cdot)$ and Z_i, we can compute $V_i^0(\cdot)$, $\kappa_i(\cdot)$ and a $\mu^0_{N-i+1}(\cdot)$ as in the proof of Proposition 2.7 or Theorem 2.8, either by using multi-parametric linear

[1] We denote with C_j the j^{th} row of a matrix C.

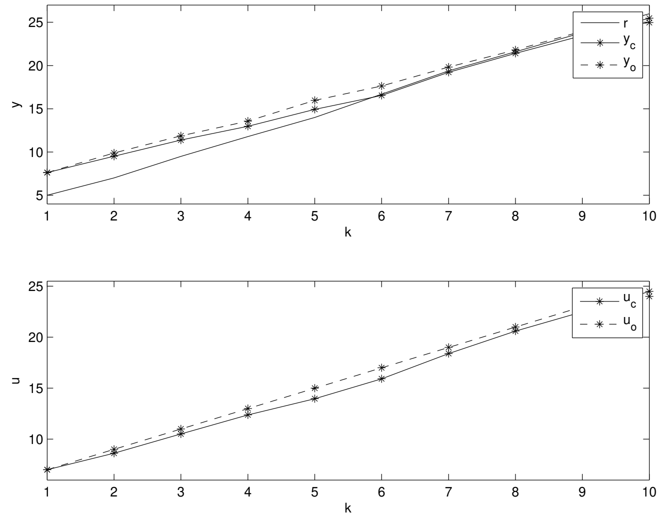

Fig. 1. The results obtained from feedback controller (full star) and the open-loop controller (dashed star). The full line represents the due dates.

programming algorithms [4], [7], [14] or projection algorithms [7].

Note that if Z_i is non-empty the problem is infeasible anyway; furthermore, as explained before, V^0 will always be proper.

III. Example

We consider the following example:

$$x(k) = \begin{bmatrix} -w_2(k)+2 & \varepsilon \\ -w_1(k)+3w_2(k) & -w_1(k)+3 \end{bmatrix} \otimes x(k-1) \oplus \begin{bmatrix} -w_1(k)+w_2(k)+2 \\ w_2(k) \end{bmatrix} \otimes u(k)$$

$$y(k) = [0 \quad \varepsilon] \otimes x(k).$$

We assume a bounded disturbance: $W = \{[w_1, w_2]^T : 2 \le w_1 \le 3, 1 \le w_2 \le 2, w_1 + w_2 \le 4\}$.

We consider a finite horizon $N = 10$. The due date signal is $\mathbf{r} = [4.5\ 6\ 9\ 12\ 14\ 16.7\ 19\ 21.5\ 23.5\ 26]^T$ and the initial state is $x(0) = [7\ \ 9]^T$.

The system is subject to input-output constraints: $x_2(k) - u(k) \le 2, x_1(N) + x_2(N) \le 2r_N, u(k+1) - u(k) \ge 0, -5 + r_k \le u(k) \le 5 + r_k$.

We use the stage cost defined in (12) with $\gamma = 0.2$, and a random sequence of disturbances.

In Figure 1 we have plotted the output (top) and the input (bottom) obtained from the open-loop controller (using the approach of [15]) and from the feedback controller presented in this paper. The top figure shows that the feedback controller gives a better tracking than the open-loop controller. Since $\gamma = 0.2$, it follows that for the stage cost (12) tracking has priority compared to the latest input. Therefore, $V_N^0(x(0), \mathbf{r}) < V_N^{0,\mathrm{ol}}(x(0), \mathbf{r})$, where $V_N^{0,\mathrm{ol}}(x(0), \mathbf{r})$ is the optimal value function of (11) when we optimize over open-loop input sequences (see [15]).

IV. Conclusions

In this paper we have extended the min-max optimal control problem traditionally applied to linear and nonlinear systems to the class of discrete event max-plus-linear systems. We have provided sufficient conditions that guarantee that the value function of the finite-horizon min-max problem for max-plus-linear systems is convex, continuous and piecewise affine, and that the optimal control policy is continuous and piecewise affine on a polyhedral domain. Furthermore, we have shown that we can compute an optimal control policy over a prediction horizon of N steps by solving N multi-parametric linear programming problems.

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