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Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
phone: +31-15-278.24.73 (secretary)
URL: https://www.dcsc.tudelft.nl

Distributed Kalman Filtering for Multiagent Systems

Zs. Lendek R. Babuška B. De Schutter

Abstract— **For naturally distributed systems, such as multi-agent systems, the construction and tuning of a centralized observer may be computationally expensive or even intractable. An important class of distributed systems can be represented as cascaded subsystems. For this class of systems, observers may be designed separately for the subsystems. If the subsystems are linear, the Kalman filter provides an efficient means to estimate the states, so that it minimizes the mean squared estimation error. Kalman-like filters may be used for the whole system or the individual subsystems. In this paper, both a theoretical comparison and simulation examples are presented. The theoretical results show that the distributed observers, except for special cases, do not minimize the overall error covariance, and so the distributed observer system is suboptimal. However, in practice, the performance achieved by the cascaded observers is comparable and in certain cases outperforms that of the centralized one. Moreover, a distributed observer system leads to increased modularity, reduced complexity, and lower computational costs.**

Index Terms— **State estimation, Kalman filters, multi-agent systems**

I. Introduction

Many problems in decision making, control, and monitoring require the estimation of states and possibly uncertain parameters, based on a dynamic system model and a sequence of noisy measurements. For such a purpose, dynamic systems are often modeled in the state-space framework, either in deterministic or stochastic form.

For a system with a large number of states, or for naturally distributed systems, the construction and tuning of a centralized observer may be computationally expensive or even intractable. Decentralized state estimation has been studied in the context of large-scale processes and distributed systems. The architecture in general takes the form of a network of sensor nodes, each with its own processing facility. Each node shares information with other nodes and computes a local state estimate. Computation and communication is distributed over the network so that a global estimate can be computed. Several topologies have been proposed, depending on the particular application. In case of large scale processes [1], [2], the network is in general in a hierarchical form, with several intermediate and one final fusion node, i.e., node where the estimates are combined. For distributed systems, such as multiagent societies [3]–[5], several fusion nodes exist, which process the data and send the information to the rest of the nodes.

The authors are with the Delft Center for Systems and Control, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands {zs.lendek, r.babuska}@tudelft.nl

B. De Schutter is also with the Marine and Transport Technology Department of the Delft University of Technology (email: b@deschutter.info).

An important class of distributed systems, such as hierarchical large-scale system, can be represented as cascaded subsystems. In several cases, conclusions referring to the overall system can be drawn based on the study of the individual subsystems. For instance, for linear time-invariant systems, the stability of the subsystems implies the stability of the cascaded system [6].

If a system model can be decomposed into cascaded subsystems, separate estimators may be designed for the individual subsystems. The idea behind this type of estimation is that many systems can be represented as cascaded, observable subsystems, which are less complex than the original system. This makes the tuning easier. Moreover, different types of observers can be combined, depending on the subsystems considered. Such a setting can be perceived as a cooperative multi-agent system. Each agent has the task of observing one of the subsystems, possibly using different methods and relying on its own measurements and the information gathered from other agents. In turn, each agent communicates its own results to other agents. If all the agents in a system use the same observer method, then such an observer system can be designed and implemented in a modular manner. However, currently, no results are available on the performance analysis of the local observers versus a centralized observer.

The most well-known and widely used probabilistic estimation methods are the Kalman filter and its extension to nonlinear systems, the Extended Kalman Filter [7], [8]. While the Kalman filter has severe limitations and becomes unstable for highly nonlinear processes, for a linear process, it provides an efficient means to estimate the states so that it minimizes the mean squared error. The filter supports the estimation of past, present and future states, even if a precise model of the system considered is unknown.

Since the publication of the Kalman's seminal paper in 1960 [7], the Kalman filter has been the subject of extensive research and applications, particularly in the area of autonomous robots, assisted navigation and sensor data fusion [9]–[11]. A wide variety of Kalman filters have also been developed from the Kalman's original formulation: the extended Kalman filter, the information filter and the family of sigma-point Kalman filters [12].

In this paper, we design Kalman-type filters for cascaded subsystems and study the performance of the cascaded filters. We present a theoretical comparison of the centralized and cascaded Kalman filter and also compare their performance on several examples.

The structure of the paper is as follows. Section 2 presents the proposed cascaded observer setting, Section 3 reviews the

Kalman Filter methodology. The distributed Kalman filters are presented in Section 4, with three illustrative examples in Section 5. Section 6 concludes the paper.

II. Cascaded Subsystems

Consider the following observable linear MIMO system:

$$\begin{aligned} \mathbf{x}(k) &= A\mathbf{x}(k-1) + B\mathbf{u}(k-1) \\ \mathbf{y}(k) &= C\mathbf{x}(k) \end{aligned} \tag{1}$$

and assume that this system can be partitioned into subsystems. For the ease of notation, only two subsystems are considered, $\mathbf{x} = [\mathbf{x}^{1^T} \ \ \mathbf{x}^{2^T}]^T$ and $\mathbf{y} = [\mathbf{y}^{1^T} \ \ \mathbf{y}^{2^T}]^T$:

$$\begin{aligned} \mathbf{x}^1(k) &= A_{11}\mathbf{x}^1(k-1) + B_1\mathbf{u}(k-1) \\ \mathbf{y}^1(k) &= C_{11}\mathbf{x}^1(k) \end{aligned} \tag{2}$$

and

$$\begin{aligned} \mathbf{x}^2(k) &= A_{22}\mathbf{x}^2(k-1) + B_2\mathbf{u}(k-1) + A_{21}\mathbf{x}^1(k-1) \\ \mathbf{y}^2(k) &= C_{22}\mathbf{x}^2(k) + C_{21}\mathbf{x}^1(k) \end{aligned} \tag{3}$$

so that (2) is observable. Note that, since both systems (1) and (2) are observable, this also means that the subsystem (3) is observable for given $\mathbf{x}^1(k)$ and $\mathbf{x}^1(k-1)$. In fact, for subsystem (3), $\mathbf{x}^1(k-1)$ is an input.

In general, such a partition of the model does not necessarily exist. The necessary and sufficient condition for the existence of a partition is that the A and C matrices can be transformed into block lower-triangular forms. If the partition exists, it might not be unique. Consider, for instance, the system

$$\begin{aligned} &x_1(k) = x_1(k-1) + x_3(k-1) && y_1(k) = x_1(k) \\ &x_2(k) = x_2(k-1) + x_3(k-1) && y_2(k) = x_2(k) \\ &x_3(k) = u(k-1) \end{aligned}$$

This system is observable, and there are two possible ways to partition it: by using as the first subsystem

$$\begin{aligned} &x_1(k) = x_1(k-1) + x_3(k-1) && y_1(k) = x_1(k) \\ &x_3(k) = u(k-1) \end{aligned}$$

or, by using as the first subsystem

$$\begin{aligned} &x_2(k) = x_2(k-1) + x_3(k-1) && y_2(k) = x_2(k) \\ &x_3(k) = u(k-1) \end{aligned}$$

Both subsystems are observable.

Given the above partitioning, observers may be designed for the two subsystems separately, with the second observer using the results of the first observer. Such a structure is depicted in Figure 1.

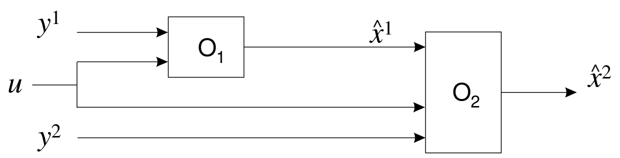


Fig. 1. Cascaded observers.

Currently, a general analysis of the joint performance (convergence, convergence rate, optimality) of the two observers and a centralized observer designed for the system (1) does not exist. In the remainder of the paper we study the conditions under which Kalman-type filters can be designed for the two subsystems so that the performance of the cascaded filters is the same as that of a single Kalman filter for system (1).

III. Kalman Filter

The Kalman filter addresses the problem of estimating the state $\mathbf{x} \in \mathbb{R}^n$ of a linear discrete-time process:

$$\begin{aligned} \mathbf{x}(k) &= A\mathbf{x}(k-1) + B\mathbf{u}(k-1) + \mathbf{w}(k-1) \\ \mathbf{y}(k) &= C\mathbf{x}(k) + \mathbf{v}(k) \end{aligned} \tag{4}$$

with $\mathbf{x}_0$ (initial state) and P_0 (initial covariance of the states) known or previously estimated.

The inputs $\mathbf{w}(k)$ and $\mathbf{v}(k)$ are random variables, representing the process and measurement noise, respectively. These noises are assumed to be independent, white and with normal probability distributions $\mathbf{w}(k) \sim \mathcal{N}(0, Q)$ and $\mathbf{v}(k) \sim \mathcal{N}(0, R)$. In general, the process noise covariance and measurement noise covariance matrices (Q and R), the state transition matrix A and the measurement matrix C can change at every time step; however, here, they are assumed constant to simplify the notation. The objective is to recursively estimate or filter the state $\mathbf{x}_k$ based on the available measurements.

The Kalman filter works in two steps: prediction:

$$\begin{aligned} \widehat{\mathbf{x}}(k|k-1) &= A\mathbf{x}(k-1) + B\mathbf{u}(k-1) \\ P(k|k-1) &= AP(k-1)A^T + Q \end{aligned} \tag{5}$$

and update or correction:

$$\begin{aligned} \widehat{\mathbf{x}}(k) =& \widehat{\mathbf{x}}(k|k-1) + K(k)(\mathbf{y}(k) - C\widehat{\mathbf{x}}(k|k-1)) \\ P(k) =& (I - K(k)C)P(k|k-1)(I - K(k)C)^T + \\ & K(k)RK^T(k) \end{aligned} \tag{6}$$

where $\widehat{\mathbf{x}}(k)$ ($P(k)$) refers to the estimate of the states (covariance) obtained by using all the measurements up to k. The Kalman gain $K(k)$ is computed at each step k so that it minimizes the error covariance $P(k)$. This is obtained by minimizing the trace of $P(k)$ at every step, as given by (7).

Then, assuming that at step $k-1$ the error covariance matrix is $P(k-1)$, the covariance and the Kalman gain at step k is expressed by (8).

IV. Distributed Kalman Filters

Consider the linear system (4), corrupted with zero-mean Gaussian noise and assume that the system can be written in the form (9), i.e., as two cascaded subsystems. Our goal is to design separate observers for the two subsystems, so that the cascaded observers have the same performance (error covariance) as the Kalman filter designed for the joint system. Note that for the system to be cascaded without losing available information (e.g., cross-covariances of states

$$\begin{aligned} \frac{\partial(\mathrm{tr}(P(k)))}{\partial K(k)} &= -2CP(k|k-1) + 2(CP(k|k-1)C^T + R)K^T(k) = 0 \\ &\implies K(k) = P(k|k-1)C^T(CP(k|k-1)C^T + R)^{-1} \end{aligned} \tag{7}$$

$$\begin{aligned} P(k) &= (I - K(k)C)(AP(k-1)A^T + Q)(I - K(k)C)^T + K(k)RK^T(k) \\ K(k) &= (AP(k-1)A^T + Q)C^T(C(AP(k-1)A^T + Q)C^T + R)^{-1} \end{aligned} \tag{8}$$

$$\begin{aligned} \begin{pmatrix} \mathbf{x}^1(k) \\ \mathbf{x}^2(k) \end{pmatrix} &= \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \mathbf{x}^1(k-1) \\ \mathbf{x}^2(k-1) \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \mathbf{u}(k-1) + \begin{pmatrix} \mathbf{w}^1(k-1) \\ \mathbf{w}^2(k-1) \end{pmatrix} \\ \begin{pmatrix} \mathbf{y}^1(k) \\ \mathbf{y}^2(k) \end{pmatrix} &= \begin{pmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} \mathbf{x}^1(k) \\ \mathbf{x}^2(k) \end{pmatrix} + \begin{pmatrix} \mathbf{v}^1(k) \\ \mathbf{v}^2(k) \end{pmatrix} \end{aligned} \tag{9}$$

belonging to different subsystems), the covariance matrices should also be block-diagonal, i.e., $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$ and $R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$. While this condition appears restrictive, in practice one rarely knows the true cross-covariances and it is often assumed that the covariance matrix is diagonal [13], [14].

Since our goal is to design separate observers for the two subsystems, but still minimize the estimation error covariance of the joint filter, we use separate Kalman filters for each subsystem. The first subsystem can be expressed as:

$$\begin{aligned} \mathbf{x}^1(k) &= A_{11}\mathbf{x}^1(k-1) + B_1\mathbf{u}(k-1) + \mathbf{w}^1(k-1) \\ \mathbf{y}^1(k) &= C_{11}\mathbf{x}^1(k) + \mathbf{v}^1(k) \end{aligned} \tag{10}$$

which is a linear system, with $\mathbf{w}^1(k) \sim \mathcal{N}(0, Q_1)$ and $\mathbf{v}^1(k) \sim \mathcal{N}(0, R_1)$ and the deterministic input $\mathbf{u}$. In order to minimize the error covariance for the first subsystem, the Kalman filter presented in Section III is used. Then, for the first subsystem (with the deterministic input $\mathbf{u}$), the covariance and the gain at each time step can be written as (11), The second subsystem can be expressed as:

$$\begin{aligned} \mathbf{x}^2(k) =& A_{22}\mathbf{x}^2(k-1) + B_2\mathbf{u}(k-1) + \\ & A_{21}\mathbf{x}^1(k-1) + \mathbf{w}^2(k-1) \\ \mathbf{y}^2(k) =& C_{22}\mathbf{x}^2(k) + C_{21}\mathbf{x}^1(k) + \mathbf{v}^2(k) \end{aligned} \tag{12}$$

with $\mathbf{w}^2(k) \sim \mathcal{N}(0, Q_2)$ and $\mathbf{v}^2(k) \sim \mathcal{N}(0, R_2)$, the deterministic input $\mathbf{u}$ and the stochastic variable $\mathbf{x}^1$. In a multi-agent setting, agents may communicate only the state estimate, and not the covariance. In such a case, $\mathbf{x}^1$ can also be considered a deterministic input. Thus, two cases can be distinguished.

Case 1: Use $\mathbf{x}^1$ as another **deterministic input** besides $\mathbf{u}$ for the second subsystem. This will be the case in a multi-agent system, if the agent entirely trusts the estimate of another agent, considers it correct and does not take into account possible errors, or that only a distribution of the estimate is available. In this case, the Kalman filter can be used also for this subsystem, and the expression for the covariance and the gain are given by (13). However, in this case, the computed error covariance is not equal to the true error covariance for the second subsystem.

Case 2: If the covariance of the estimates is also available, then $\mathbf{x}^1$ can be considered as a **stochastic input**, with estimated covariance $P_1(k)$, for the second subsystem. For this case, a Kalman-type gain can be computed by minimizing the trace of the error covariance for the second subsystem, assuming that $\mathbf{x}^1$ is a stochastic variable with a known covariance matrix P_1 (14). The covariance for $\mathbf{x}^2$ is calculated as (15), where $P_2(k)$ is the true covariance obtained for the states of the second subsystem.

In both cases, the observer gain and the covariance matrix for the whole system are expressed as:

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \qquad P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \tag{16}$$

However, only in the second case (if $\mathbf{x}^1$ is considered a stochastic input), the covariance matrix for the joint system equals the true covariance obtained by the observers.

Proposition 1: The cascaded setting achieves the same error covariance as the centralized Kalman filter if and only if the subsystems are independent, i.e., in (9), $A_{21} = 0$, $C_{21} = 0$, $R_{12} = 0$ and $Q_{12} = 0$.

Proof: Assume that the joint form of the cascaded Kalman filters is equivalent to that of the centralized Kalman filter. If this assumption holds, then it is also possible to decompose the error system and the Kalman gain obtained for the joint system. Let

$$P(k|k-1) = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \tag{17}$$

Then, $CP(k|k-1)C^T + R$ can be expressed as (18). The conditions for the observer to be partitioned without losing optimality, are given by (19). Moreover, $P(k|k-1)$ is expressed as (20), and it is also required that $P_{21} = P_{12}^T = 0$ (due to the form of the covariance matrix obtained in (16)). Under these conditions, the requirements expressed by (19) will only be fulfilled if the two subsystems are independent, i.e., $A_{21} = 0$, $C_{21} = 0$, $R_{12} = 0$ and $Q_{12} = 0$. Only in this

$$\begin{aligned} P_1(k) &= (I - K_1(k)C_{11})(A_{11}P_1(k-1)A_{11}^T + Q_1)(I - K_1(k)C_{11})^T + K_1(k)R_1K_1^T(k) \\ K_1(k) &= (A_{11}P_1(k-1)A_{11}^T + Q_1)C_{11}^T(C_{11}(A_{11}P_1(k-1)A_{11}^T + Q_1)C_{11}^T + R_1)^{-1} \end{aligned} \tag{11}$$

$$\begin{aligned} P_2(k) &= (I - K_2(k)C_{22})(A_{22}P_2(k-1)A_{22}^T + Q_2)(I - K_2(k)C_{22})^T + K_2(k)R_2K_2^T(k) \\ K_2(k) &= (A_{22}P_2(k-1)A_{22}^T + Q_2)C_{22}^T(C_{22}(A_{22}P_2(k-1)A_{22}^T + Q_2)C_{22}^T + R_2)^{-1} \end{aligned} \tag{13}$$

$$\begin{aligned} 0 &= -2C_{22}(A_{22}P_2(k-1)A_{22}^T + A_{21}P_1(k-1)A_{21}^T + Q_2) + 2(C_{22}(A_{22}P_2(k-1)A_{22}^T + \\ &\quad A_{21}P_1(k-1)A_{21}^T + Q_{22})C_{22}^T + R_2)K_2(k)^T + 2C_{21}P_1(k-1)C_{21}^TK_2^T(k) \\ K_2(k) &= (C_{22}(A_{22}P_2(k-1)A_{22}^T + A_{21}P_1(k-1)A_{21}^T + Q_2))^T \cdot ((C_{22}(A_{22}P_2(k-1)A_{22}^T + \\ &\quad A_{21}P_1(k-1)A_{21}^T + Q_{22})C_{22}^T + R_2 + C_{21}P_1(k-1)C_{21}^T)^{-1})^T \end{aligned} \tag{14}$$

$$\begin{aligned} P_2(k) &= (I - K_2(k)C_{22})(A_{22}P_2(k-1)A_{22}^T + A_{21}P_1(k-1)A_{21}^T + Q_2)(I - K_2(k)C_{22})^T + \\ &\quad K_2(k)R_2K_2^T(k) + K_2(k)C_{21}P_1(k-1)(K_2(k)C_{21})^T \end{aligned} \tag{15}$$

$$\begin{aligned} &CP(k|k-1)C^T + R = \\ &= \begin{pmatrix} C_{11}P_{11}C_{11}^T + R_{11} & C_{11}P_{11}C_{21}^T + C_{11}P_{12}C_{22}^T + R_{12} \\ C_{21}P_{11}C_{11}^T + C_{22}P_{21}C_{11}^T + R_{21} & C_{21}(P_{11}C_{21}^T + P_{12}C_{22}^T) + C_{22}(P_{21}C_{21}^T + P_{22}C_{22}^T) + R_{22} \end{pmatrix} \end{aligned} \tag{18}$$

$$\begin{aligned} P_{11}C_{11}^T &= K_1(k)(C_{11}P_{11}C_{11}^T + R_{11}) \\ P_{21}C_{21}^T + P_{22}C_{22}^T &= K_2(k)(C_{21}(P_{11}C_{21}^T + P_{12}C_{22}^T) + C_{22}(P_{21}C_{21}^T + P_{22}C_{22}^T) + R_{22}) \\ P_{11}C_{21}^T + P_{12}C_{22}^T &= K_1(k)(C_{11}P_{11}C_{21}^T + C_{11}P_{12}C_{22}^T + R_{12}) \\ P_{21}C_{11}^T &= K_2(k)(C_{11}P_{11}C_{21}^T + C_{11}P_{12}C_{22}^T + R_{12})^T \end{aligned} \tag{19}$$

$$\begin{aligned} P(k|k-1) &= \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix} \\ &= \begin{pmatrix} A_{11}P_{11}(k-1)A_{11}^T + Q_{11} & A_{11}P_{11}(k-1)A_{21}^T + A_{11}P_{12}(k-1)A_{22}^T + Q_{12} \\ \begin{pmatrix} A_{11}P_{11}(k-1)A_{21}^T + \\ +A_{11}P_{12}(k-1)A_{22}^T + Q_{12})^T \end{pmatrix} & \begin{pmatrix} A_{21}(P_{11}(k-1)A_{21}^T + P_{12}(k-1)A_{22}^T) \\ +A_{22}(P_{21}(k-1)A_{21}^T + P_{22}(k-1)A_{22}^T) + Q_{22} \end{pmatrix} \end{pmatrix} \end{aligned} \tag{20}$$

case, the cross-covariances $P_{12}(k|k-1)$ and $P_{12}(k)$ and their transpose will also be zero. ■

Since the distributed filters obtain the same performance as the centralized Kalman filter if and only if the subsystems are independent, in general, the distributed observers will not minimize the joint covariance. However, in practice, the performance of the centralized and distributed observers is comparable, as demonstrated in the following section.

V. Examples

In the previous sections, the basic form of the Kalman filter and the proposed distributed version were given. Here, three examples are presented to compare the performance of the distributed and centralized observers, both in open-loop and closed-loop control.

A. Distributed Kalman Filter in Open-Loop

Example 1: Consider the following, randomly generated discrete-time system:

$$\begin{aligned} \mathbf{x}(k) &= A\mathbf{x}(k-1) + Bu(k-1) + \mathbf{w}(k-1) \\ \mathbf{y}(k) &= C\mathbf{x}(k) + \mathbf{v}(k) \end{aligned}$$

with

$$\begin{aligned} A &= \begin{pmatrix} 0.1 & 0 & 0 \\ 0.5 & 0.6 & -0.9 \\ -1.1 & 2.0 & -0.3 \end{pmatrix} \\ B &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{w}(k) \sim \mathcal{N}(0, Q) \quad Q &= \begin{pmatrix} 0.68 & 0.22 & 0.08 \\ 0.22 & 0.28 & 0.11 \\ 0.08 & 0.11 & 0.22 \end{pmatrix} \\ \mathbf{v}(k) \sim \mathcal{N}(0, R) \quad R &= \begin{pmatrix} 0.17 & 0.06 \\ 0.06 & 0.12 \end{pmatrix} \end{aligned} \tag{21}$$

It can be easily seen that the deterministic part of the system can be cascaded. Two cases are distinguished:

a) Discard the cross-covariance between the subsystems: since the cascaded filters do not take into account the cross-covariance between the subsystems, in order to ensure the exact same conditions, consider for both the Kalman filter and the cascaded filters the following approximate noise covariances:

$$\bar{Q} = \begin{pmatrix} 0.68 & 0 & 0 \\ 0 & 0.28 & 0.11 \\ 0 & 0.11 & 0.22 \end{pmatrix} \quad \bar{R} = \begin{pmatrix} 0.17 & 0 \\ 0 & 0.12 \end{pmatrix} \tag{22}$$

The input signal is presented in Figure 2. Using the centralized Kalman filter, after 300 steps, we obtain:

$$P = \begin{pmatrix} 0.1361 & 0.0002 & -0.0034 \\ 0.0002 & 0.1062 & 0.0339 \\ -0.0034 & 0.0339 & 0.7498 \end{pmatrix}$$

$$K = \begin{pmatrix} 0.8003 & 0.0014 \\ 0.0010 & 0.8853 \\ -0.0198 & 0.2824 \end{pmatrix}$$

while for the cascaded subsystems:

$$P_c = \begin{pmatrix} 0.1361 & 0 & 0 \\ 0 & 0.1030 & 0.0430 \\ 0 & -0.0430 & 0.5245 \end{pmatrix}$$

$$K_c = \begin{pmatrix} 0.8003 & 0 \\ 0 & 0.8811 \\ 0 & 0.3689 \end{pmatrix}$$

if $\mathbf{x}^1$ is considered to be a deterministic input (Case 1) and

$$P_c = \begin{pmatrix} 0.1361 & 0 & 0 \\ 0 & 0.1062 & 0.0342 \\ 0 & 0.0342 & 0.7511 \end{pmatrix}$$

$$K_c = \begin{pmatrix} 0.8003 & 0 \\ 0 & 0.8850 \\ 0 & 0.2852 \end{pmatrix}$$

if $\mathbf{x}^1$ is considered to be a stochastic input (Case 2).

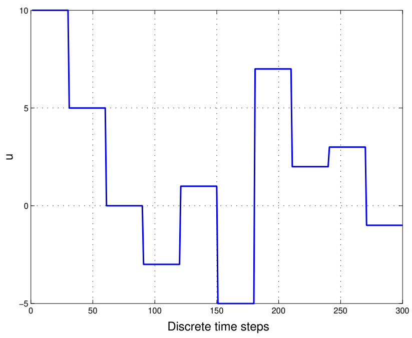


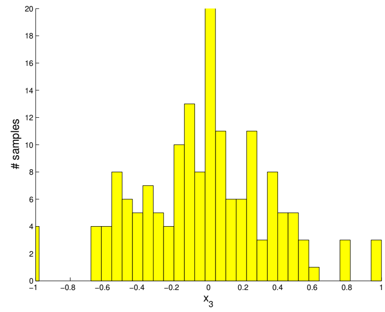
Fig. 2. Input used for the distributed filters in open-loop for example 1.

Histograms of the residuals obtained for x_3 (the state which is not measured) with the centralized Kalman filter, and for both cases of the distributed filters are presented in Figure 3. The statistics of the distributions of the residuals for all states and observers are given in Table I. It can be seen that the performance of the cascaded observers is comparable with that of the centralized observer.

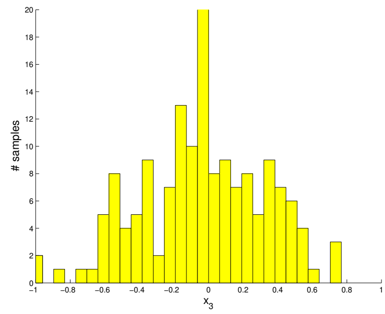
TABLE I

STATISTICS OF THE RESIDUALS WHEN THE CENTRALIZED AND DISTRIBUTED OBSERVERS USE THE SAME COVARIANCE MATRIX.

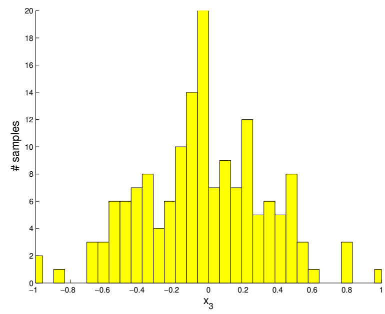
State	Method	Mean	Standard deviation
x_1	centralized	-0.0015	0.1806
	cascaded	-0.0014	0.1806
x_2	centralized	-0.0004	0.0798
	cascaded deterministic	-0.0004	0.0807
	cascaded stochastic	-0.0002	0.0797
x_3	centralized	0.0041	0.2487
	cascaded deterministic	-0.0033	0.2397
	cascaded stochastic	-0.0032	0.2423



(a) Residuals for x_3 with the centralized Kalman filter.



(b) Residuals for x_3 with the cascaded Kalman filter and deterministic input (case 1).



(c) Residuals for x_3 with the cascaded Kalman filter and stochastic input (case 2).

Fig. 3. Results when the centralized and cascaded filters use the same covariance matrix, for example 1.

b) Use true covariance: the Kalman filter uses the true noise covariances (21), while the cascaded filters neglect the cross-covariance between the subsystems and consider only (22). The same input is used as that in the previous case. In terms of the standard deviation, the centralized filter performs slightly better than the cascaded one.

The histogram of the residuals obtained for x_3 is presented in Figure 4. The statistics of the distributions of the residuals for all states and observers are given in Table II.

TABLE II

STATISTICS OF RESIDUALS WITH THE CASCADED KALMAN FILTER DISCARDING THE CROSS-COVARIANCE.

State	Method	Mean	Standard deviation
x_1	centralized	-0.0119	0.2235
	cascaded	-0.0119	0.2237
x_2	centralized	-0.0012	0.1405
	deterministic	-0.0028	0.1360
	stochastic	-0.0020	0.1359
x_3	centralized	-0.0067	0.2065
	deterministic	-0.0063	0.2156
	stochastic	-0.0067	0.2125

For this case, the final covariance matrix and the Kalman gain obtained after 300 steps by the centralized Kalman filter are

$$P = \begin{pmatrix} 0.1360 & 0.0477 & 0.0155 \\ 0.0477 & 0.1029 & 0.0354 \\ 0.0155 & 0.0354 & 0.5241 \end{pmatrix}$$

$$K = \begin{pmatrix} 0.8014 & -0.0031 \\ -0.0269 & 0.8713 \\ -0.0154 & 0.3026 \end{pmatrix}$$

while those obtained by the cascaded observers are the same as in the previous case.

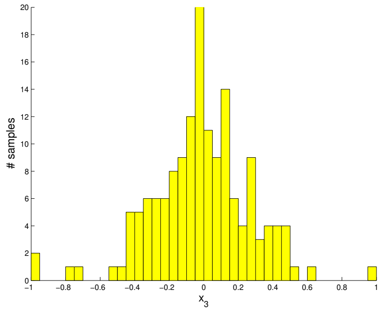
The statistics of the residuals confirm that the cascaded filters are suboptimal. However, the difference between the residuals is minimal, even if $\mathbf{x}^1$ obtained from the first subsystem is considered as a deterministic input, and the computed covariance is not the correct one.

B. Distributed Kalman Filter in Closed-Loop

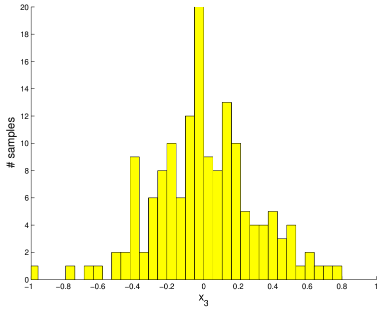
In this section, two examples are presented to compare the performance of the distributed and centralized observers, in closed-loop control. For this purpose, a state-feedback control is designed based on the system model. However, not all the states are measured, and the control input is computed based on the estimated states. Such a setting is depicted in Figure 5.

Example 2: Consider the following, randomly generated discrete-time system:

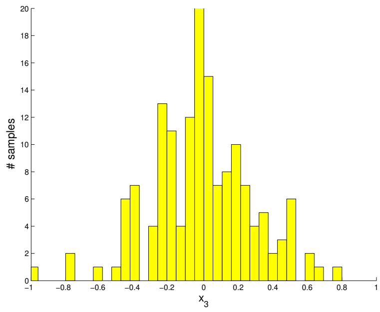
$$\begin{aligned} \mathbf{x}(k) &= A\mathbf{x}(k-1) + Bu(k-1) + \mathbf{w}(k-1) \\ \mathbf{y}(k) &= C\mathbf{x}(k) + \mathbf{v}(k) \end{aligned}$$



(a) Residuals for x_3 with centralized Kalman filter.



(b) Residuals for x_3 with cascaded Kalman filter and deterministic input.



(c) Residuals for x_3 with cascaded Kalman filter and stochastic input.

Fig. 4. Results with discarded cross-covariances, for example 1.

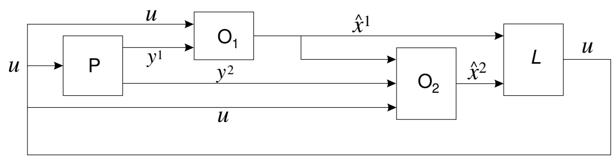


Fig. 5. Cascaded observers in closed-loop.

$$A = \begin{pmatrix} 1.05 & 0 & 0 \\ 0.05 & 1.17 & 0.07 \\ -0.076 & 0.14 & 0.77 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.1 \\ 0 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{w}(k) \sim \mathcal{N}(0, Q) \qquad Q = \begin{pmatrix} 0.0097 & 0.0026 & 0.0032 \\ 0.0026 & 0.0066 & 0.0002 \\ 0.0032 & 0.0002 & 0.0128 \end{pmatrix}$$

$$\mathbf{v}(k) \sim \mathcal{N}(0, R) \qquad R = \begin{pmatrix} 0.0035 & 0.0078 \\ 0.0078 & 0.0118 \end{pmatrix}$$

for which a state feedback control with constant gain $L = [7.4000\ 51.4363\ 8.5107]$ has been computed by pole placement.

The deterministic part of the system is decomposed. The cascaded filters do not take into account the noise covariances between the subsystems. Now the control is applied for four different cases:

1) the states are known, and the controller is applied directly;
2) the first two states are measured, and the control input is computed based on the estimate given by a centralized Kalman filter;
3) the first two states are measured, and the control input is computed based on the estimate given by a cascaded Kalman-type filter, with the second subsystem considering the estimates of the first subsystem as stochastic inputs;
4) the same as 3), but with the second subsystem using the estimates of the first subsystem as deterministic inputs.

The results obtained can be seen in Figure 6. The estimation error for all states, is very small, and the estimates converge at approximately the same speed.

Example 3: Consider the following, randomly generated discrete-time system:

$$\mathbf{x}(k) = A\mathbf{x}(k-1) + Bu(k-1) + \mathbf{w}(k-1)$$
$$\mathbf{y}(k) = C\mathbf{x}(k) + \mathbf{v}(k)$$
$$A = \begin{pmatrix} 1.05 & 0 & 0 \\ -0.17 & 0.91 & 0.23 \\ -0.02 & 0.18 & 0.94 \end{pmatrix}$$
$$B = \begin{pmatrix} 0.1 \\ 0 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

for which a state feedback control with constant gain $L = [6.5000\ \ -6.7391\ \ -8.2180]$ has been computed. The state and measurement noise have the same covariance matrices as in the previous example. Note that this system with this control law, when applied the centralized Kalman filter, becomes unstable.

The estimates of the states using the distributed observers can be seen in Figure 7. In this case, the system does not become unstable.

VI. Conclusions

In many real-life applications, a complex process model can be decomposed into cascaded subsystems, and observers can be designed for these individual subsystems. This partitioning of a process and observer leads to increased modularity and reduced complexity of the problem, with reduced computational costs and more straightforward tuning.

For such cascaded systems, distributed, Kalman-like filters can be designed. The observers are optimal for the individual subsystems, and the error system will converge to a zero-mean Gaussian. However, the overall filter will not necessarily be optimal. The theoretical results show that the distributed Kalman filters can be jointly optimal, if and only if the subsystems are decoupled (i.e., the second subsystem does not depend on the states of the first one).

Based on the examples, however, the performance of the centralized Kalman filter and cascaded filters are comparable. Moreover, our simulations show that for certain cases, in closed-loop the cascaded observers perform better than the Kalman filter.

In our future research, we will investigate the conditions under which such a distribution of the process and the estimation is possible for other types of observers while maintaining the performance (convergence, convergence speed) of a centralized one.

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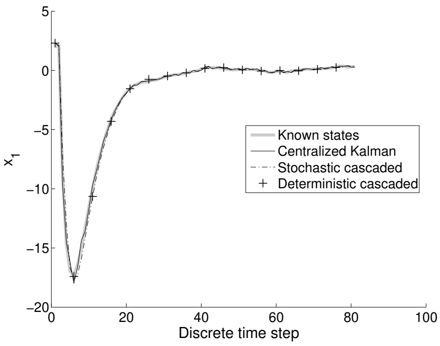
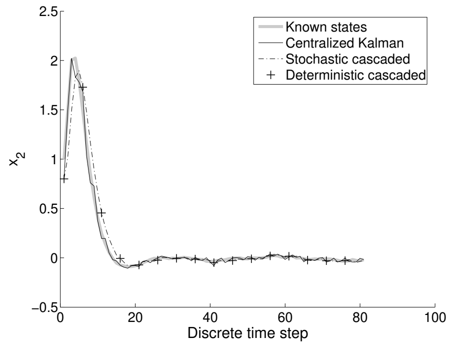
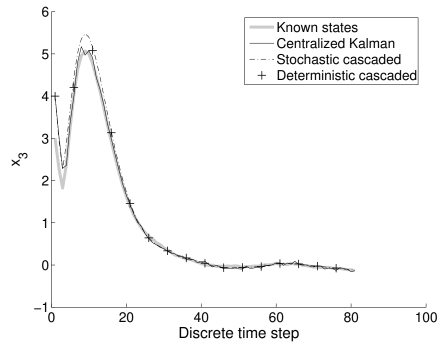


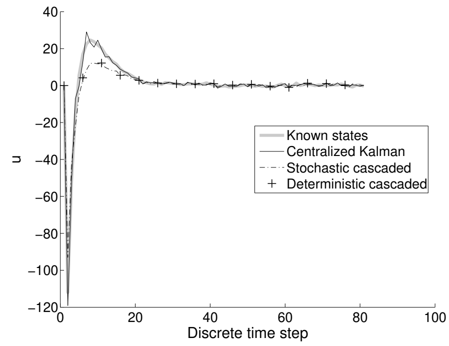
(a) x_1



(b) x_2

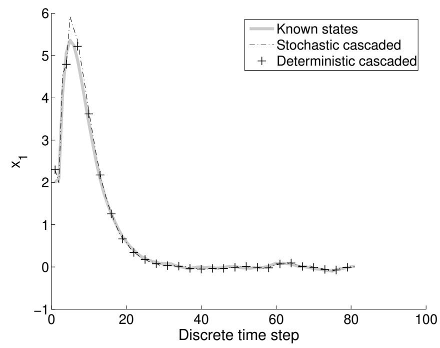


(c) x_3

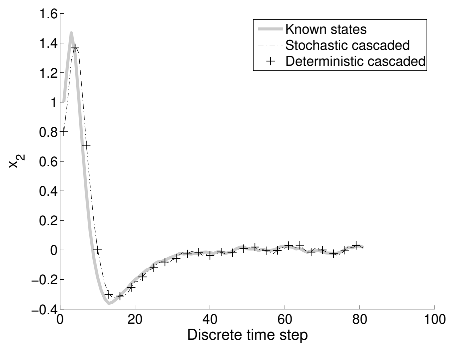


(d) u

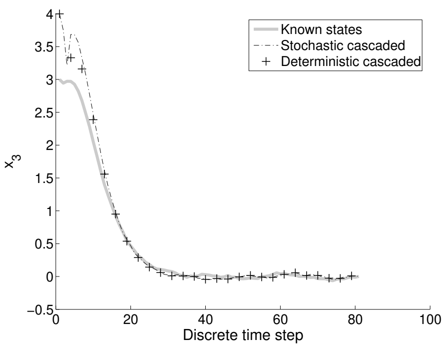
Fig. 6. State estimates in closed-loop with different observers (state feedback without observer, Kalman, stochastic cascaded, deterministic cascaded) for example 2.



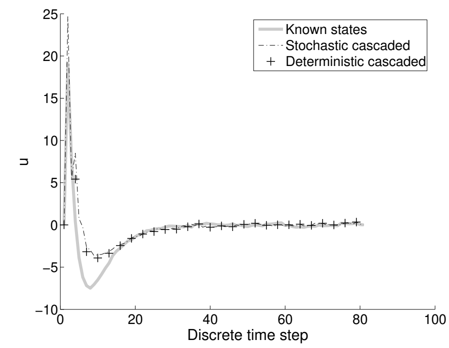
(a) x_1



(b) x_2



(c) x_3



(d) u

Fig. 7. State estimates in closed-loop with different observers (state feedback without observer, stochastic cascaded, deterministic cascaded) for example 3. The system using the centralized Kalman filter is unstable in this case.