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Technical report 08-001

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If you want to cite this report, please use the following reference instead:
Zs. Lendek, R. Babuška, and B. De Schutter, "Stability of cascaded fuzzy systems and observers," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 3, pp. 641–653, June 2009. doi:10.1109/TFUZZ.2008.924353

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Stability of Cascaded Fuzzy Systems and Observers

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***Abstract*—A large class of nonlinear systems can be well approximated by Takagi-Sugeno (TS) fuzzy models, with linear or affine consequents. It is well-known that the stability of these consequent models does not ensure the stability of the overall fuzzy system. Therefore, several stability conditions have been developed for TS fuzzy systems. We study a special class of nonlinear dynamic systems that can be decomposed into cascaded subsystems, represented as TS fuzzy models. We analyze the stability of the overall TS system based on the stability of the subsystems, and prove that the stability of the subsystems implies the stability of the overall system. The main benefit of this approach is that it relaxes the conditions imposed when the system is globally analyzed, thereby solving some of the feasibility problems. Another benefit is that by using this approach, the dimension of the associated linear matrix inequality (LMI) problem can be reduced. For naturally distributed applications, such as multi-agent systems, the construction and tuning of a centralized observer may not be feasible. Therefore, we extend the cascaded approach also to observer design, and use fuzzy observers to individually estimate the states of these subsystems. A theoretical proof of stability and simulation examples are presented. The results show that the distributed observer achieves the same performance as the centralized one, while leading to increased modularity, reduced complexity, lower computational costs, and easier tuning. Applications of such cascaded systems include multi-agent systems, distributed process control, and hierarchical large-scale systems.**

***Index Terms*—fuzzy systems, fuzzy observers, cascaded systems, Lyapunov stability.**

I. Introduction

Many problems in decision making, control, and monitoring require the estimation of states and possibly uncertain parameters, based on a dynamic system model and a sequence of noisy measurements. For such a purpose, dynamic systems are often modeled in the state-space framework, using the state-transition model, which describes the evolution of states over time, and the sensor model, which relates the measurements to the states.

Traditionally, the class of linear, time-invariant systems has dominated control theory. The linearity and time-invariance make this type of systems easy to analyze. The disadvantages are that such systems fail to describe nonlinear systems globally. An accurate approximation of a nonlinear system can only be expected in the vicinity of an equilibrium point.

A generic method for the design of an observer valid for all types of nonlinear systems has not yet been developed. A large class of nonlinear systems can be well approximated by Takagi-Sugeno fuzzy models [1], which in theory can approximate a general nonlinear system to an arbitrary degree of accuracy [2]. Stability conditions have been derived for Takagi-Sugeno fuzzy systems, most of them relying on the feasibility of an associated system of linear matrix inequalities (LMIs) [3]–[5]. A comprehensive survey on the analysis of fuzzy systems can be found in [6].

For a general nonlinear system represented by a fuzzy model, well-established methods and algorithms can be used to design fuzzy observers, therefore the analysis and design becomes much easier. Several types of observers have been developed for Takagi-Sugeno fuzzy systems, among which: fuzzy Thau-Luenberger observers [3], [7], reduced-order observers [5], [8], and sliding-mode observers [9]. In general, the design methods for observers also lead to an LMI feasibility problem. However, the complexity of the system grows exponentially with the number of antecedents and the stability analysis problem eventually becomes intractable for a large number of rules.

Decentralized state estimation has been studied in the context of large-scale processes and distributed systems. The decentralized architecture generally has the form of a network of sensor nodes, each with its own computing capability. In case of a fully decentralized system, computations are performed locally and communication takes place between any two nodes. Each node shares information with other nodes and computes a local estimate. Computation and communication is distributed over the network and the global estimate is computed by fusing the local results. Several topologies have been proposed, depending on the particular application. In case of large-scale processes [10], [11], the network is generally in a hierarchical form, with several intermediate nodes and one final fusion node. For distributed systems, such as multi-agent societies [12]–[14], several fusion nodes are used, which process the data and send the information to the rest of the nodes. Observers for distributed estimation include, but are not limited to the decentralized Kalman and the Extended Kalman filter [15], the information filter and several types of particle filters [16], [17].

An important class of distributed systems can be represented as cascaded subsystems (e.g., material processing systems, chemical processes). In several cases, conclusions referring to the overall system can be drawn based on the study of the individual subsystems. For instance, for linear systems, the stability of the subsystems implies the stability of the cascaded system [18]. However, this property in general does not hold for nonlinear or time-varying systems. Even global asymptotic stability of the individual subsystems does not necessarily imply stability of the cascade.

In the literature, the stability of several types of cascaded systems has been studied. The main motivation came from the linear-nonlinear cascade [19], resulting from input-output linearization. Conditions to ensure the overall stability of more

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general cascades, in which both subsystems are nonlinear, were derived in [18], [20], [21].

The contribution of this paper is twofold. First, we study a special class of systems, represented as Takagi-Sugeno fuzzy systems, which can be decomposed into cascaded subsystems, and analyze the stability of the whole system based on the stability of the subsystems. The idea behind this type of stability analysis is that many systems are naturally distributed (e.g., traffic control systems) or cascaded (e.g., hierarchical large-scale systems), while others may be represented as cascaded, observable subsystems, which are less complex than the original system. The main benefit of this approach is that it relaxes the conditions imposed by analyzing the system globally. Global analysis may lead to infeasible LMI conditions, even if the system is stable. We propose relaxed stability conditions, which may render the associated LMI problem feasible. Moreover, the dimension of the associated LMI problem is generally reduced.

Second, the results are extended to observer design. We analyze the joint performance of fuzzy observers individually designed for the subsystems. The benefit of this type of estimation is that separate observers can be designed for the individual subsystems, which makes their tuning easier. Moreover, different types of observers can be combined, depending on the subsystem considered. Such a cascaded system can be regarded as a cooperative multi-agent system, where each agent observes at least its own states and makes decisions based on these observations. The agents rely on their own measurements and the information gathered from other agents. In turn, each agent communicates its own results to other agents. If all the agents in a system use the same observer method, then such an observer system can be designed and implemented in a modular manner, i.e., after identifying the coupling among the subsystems, the observers can be designed in a similar fashion. We present a theoretical comparison of the centralized and cascaded fuzzy observers and also compare their performance on two examples.

The structure of the paper is as follows. Section II introduces the cascaded setting for nonlinear systems and gives stability conditions for such systems. Section III reviews stability conditions for TS fuzzy systems and observers. The proposed stability conditions for cascaded fuzzy systems and observers are presented in Section IV. Examples are given in Section V. Finally, Section VI concludes the paper.

II. Stability of Cascaded Dynamic Systems

In the literature, the main motivation to consider cascaded dynamical systems came from the analysis of the models obtained after input-output linearization [18], [19]. Several stability conditions were derived for different types of subsystems. In this section, the cascaded setting for general nonlinear systems and observers is presented, together with the relevant stability conditions.

A. Preliminaries

Consider the following general, observable nonlinear system:

$$\begin{aligned} \dot{x}_1 &= f_1(\boldsymbol{x}, \boldsymbol{u}) & y_1 &= h_1(\boldsymbol{x}, \boldsymbol{u}) \\ \dot{x}_2 &= f_2(\boldsymbol{x}, \boldsymbol{u}) & y_2 &= h_2(\boldsymbol{x}, \boldsymbol{u}) \\ &\vdots & &\vdots \\ \dot{x}_n &= f_n(\boldsymbol{x}, \boldsymbol{u}) & y_m &= h_m(\boldsymbol{x}, \boldsymbol{u}) \end{aligned} \tag{1}$$

where $\boldsymbol{x} = [x_1, ..., x_n]^T$ and $\boldsymbol{u} = [u_1, ..., u_l]^T$ and assume that this system can be partitioned into subsystems. For the ease of notation, only two subsystems are considered, without a loss of generality:

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{u}) \\ \boldsymbol{y}_1 &= \boldsymbol{h}_1(\boldsymbol{x}_1, \boldsymbol{u}) \end{aligned} \tag{2}$$

and

$$\begin{aligned} \dot{\boldsymbol{x}}_2 &= \boldsymbol{f}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}) \\ \boldsymbol{y}_2 &= \boldsymbol{h}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}) \end{aligned} \tag{3}$$

where $\boldsymbol{x} = [\boldsymbol{x}_1^T \ \ \boldsymbol{x}_2^T]^T$, $\boldsymbol{y} = [\boldsymbol{y}_1^T \ \ \boldsymbol{y}_2^T]^T$ (with possible reordering), so that (2) is observable. Note that, since both system (1) and (2) are observable, subsystem (3) is also observable. In fact, for subsystem (3), $\boldsymbol{x}_1$ is an input. In general, such a partition of the model does not necessarily exist. Moreover, if a partition exists, it might not be unique.

Given a particular nonlinear system of the form (1), with at least two measurements, a partitioning into two subsystems can be easily constructed, while also deciding whether the partitioning is possible. For two subsystems, the cascaded structure is depicted in Figure 1.

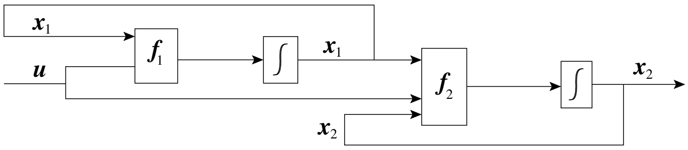

Fig. 1. Cascaded subsystems.

If such a partition exists, observers may be designed for the subsystems separately, with some observers using the estimates obtained by other observers. For two subsystems, the cascaded observer structure is depicted in Figure 2.

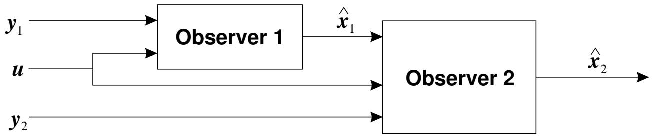

Fig. 2. Cascaded observers.

B. Stability of Cascaded Systems

It is well-known that the cascade of stable linear systems is stable, since the eigenvalues of the joint system are determined only by the eigenvalues of the individual subsystems [18]. Therefore, the stability of the joint system is determined by the stability of the subsystems. However, the same reasoning does not necessarily hold for nonlinear or time-varying systems. Even global asymptotic stability of the decoupled subsystems does not necessarily imply the stability of the cascade.

In the literature, the stability of several special cases has been studied. The main motivation comes from the linear-nonlinear cascade [19], resulting from input-output linearization. More general cascades, in which both subsystems are nonlinear, were studied and conditions to ensure overall stability were derived in [18]. A selection of relevant results is presented below.

Definition 1: A continuous function $\alpha : \mathcal{R}^+ \to \mathcal{R}^+$ belongs to class $\mathcal{K}$ *if it is strictly increasing and $\alpha(0) = 0$. If $\alpha(s) \to \infty$ when $s \to \infty$ then α is said to be of* class $\mathcal{K}_\infty$. □

Definition 2: A system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ is input-to-state stable (ISS) if and only if there exist a positive definite proper function $V(\boldsymbol{x})$ and two class $\mathcal{K}$ functions α_1 and α_2 such that

$$\{\|\boldsymbol{x}\| \geq \alpha_1(\|\boldsymbol{u}\|)\} \Rightarrow \left\{ \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \leq -\alpha_2(\|\boldsymbol{x}\|) \right\} \quad (4)$$

where $\|\cdot\|$ represents the Euclidian norm. □

Consider the nonlinear, cascaded, autonomous system

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{f}_1(\boldsymbol{x}_1) \quad (5)$$
$$\dot{\boldsymbol{x}}_2 = \boldsymbol{f}_2(\boldsymbol{x}_1, \boldsymbol{x}_2). \quad (6)$$

It has been shown in [22] that if

- the functions f_1 and f_2 are sufficiently smooth in their arguments,
- system (6) is input-to-state-stable with regard to the input $\boldsymbol{x}_1$, and
- system (5) and the system

$$\dot{\boldsymbol{x}}_2 = \boldsymbol{f}_2(0, \boldsymbol{x}_2) \quad (7)$$

  are globally asymptotically stable (GAS),

then the cascade (5)-(6) is GAS. An equivalent sufficient stability condition is presented in [21]: the cascaded system is GAS, if both subsystems are GAS and all solutions are bounded. The main difficulty with this approach is that in general, boundedness of all the solutions is not easy to determine and the conditions to ensure boundedness may be very conservative.

More relaxed sufficient stability conditions have been derived for systems of the form:

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{f}_1(\boldsymbol{x}_1) \\ \dot{\boldsymbol{x}}_2 &= \boldsymbol{f}_2(\boldsymbol{x}_2) + \boldsymbol{g}(\boldsymbol{x}_1, \boldsymbol{x}_2) \end{aligned} \quad (8)$$

assuming that the individual subsystems are GAS and, additionally, certain restrictions related to the continuity and/or slope, apply for the interconnection term $\boldsymbol{g}$ [19], [23], [24]. A theorem for ensuring that the cascaded system (8) is uniformly GAS (UGAS) [18] is presented below.

Assumptions:

1) System (7) is UGAS.
2) There exist constants $c_1, c_2, \mu > 0$ and a Lyapunov function $V(t, \boldsymbol{x}_2)$ for (7) such that $V : \mathcal{R}^+ \times \mathcal{R}^n \to \mathcal{R}^+$ is positive definite, radially unbounded, $\dot{V}(t, \boldsymbol{x}_2) \leq 0$ and

$$\begin{aligned} &\left\| \frac{\partial V}{\partial \boldsymbol{x}_2} \right\| \|\boldsymbol{x}_2\| \leq c_1 V(t, \boldsymbol{x}_2) && \forall \boldsymbol{x}_2 : \|\boldsymbol{x}_2\| > \mu \\ &\left\| \frac{\partial V}{\partial \boldsymbol{x}_2} \right\| \leq c_2 && \forall \boldsymbol{x}_2 : \|\boldsymbol{x}_2\| \leq \mu \end{aligned} \quad (9)$$

3) There exist two continuous functions $\theta_1, \theta_2 : \mathcal{R}^+ \to \mathcal{R}^+$ such that $\boldsymbol{g}(\boldsymbol{x})$ satisfies:

$$\|\boldsymbol{g}(\boldsymbol{x})\| \leq \theta_1(\|\boldsymbol{x}_1\|) + \theta_2(\|\boldsymbol{x}_1\|)\|\boldsymbol{x}_2\| \quad (10)$$

4) There exists a class $\mathcal{K}$ function $\alpha(\cdot)$ so that for all $t_0 \geq 0$, the trajectories of the system (5) satisfy

$$\int_{t_0}^{\infty} \|\boldsymbol{x}_1(t; t_0, \boldsymbol{x}_1(t_0))\| dt \leq \alpha(\|\boldsymbol{x}_1(t_0)\|) \quad (11)$$

Theorem 1: Let Assumption 1 hold and suppose that the trajectories of (5) *are uniformly globally bounded. If, in addition, Assumptions 2–4 are satisfied, then the solutions of system* (8) *are uniformly globally bounded. If furthermore, system* (5) *is UGAS, then so is the cascaded system* (8). □

Proposition 1: If in addition to the above assumptions systems (5) *and* (7) *are exponentially stable, then the cascaded system* (8) *is also exponentially stable.* □

The proof of Theorem 1, Proposition 1, and the study of different cases of interconnection terms can be found in [18], [19] and stabilizability conditions were derived in [24], [25]. For observer design for a special type of cascaded SISO system see [26].

III. Stability of Fuzzy Systems

Though the consequents of TS fuzzy systems are usually chosen linear or affine, it is well-known that the stability of these local models does not imply the stability of the overall fuzzy system. During operation of the full fuzzy model, the local models are blended. The particular blending of several local models may be strongly nonlinear, which influences the stability. The switching surfaces between the models depend on the operators used for intersection, union, and implication in the antecedent of the rules. Several such operators can be used. Some of them are continuous like the product or sum, others, however, are not (e.g., MIN and MAX). MIN and MAX-operators cause rapid switching in fuzzy models, and therefore the model surface is not smooth. The smoothness of the premise membership functions influences the smoothness of the switching of the local models and consequently the general stability of the system. It is theoretically a very difficult problem to establish stability of a fuzzy system considering the possible switching surfaces and therefore, several stability conditions were derived that ensure the stability of the system for any normalized membership functions, independent of the operators used in the antecedent rules. Most of these conditions depend on the feasibility of an associated LMI problem. Some of them are reviewed below. Throughout the paper it is assumed that the membership functions are normalized.

A. Autonomous Fuzzy Systems

Consider the autonomous fuzzy system expressed as:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{m} w_i(\boldsymbol{z}) A_i \boldsymbol{x} \tag{12}$$

where A_i, $i = 1, 2, \ldots, m$ represents the ith local linear model, w_i is the corresponding normalized membership function, and $\boldsymbol{z}$ the vector of the scheduling parameters. System (12) can also be written as:

$$\dot{\boldsymbol{x}} = A(\boldsymbol{z})\boldsymbol{x} \tag{13}$$

with $A(\boldsymbol{z}) = \sum_{i=1}^{m} w_i(\boldsymbol{z}) A_i$.

For system (12), several stability conditions were derived. Among them, a well-known and frequently used condition is formulated below [3].

Theorem 2: System (12) *is exponentially stable if there exists* $P = P^T > 0$ *so that* $A_i^T P + P A_i^T < 0$*, for* $i = 1, 2, \ldots, m$. □

A condition on the convergence rate of system (12), was also derived from Theorem 2 [3].

Theorem 3: The decay rate of system (12) *is at least* α*, if there exists* $P = P^T > 0$*, so that*

$$A_i^T P + P A_i + 2\alpha P < 0 \qquad i = 1, 2, \ldots, m$$

□

Stability conditions similar to those of Theorem 2 can be used if the system considered is subjected to vanishing disturbances. Consider the following perturbed fuzzy system:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{m} w_i(\boldsymbol{z}) A_i \boldsymbol{x} + D\boldsymbol{f}(t, \boldsymbol{x}) \tag{14}$$

where D is a perturbation distribution matrix and $\boldsymbol{f}$ is a vanishing disturbance, i.e., $\boldsymbol{f}(t, \boldsymbol{x}) \to 0$ when $t \to \infty$, and assume that $\boldsymbol{f}$ is Lipschitz, i.e., there exists $\mu > 0$ so that $\|\boldsymbol{f}(t, \boldsymbol{x})\| \leq \mu \|\boldsymbol{x}\|$, for all t and $\boldsymbol{x}$. With these assumptions, a sufficient stability condition can be formalized by the following theorem [27].

Theorem 4: System (14) *is exponentially stable if there exist matrices* $P = P^T$*,* $Q = Q^T$*, so that*

$$\begin{aligned} & P > 0 \quad Q > 0 \\ & \mu \leq \frac{\lambda_{\min}(Q)}{\|PD\|} \\ & A_i^T P + P A_i^T < -2Q \qquad i = 1, 2, \ldots, m \end{aligned} \tag{15}$$

where $\lambda_{\min}$ *is the eigenvalue with the smallest absolute value.* □

Several variants of the above theorem exist, together with algorithms to compute robustness measures [27]. However, these approaches are conservative by disregarding the fact that the rules are valid only in a region of the state space. For fuzzy systems, the membership functions often have bounded support. Therefore, it is sufficient that $\boldsymbol{x}^T(A_i^T P + P A_i^T)\boldsymbol{x} < 0$ only where $w_i(\boldsymbol{z}) > 0$. Stability conditions for the case when the support of each membership function can be bounded were derived in [4].

Another approach, based on partitioning the state space into operating and interpolation regimes X_k, $k = 1, 2, \ldots, K$, K being the number of regimes, and K_i the index set of the local models active in the regime i, is described in [28]. Assuming that in (12) $\boldsymbol{z}$ can be expressed as some function of $\boldsymbol{x}$, the system can be written as:

$$\dot{\boldsymbol{x}} = \sum_{i \in K_k} w_i(\boldsymbol{x}) A_i \boldsymbol{x} \qquad \boldsymbol{x} \in X_k \tag{16}$$

where K_k is the index set of the linear subsystems active in the region X_k. Then, one can use a Lyapunov function $V_k(\boldsymbol{x}) = \boldsymbol{x}^T P_k \boldsymbol{x}$ for region X_k, i.e., the corresponding part of the Lyapunov function $V(\boldsymbol{x}) = \sum_{k=1}^{K} w_k(\boldsymbol{x}) \boldsymbol{x}^T P_k \boldsymbol{x}$ for the whole system. The system (16) is stable, under the conditions expressed by the following theorem [28].

Theorem 5: The system (16) *is GAS, if there exist matrices* $P_i = P_i^T$*,* $H = H^T > 0$*,* F_i*,* $i = 1, 2, \ldots, K$ *so that*

$$\begin{aligned} & P_i = F_i^T H F_i \\ & P_i > 0 \\ & F_i \boldsymbol{x} = F_j \boldsymbol{x} && \forall \boldsymbol{x} \in X_i \cap X_j \\ & A_k^T P_i + P_i A_k < 0 && \forall k \in K_i \end{aligned} \tag{17}$$

□

For more relaxed conditions, and how to compute the corresponding matrices see [29], [30]. Similar conditions for the discrete-time case are described in [31].

Note that all the above conditions rely on the feasibility of a derived LMI problem. Since efficient algorithms exist for solving LMIs, they can be easily verified. However, two shortcomings of the above theorems must be mentioned: 1) the conditions are conservative and often lead to infeasible LMIs and 2) the number of LMIs associated in particular to the conditions of Theorem 5, is in the worst case exponential in the number of local models.

B. Fuzzy Observers

Consider now the affine fuzzy system

$$\begin{aligned} \dot{\boldsymbol{x}} &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_i \boldsymbol{x} + B_i \boldsymbol{u} + a_i) \\ \boldsymbol{y} &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(C_i \boldsymbol{x} + d_i) \end{aligned} \tag{18}$$

and an observer of the form

$$\begin{aligned} \dot{\widehat{\boldsymbol{x}}} &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_i \widehat{\boldsymbol{x}} + B_i \boldsymbol{u} + a_i + L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}})) \\ \widehat{\boldsymbol{y}} &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(C_i \widehat{\boldsymbol{x}} + d_i). \end{aligned} \tag{19}$$

As before, it is assumed that the membership functions are normal. Depending on the explicit form of the error system given by $\dot{\boldsymbol{e}} = \dot{\boldsymbol{x}} - \dot{\widehat{\boldsymbol{x}}}$, the theorems presented in Section III-A can be directly applied, or similar conditions may be derived to ensure the stability of the observer. For the analysis, two cases are distinguished: 1) the scheduling vector $\boldsymbol{z}$ does not depend on the estimated states and 2) $\boldsymbol{z}$ depends on (some of) the estimated states, that is $\boldsymbol{z} = \widehat{\boldsymbol{z}}$.

1) The scheduling vector does not depend on the estimated states: In this case the error system can be written as:

$$\dot{\boldsymbol{e}} = \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\boldsymbol{z})w_j(\boldsymbol{z})(A_i - L_iC_j)\boldsymbol{e}. \tag{20}$$

Using a Lyapunov function of the form $V(t) = \boldsymbol{e}^T P\boldsymbol{e}$, with $P = P^T > 0$, basic sufficient stability conditions for this system were derived in [3]:

Theorem 6: The system (20) *is GAS, if there exists* $P = P^T > 0$ *so that for* $i = 1, 2, \dots, m$*:*

$$\begin{aligned}&(A_i - L_iC_i)^T P + P(A_i - L_iC_i) < 0\\ &(G_{ij} + G_{ji})^T P + P(G_{ij} + G_{ji}) \le 0\\ &G_{ij} = A_i - L_iC_j\\ &\text{for } \ j = 1, 2, \dots, m \ \text{ for } \ w_i(\boldsymbol{z})w_j(\boldsymbol{z}) > 0\end{aligned} \tag{21}$$

□

A well-known condition on the design of the observer for the system (12), so that a desired convergence rate α is guaranteed, is presented below [3].

Theorem 7: The decay rate of the error system (20) *is at least* α*, if there exists* $P = P^T > 0$*, so that*

$$\begin{aligned}&(A_i - L_iC_i)^T P + P(A_i - L_iC_i) + 2\alpha P < 0\\ &i = 1, 2, \dots, m\\ &(A_i - L_iC_j)^T P + P(A_i - L_iC_j) + 2\alpha P < 0\\ &i, j = 1, 2, \dots, m \ \text{ for } \ w_i(\boldsymbol{z})w_j(\boldsymbol{z}) \neq 0\end{aligned}$$

□

2) The scheduling vector depends on the estimated states: The second case is when the scheduling vector depends on the states to be estimated, i.e., $\boldsymbol{z} = \widehat{\boldsymbol{z}}$. For simplicity, only the case with common measurement matrices will be considered. Then, the observer (19) becomes:

$$\begin{aligned}\dot{\widehat{\boldsymbol{x}}} &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{z}})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + a_i + L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}}))\\ \widehat{\boldsymbol{y}} &= C\widehat{\boldsymbol{x}}\end{aligned} \tag{22}$$

and the error dynamics can be expressed as:

$$\begin{aligned}\dot{\boldsymbol{e}} =& \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{z}})(A_i - L_iC)\boldsymbol{e}\\ &+ \sum_{i=1}^{m}(w_i(\boldsymbol{z}) - w_i(\widehat{\boldsymbol{z}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i)\end{aligned}$$

For such a system, sufficient stability conditions are given by the following theorem [27].

Theorem 8: The error system (22) *is exponentially stable, if there exist* $\mu > 0$*,* $P = P^T > 0$*,* $Q = Q^T > 0$ *so that for all* $i = 1, \dots, m$

$$\begin{aligned}(A_i - L_iC)^T P + P(A_i - L_iC) &\le Q\\ \begin{pmatrix} Q - \mu^2 & P\\ P & I\end{pmatrix} &> 0\\ \|(w_i(\boldsymbol{z}) - w_i(\widehat{\boldsymbol{z}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i)\| &\le \mu\|e\|\end{aligned} \tag{23}$$

i.e., $(w_i(\boldsymbol{z}) - w_i(\widehat{\boldsymbol{z}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i)$ *is bounded by a linear growth of* $\boldsymbol{e}$*.* □

IV. Stability of Cascaded Fuzzy Observers

In this section, stability conditions for cascaded TS fuzzy systems and observers are derived. We prove that, thanks to the special form of the TS fuzzy system, the stability of the individual subsystems implies the stability of the global cascaded system.

A. Cascaded Fuzzy Systems

Consider the case when the system matrices of the model (12) for each rule $i = 1, 2, \dots, m$ can be written as:

$$A_i = \begin{pmatrix} A_1 & 0\\ A_{21} & A_2\end{pmatrix}_i = \begin{pmatrix} A_{1i} & 0\\ A_{21i} & A_{2i}\end{pmatrix}$$

i.e., system (12) can be expressed as the cascade of two fuzzy systems:

$$\begin{aligned}\dot{\boldsymbol{x}}_1 &= \sum_{i=1}^{m} w_{1i}(\boldsymbol{z}_1)A_{1i}\boldsymbol{x}_1\\ \dot{\boldsymbol{x}}_2 &= \sum_{i=1}^{m} w_{2i}(\boldsymbol{z}_1, \boldsymbol{z}_2)(A_{21i}\boldsymbol{x}_1 + A_{2i}\boldsymbol{x}_2)\end{aligned} \tag{24}$$

or, equivalently:

$$\begin{aligned}\dot{\boldsymbol{x}}_1 =& A_1(\boldsymbol{z}_1)\boldsymbol{x}_1\\ \dot{\boldsymbol{x}}_2 =& A_{21}(\boldsymbol{z}_1, \boldsymbol{z}_2)\boldsymbol{x}_1 + A_2(\boldsymbol{z}_1, \boldsymbol{z}_2)\boldsymbol{x}_2\end{aligned} \tag{25}$$

with normalized membership functions w_{1i} and w_{2i}, $\boldsymbol{x} = [\boldsymbol{x}_1^T, \boldsymbol{x}_2^T]^T$, $\boldsymbol{z} = [\boldsymbol{z}_1^T, \boldsymbol{z}_2^T]^T$, $A_1(\boldsymbol{z}_1) = \sum_{i=1}^{m} w_i(\boldsymbol{z}_1)A_{1i}$, $A_2(\boldsymbol{z}) = \sum_{i=1}^{m} w_{2i}(\boldsymbol{z})A_{2i}$, etc.

Below, we prove that, if the subsystems

$$\dot{\boldsymbol{x}}_1 = A_1(\boldsymbol{z}_1)\boldsymbol{x}_1 \tag{26}$$

and

$$\dot{\boldsymbol{x}}_2 = A_2(\boldsymbol{z}_1, \boldsymbol{z}_2)\boldsymbol{x}_2 \tag{27}$$

are UGAS, then it is possible to apply Theorem 1 to fuzzy systems of the form (24).

Theorem 9: If there exist two Lyapunov functions of the form $V_1(\boldsymbol{x}_1) = \boldsymbol{x}_1^T P_1\boldsymbol{x}_1$ *and* $V_2(\boldsymbol{x}_2) = \boldsymbol{x}_2^T P_2\boldsymbol{x}_2$ *so that the subsystems* (26) *and* (27) *are UGAS, then the cascaded system* (25) *is also UGAS.*

Proof: Note that the Lyapunov functions $V_1(\boldsymbol{x}_1) = \boldsymbol{x}_1^T P_1\boldsymbol{x}_1$ and $V_2(\boldsymbol{x}_2) = \boldsymbol{x}_2^T P_2\boldsymbol{x}_2$ for the subsystems (26) and (27) satisfy Assumptions 1 and 4 and also ensure exponential stability of the individual subsystems. Equations (14) and also (24) and (25) are special cases of (8), where the individual subsystems $\boldsymbol{f}_1(\boldsymbol{x}_1)$ and $\boldsymbol{f}_2(\boldsymbol{x}_2)$ are represented by fuzzy models. The interconnection term $\boldsymbol{g}$ is a nonlinear combination of local linear models.

- Assumption 2 is satisfied as: $\forall\boldsymbol{x}_2 : \|\boldsymbol{x}_2\| > \mu$,

$$\begin{aligned}\left\|\frac{\partial V_2}{\partial \boldsymbol{x}_2}\right\| \|\boldsymbol{x}_2\| &= 2\|\boldsymbol{x}_2^T\|\|P_2\|\|\boldsymbol{x}_2\|\\ &\le 2\lambda_{\max}(P_2)\|\boldsymbol{x}_2\|^2 \le c_1V_2(\boldsymbol{x}_2)\end{aligned}$$

  for any $c_1 \ge \frac{2\lambda_{\max}(P_2)}{\lambda_{\min}(P_2)}$. For the second condition of Assumption 2, we have $\forall\boldsymbol{x}_2 : \|\boldsymbol{x}_2\| \le \mu$

$$\left\|\frac{\partial V_2}{\partial \boldsymbol{x}_2}\right\| = \|2\boldsymbol{x}_2^T P_2\| \le 2\|\boldsymbol{x}_2\|\|P_2\| \le 2\mu\lambda_{\max}(P_2) = c_2$$

- Consider continuous, positive functions $\theta_1(\|\boldsymbol{x}_1\|) = \max_{\boldsymbol{z}} \|A_{21}(\boldsymbol{z})\|\|\boldsymbol{x}_1\|$ and $\theta_2(\|\boldsymbol{x}_1\|) = 0$. By choosing these functions, we can ensure that $\|\boldsymbol{g}(\boldsymbol{x})\| = \sum_{i=1}^{m} w_i(\boldsymbol{z})A_{21}\boldsymbol{x}_1 \leq \theta_1(\|\boldsymbol{x}_1\|) + \theta_2(\|\boldsymbol{x}_1\|)\|\boldsymbol{x}_2\|$ and therefore Assumption 3 is satisfied.

Since the conditions of Theorem 1 are satisfied, the cascaded system is UGAS. Furthermore, since these Lyapunov functions ensure exponential stability of the subsystems, based on Proposition 1, the cascaded system is also exponentially stable. □

While it is true that the cascaded system is stable under the above conditions, finding a Lyapunov function valid for the cascaded system is not trivial. A global Lyapunov function of the form:

$$V_0(\boldsymbol{x}_1, \boldsymbol{x}_2) = V_1(\boldsymbol{x}_1) + V_2(\boldsymbol{x}_2) + \Psi(\boldsymbol{x}_1, \boldsymbol{x}_2) \tag{28}$$

has been proposed in [23] with V_1 and V_2 being Lyapunov functions for the systems (26) and (27), respectively. The cross-term $\Psi(\boldsymbol{x}_1, \boldsymbol{x}_2)$ has been constructed by the authors of [23], under the condition that the cascaded system satisfies Assumptions 2 and 3.

For the case when the first subsystem is linear and time-invariant, the authors of [23] proved that the cross-term exists and is continuous, and V_0 is positive definite and radially unbounded. If (26) is globally exponentially stable, the result from [23] can be extended to the system (25). The cross-term Ψ is then given by:

$$\Psi(\boldsymbol{x}_1, \boldsymbol{x}_2) = \int_0^\infty \frac{\partial V_2}{\partial \boldsymbol{x}_2}(\tilde{\boldsymbol{x}}_2(s))A_{21}(\boldsymbol{z}(s))\tilde{\boldsymbol{x}}_1(s)ds$$

where $\tilde{\boldsymbol{x}}_1$ and $\tilde{\boldsymbol{x}}_2$ are the trajectories of systems (26) and (27), respectively.

It has to be noted that in general, the sum of the individual Lyapunov function is not a Lyapunov function for the cascaded system. To use the sum of Lyapunov functions would require additional, unnecessary constraints.

In the same way, for Theorems 4 and 5, the stability conditions presented in Section III can be relaxed. The new conditions are presented below.

The conditions of Theorem 4 can be replaced as follows.

Theorem 10: Consider system (24) *expressed as:*

$$\dot{\boldsymbol{x}} = \begin{pmatrix} A_1(\boldsymbol{z}_1) & 0 \\ 0 & A_2(\boldsymbol{z}_1, \boldsymbol{z}_2) \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 0 \\ A_{21}(\boldsymbol{z}_1, \boldsymbol{z}_2) \end{pmatrix}\boldsymbol{x}_1$$

where $A_1(\boldsymbol{z}_1) = \sum_{i=1}^{m} w_i(\boldsymbol{z}_1)A_{1i}$, $A_2(\boldsymbol{z}_1, \boldsymbol{z}_2) = \sum_{i=1}^{m} w_i(\boldsymbol{z}_1, \boldsymbol{z}_2)A_{2i}$, *etc. This system is UGAS, if there exist* $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, *so that:*

$$\begin{aligned} A_{1i}^T P_1 + P_1 A_{1i} < 0 \\ A_{2i}^T P_2 + P_2 A_{2i} < 0 \end{aligned}$$

for $i = 1, 2, \ldots, m$. □

The proof is similar to that of Theorem 9.

In order to relax the conditions of Theorem 5, let K_1 and K_2 be the number of operating and interpolation regimes for the individual subsystems, with K_1^i and K_2^j the index sets corresponding to the local models of the subsystems active in the matching region. Note that in general, the number of regions generated in such a way is smaller than the number of regions for the global system, i.e., $K_1 + K_2 \leq K$. Then, the conditions can be expressed as:

Proposition 2: The system (16) *is UGAS, if there exist matrices* $P_1^i = (P_1^i)^T > 0$, $P_2^j = (P_2^j)^T > 0$, $H_1 = H_1^T > 0$, $H_2 = H_2^T > 0$, F_1^i, *and* F_2^j, $i = 1, 2, \ldots, K_1$, $j = 1, 2, \ldots, K_2$, *so that:*

$$\begin{aligned} & P_1^i = (F_1^i)^T H_1 F_1^i \\ & P_2^j = (F_2^j)^T H_2 F_2^j \\ & F_1^i \boldsymbol{x}_1 = F_1^t \boldsymbol{x}_1 && \forall \boldsymbol{x}_1 \in X_1^i \cap X_1^t \\ & F_2^j \boldsymbol{x}_2 = F_2^l \boldsymbol{x}_2 && \forall \boldsymbol{x}_2 \in X_2^j \cap X_2^l \\ & A_{1k}^T P_1^i + P_1^i A_{1k} < 0 && \forall k \in K_1^i \\ & A_{2k}^T P_2^j + P_2^j A_{2k} < 0 && \forall k \in K_2^j \end{aligned} \tag{29}$$

□

Note that the proposed conditions are still only sufficient conditions for the stability of cascaded fuzzy systems. However, by taking advantage of the special form of the system, i.e., studying the subsystems instead of the overall fuzzy system, the complexity of the associated LMI problem is reduced with respect to Theorems 2, 4, and 5.

In the remainder of this section, we study the convergence rate of the system (25) with respect to the convergence rate of the individual subsystems (26) and (27).

Consider the case, when both subsystems are exponentially stable, i.e., there exist $\beta_1, \beta_2, \gamma_1, \gamma_2 > 0$ so that

$$\|\boldsymbol{x}_1\| \leq \beta_1 \|\boldsymbol{x}_{10}\| e^{-\gamma_1 t} \tag{30}$$

$$\|\boldsymbol{x}_2\| \leq \beta_2 \|\boldsymbol{x}_{20}\| e^{-\gamma_2 t} \tag{31}$$

It has been an open question whether this also meant that the convergence rate of the system (25) is $\min\{\gamma_1, \gamma_2\}$: this conjecture is indeed valid for linear systems; however, it cannot be proven with a Lyapunov function of the form $V(\boldsymbol{x}) = V_1(\boldsymbol{x}_1) + V_2(\boldsymbol{x}_2)$, where V_1 and V_2 are Lyapunov functions for the individual subsystems, as presented below.

Consider the joint system (25), and assume that there exist a Lyapunov function, $V = \boldsymbol{x}^T P \boldsymbol{x}$, $P = P^T > 0$, and $\gamma > 0$, so that:

$$\begin{aligned} & \alpha\|\boldsymbol{x}\|^2 \leq V \leq \beta\|\boldsymbol{x}\|^2 \\ & \dot{V} = \boldsymbol{x}^T(A(\boldsymbol{z})^T P + A(\boldsymbol{z}))\boldsymbol{x} = -\boldsymbol{x}^T Q(\boldsymbol{z})\boldsymbol{x} \leq -\gamma\|\boldsymbol{x}\|^2 \end{aligned}$$

Then the convergence rate of the system is at least γ/β. To have the same convergence rate for the subsystems, it is necessary (in terms of the above conditions) that there exist $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$ so that

- $\alpha \leq \lambda_{\min}(P_1, P_2)$,
- $\beta \geq \lambda_{\max}(P_1, P_2)$, and
- $\lambda_{\max}(\mathrm{diag}[A_1(\boldsymbol{z}_1)^T P_1 + P_1 A_1(\boldsymbol{z}_1), A_2(\boldsymbol{z}_2)^T P_2 + P_2 A_2(\boldsymbol{z}_2)] \leq -\gamma$

Now it will be proven that, if the subsystems are exponentially stable, the convergence rate of the system (24) is also determined by the convergence rate of the individual subsystems (26) and (27).

Theorem 11: The convergence rate of the system (25) *is equal to* $\max\{-\alpha_1, -\alpha_2\}$, *if*

1) *system* (26) *is exponentially stable, with convergence rate* $-\alpha_1$,
2) *system* (27) *is exponentially stable, with convergence rate* $-\alpha_2$, *and*
3) *the matrix* $A_{21}(\boldsymbol{z})$ *is bounded, i.e., there exists* $M \in \mathbb{R}$, *so that* $\forall\, \boldsymbol{z}$, $\|A_{21}(\boldsymbol{z})\| \le M$.

Proof: Assumption 1 can be written as $\|\boldsymbol{x}_1(t)\| \le k_1\|\boldsymbol{x}_{10}\|e^{-\alpha_1 t}$. The solution of the system (27) is the homogeneous solution $\boldsymbol{x}_{2\mathrm{h}}(t)$ of the system

$$\dot{\boldsymbol{x}}_2 = A_{21}(\boldsymbol{z})\boldsymbol{x}_1 + A_2(\boldsymbol{z})\boldsymbol{x}_2 \tag{32}$$

and therefore it satisfies $\|\boldsymbol{x}_{2\mathrm{h}}(t)\| \le k_2\|\boldsymbol{x}_{20}\|e^{-\alpha_2 t}$. The particular solution of equation (32) can be expressed as:

$$\boldsymbol{x}_{2\mathrm{p}} = \int_{t_0}^{t} \boldsymbol{x}_{2\mathrm{h}}(t-s)A_{21}(\boldsymbol{z}(s))\boldsymbol{x}_1(s)ds.$$

Then,

$$\begin{aligned}
\|\boldsymbol{x}_{2\mathrm{p}}\| &= \|\int_{t_0}^{t} \boldsymbol{x}_{2\mathrm{h}}(t-s)A_{21}(\boldsymbol{z}(s))\boldsymbol{x}_1(s)ds\| \\
&\le \int_{t_0}^{t} \|\boldsymbol{x}_{2\mathrm{h}}(t-s)\|\|A_{21}(\boldsymbol{z}(s))\|\|\boldsymbol{x}_1(s)\|ds \\
&\le \int_{t_0}^{t} k_2\|\boldsymbol{x}_{20}\|e^{-\alpha_2(t-s)}Mk_1\|\boldsymbol{x}_{10}\|e^{-\alpha_1 s}ds \\
&= k_1k_2M\|\boldsymbol{x}_{10}\|\|\boldsymbol{x}_{20}\|e^{-\alpha_2 t}\int_{t_0}^{t} e^{(\alpha_2-\alpha_1)s}ds
\end{aligned}$$

If $\alpha_2 \ne \alpha_1$,

$$\begin{aligned}
\|\boldsymbol{x}_{2\mathrm{p}}\| &= k_1k_2M\|\boldsymbol{x}_{10}\|\|\boldsymbol{x}_{20}\||(\alpha_2-\alpha_1)|^{-1}\cdot \\
&\quad \cdot e^{-\alpha_2 t}(e^{(\alpha_2-\alpha_1)t} - e^{(\alpha_2-\alpha_1)t_0}) \\
&= k_1k_2M\|\boldsymbol{x}_{10}\|\|\boldsymbol{x}_{20}\||(\alpha_2-\alpha_1)|^{-1}|(e^{-\alpha_1 t} - c_1e^{-\alpha_2 t})|
\end{aligned}$$

where $c_1 = e^{(\alpha_2-\alpha_1)t_0}$.

A bound on the general solution of (32) is:

$$\begin{aligned}
\|\boldsymbol{x}_2\| &\le \|\boldsymbol{x}_{2\mathrm{h}}\| + \|\boldsymbol{x}_{2\mathrm{p}}\| \\
&\le k_2\|\boldsymbol{x}_{20}\|e^{-\alpha_2 t} + k_1k_2M\|\boldsymbol{x}_{10}\|\|\boldsymbol{x}_{20}\|\cdot \\
&\qquad \cdot|(\alpha_2-\alpha_1)|^{-1}|(e^{-\alpha_1 t} - c_1e^{-\alpha_2 t})| \\
&\le c_2e^{\max\{-\alpha_1,-\alpha_2\}t}
\end{aligned}$$

where $c_2 = \max\{k_2\|\boldsymbol{x}_{20}\|(1 + k_1M\|\boldsymbol{x}_{10}\||(\alpha_2 - \alpha_1)|^{-1}c_1),\ k_1k_2\|\boldsymbol{x}_{10}\|\|\boldsymbol{x}_{20}\|M|(\alpha_2-\alpha_1)|^{-1}\}$.

For $\alpha_1 = \alpha_2 = \alpha$, we have

$$\begin{aligned}
&\|\boldsymbol{x}_{2\mathrm{p}}\| \le k_1k_2M\|\boldsymbol{x}_{10}\|\|\boldsymbol{x}_{20}\|e^{-\alpha t}(t-t_0) \\
&\|\boldsymbol{x}_2\| \le \|\boldsymbol{x}_{2\mathrm{h}}\| + \|\boldsymbol{x}_{2\mathrm{p}}\| \\
&\le k_2\|\boldsymbol{x}_{20}\|e^{-\alpha t} + k_1k_2M\|\boldsymbol{x}_{10}\|\|\boldsymbol{x}_{20}\|e^{-\alpha t}(t-t_0) \\
&\le c_3e^{-\alpha t} + c_4te^{-\alpha t}
\end{aligned} \tag{33}$$

with $c_3 = k_2\|\boldsymbol{x}_{20}\|$ and $c_4 = k_1k_2\|\boldsymbol{x}_{10}\|\|\boldsymbol{x}_{20}\|M$. For the bound in (33) it has been shown that the convergence rate is α [32].

This means that the convergence rate of the system (32), and, therefore, of the system (25) is determined by the convergence rate of the individual subsystems. □

B. Cascaded Observers

This section presents the cascaded approach applied to observer design for TS fuzzy systems. As before, consider the fuzzy system with normal membership functions:

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i) \\
\boldsymbol{y} &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(C_i\boldsymbol{x} + d_i)
\end{aligned} \tag{34}$$

and a fuzzy observer of the form:

$$\begin{aligned}
\dot{\widehat{\boldsymbol{x}}} &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + a_i + L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}})) \\
\widehat{\boldsymbol{y}} &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(C_i\widehat{\boldsymbol{x}} + d_i)
\end{aligned} \tag{35}$$

Assuming that the system matrices for each rule $i = 1, 2, \ldots, m$ can be written as:

$$\begin{aligned}
A_i &= \begin{pmatrix} A_1 & 0 \\ A_{21} & A_2 \end{pmatrix}_i = \begin{pmatrix} A_{1i} & 0 \\ A_{21i} & A_{2i} \end{pmatrix} \\
C_i &= \begin{pmatrix} C_1 & 0 \\ C_{21} & C_2 \end{pmatrix}_i = \begin{pmatrix} C_{1i} & 0 \\ C_{21i} & C_{2i} \end{pmatrix}
\end{aligned}$$

observers can be designed individually for each subsystem and each rule, with the overall observer gain having the form $L_i = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}_i = \begin{pmatrix} L_{1i} & 0 \\ 0 & L_{2i} \end{pmatrix}$, where i denotes the rule number.

Again, two cases are distinguished. If the weights do not depend on the states to be estimated, the cascaded error system can be written as:

$$\begin{aligned}
\dot{\boldsymbol{e}} &= \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\boldsymbol{z})w_j(\boldsymbol{z})(A_i - L_iC_j)\boldsymbol{e} \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\boldsymbol{z})w_j(\boldsymbol{z})\begin{pmatrix} A_{1i} - L_{1i}C_{1j} & 0 \\ A_{21i} - L_{2i}C_{21j} & A_{2i} - L_{2i}C_{2j} \end{pmatrix}\boldsymbol{e}.
\end{aligned} \tag{36}$$

This system is of the form (24) for which the stability conditions from Section IV-A can be used. If the C matrix is common for all the rules, the presented theorems can be directly applied.

In the case when the scheduling vector does not depend on the states to be estimated, Theorem 11 can also be applied to the design of observers with guaranteed convergence rate.

Using the results on the convergence rate, Theorem 7 can be reformulated as follows:

Theorem 12: The decay rate of the error system (36) *is at least* α, *if there exist* $P_1 = P_1^T > 0$ *and* $P_2 = P_2^T > 0$, *so that for* $i = 1, 2, \ldots, m$

$$\begin{aligned}
&(A_{1i} - L_{1i}C_{1i})^TP_1 + P_1(A_{1i} - L_{1i}C_{1i}) + 2\alpha P_1 < 0 \\
&(A_{2i} - L_{2i}C_{2i})^TP_2 + P_2(A_{2i} - L_{2i}C_{2i}) + 2\alpha P_2 < 0 \\
&(A_{1i} - L_{1i}C_{1j})^TP_1 + P_1(A_{1i} - L_{1i}C_{1j}) + 2\alpha P_1 < 0 \\
&\forall\, j = 1, 2, \ldots, m \qquad w_{1i}(\boldsymbol{z}_1)w_{1j}(\boldsymbol{z}_1) \ne 0 \\
&(A_{2i} - L_{2i}C_{2j})^TP_2 + P_2(A_{2i} - L_{2i}C_{2j}) + 2\alpha P_2 < 0 \\
&\ j = 1, 2, \ldots, m \qquad w_{2i}(\boldsymbol{z})w_{2j}(\boldsymbol{z}) \ne 0
\end{aligned}$$

□

The proof follows directly. The above conditions explicitly state, that in order to design a global observer with a desired convergence rate, it is sufficient to design observers for the subsystems with the same convergence rate.

Now, consider the case when the parameters $\boldsymbol{z}$ depend on the states to be estimated, i.e., $\boldsymbol{z} = \widehat{\boldsymbol{z}}$. For simplicity, only the case with common measurement matrix is considered. Then, the fuzzy system is expressed as

$$\begin{aligned}\dot{\boldsymbol{x}} &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i)\\ \boldsymbol{y} &= C\boldsymbol{x},\end{aligned} \tag{37}$$

and the error system can be written as:

$$\begin{aligned}\dot{\boldsymbol{e}} =& \sum_{i=1}^{m}\begin{pmatrix} w(\widehat{\boldsymbol{z}}_1)(A_1 - L_1C_1) & 0\\ w(\widehat{\boldsymbol{z}})(A_{21} - L_2C_{21}) & w(\widehat{\boldsymbol{z}})(A_2 - L_2C_2)\end{pmatrix}_i \boldsymbol{e}\\ &+ \sum_{i=1}^{m}\begin{pmatrix} w_{1i}(\boldsymbol{z}_1) - w_{1i}(\widehat{\boldsymbol{z}}_1)\\ w_{2i}(\boldsymbol{z}) - w_{2i}(\widehat{\boldsymbol{z}})\end{pmatrix}(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i)\end{aligned} \tag{38}$$

To ensure the stability of the observer in such a case, Theorem 8 can be applied. However, using the results for cascaded systems, relaxed stability conditions are derived. These conditions can be expressed as follows.

Theorem 13: The cascaded error system (38) *is UGAS, if there exist a Lyapunov function* $V_1(\boldsymbol{x}_1)$, $P_2 = P_2^T > 0$ *and two continuous functions* θ_1, $\theta_2 : \mathcal{R}^+ \to \mathcal{R}^+$ *such that:*

1) *The Lyapunov function* V_1 *ensures exponential stability of the error system*

$$\begin{aligned}\dot{\boldsymbol{e}}_1 =& \sum_{i=1}^{m} w_{1i}(\widehat{\boldsymbol{z}}_1)(A_1 - L_1C_1)\boldsymbol{e}_1 +\\ &+ (w_{1i}(\boldsymbol{z}_1) - w_{1i}(\widehat{\boldsymbol{z}}_1))(A_{1i}\boldsymbol{x}_1 + B_{1i}\boldsymbol{u} + a_{1i}),\end{aligned} \tag{39}$$

2) P_2 *satisfies* $(A_{2i})^TP_2 + P_2A_{2i} < 0$, $i = 1, 2, \ldots, m$ *and*
3) $\|\sum_{i=1}^{m}(w_{2i}(\boldsymbol{z}_1, \boldsymbol{z}_2) - w_{2i}(\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_2))(A_{21i}\boldsymbol{x}_1 + A_{2i}\boldsymbol{x}_2 + B_{2i}\boldsymbol{u} + a_{2i})\| \le \theta_1(\|\boldsymbol{e}_1\|) + \theta_2(\|\boldsymbol{e}_1\|)\|\boldsymbol{e}_2\|$.

Proof: Since $(A_{2i})^TP_2 + P_2A_{2i} < 0$, $i = 1, 2, \ldots, m$, V_2 is a Lyapunov function for

$$\dot{\boldsymbol{e}}_2 = \sum_{i=1}^{m} w_{2i}(\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_2)(A_2 - L_2C_2)\boldsymbol{e}_2 \tag{40}$$

and this system is UGAS (Assumption 1). Let $c_1 = 2\frac{\lambda_{\max}(P_2)}{\lambda_{\min}(P_2)}$, where $\lambda_{\max}$ is the eigenvalue with the largest absolute value and $c_2 = 2\eta\lambda_{\max}(P_2)$. With these constants, Assumption 2 is satisfied. The Lyapunov function V_1 satisfies Assumption 4.

Now, the interconnection term in the second subsystem can be written as:

$$\begin{aligned}g(\boldsymbol{e}_1, \boldsymbol{e}_2) =& \sum_{i=1}^{m} w_{2i}(\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_2)(A_{21i} - L_{2i}C_{2i})\boldsymbol{e}_1 +\\ &+ \sum_{i=1}^{m}(w_{2i}(\boldsymbol{z}_1, \boldsymbol{z}_2) - w_{2i}(\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_2))\\ &\qquad (A_{21i}\boldsymbol{x}_1 + A_{2i}\boldsymbol{x}_2 + B_{2i}\boldsymbol{u} + a_{2i})\\ \|g(\boldsymbol{e}_1, \boldsymbol{e}_2)\| \le& \sum_{i=1}^{m}\|w_{2i}(\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_2)\|\|A_{21i} - L_{2i}C_{2i}\|\|\boldsymbol{e}_1\| +\\ &+ \theta_1(\|\boldsymbol{e}_1\|) + \theta_2(\|\boldsymbol{e}_1\|)\|\boldsymbol{e}_2\|\\ \|g(\boldsymbol{e}_1, \boldsymbol{e}_2)\| \le& \tau\|\boldsymbol{e}_1\| + \theta_1(\|\boldsymbol{e}_1\|) + \theta_2(\|\boldsymbol{e}_1\|)\|\boldsymbol{e}_2\|\\ \|g(\boldsymbol{e}_1, \boldsymbol{e}_2)\| \le& \theta_1'(\|\boldsymbol{e}_1\|) + \theta_2(\|\boldsymbol{e}_1\|)\|\boldsymbol{e}_2\|\end{aligned}$$

With this, Assumption 3 (see (10)) is satisfied, and based on Theorem 1, the cascaded system is UGAS. Moreover, since the first subsystem is exponentially stable, the cascaded system is also exponentially stable (see Proposition 1). □

C. Design Using LMI Regions

Designing observers based on the conditions presented in the previous sections, may not give an acceptable performance, since the poles of the observer may be placed at arbitrary locations in the left half-plane. This problem can be avoided by using LMI regions, i.e., constraining the poles of each local model to a specific region. A definition of the LMI regions can be found in [33]:

Definition 3: A subset $\mathcal{D}$ *of the complex plane is called an LMI region if there exist a symmetric matrix* $\alpha \in \mathcal{R}^{m\times m}$ *and a matrix* $\beta \in \mathcal{R}^{m\times m}$ *such that:*

$$\mathcal{D} = \{z \in \mathcal{C} : f_{\mathcal{D}}(z) < 0\}$$

where

$$f_{\mathcal{D}}(z) = \alpha + z\beta + \bar{z}\beta^T$$

is the characteristic function of the LMI region.

One can easily see, that, because of the form of the function $f_{\mathcal{D}}(z)$, LMI regions are convex and symmetric with respect to the real axis. Useful LMI regions include a vertical strip $[d_{\mathrm{l}}, d_{\mathrm{u}}]$ and a conic sector centered in the origin with inner angle θ (Figure 3). If all the eigenvalues of a matrix A are located in a region $\mathcal{D}$, then the matrix A is called $\mathcal{D}$-stable.

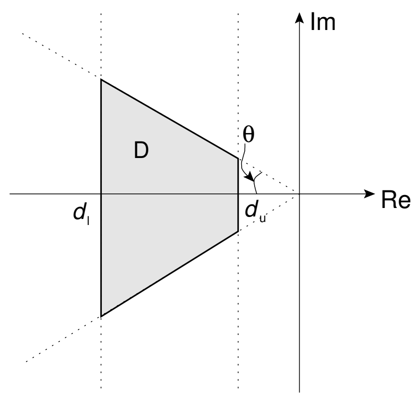

Fig. 3. LMI regions.

A theorem to ensure $\mathcal{D}$-stability of a matrix A was given in [33].

Theorem 14: The matrix A is $\mathcal{D}$-stable if and only if there exists $P = P^T > 0$ so that

$$\alpha \otimes P + \beta \otimes AP + \beta^T \otimes (AP)^T < 0$$

where $\otimes$ is the Kronecker product.

In the context of observer design, using LMI regions to ensure the specific $\mathcal{D}$-stability of the observer effectively means adding constraints to the presented LMI problems, more specifically:

$$[\alpha_{j,k} P + \beta_{j,k} P(A_i - L_i C_i) + \beta_{k,j}(A_i - L_i C_i)^T P] < 0$$
$$j, k = 1, 2, \ldots, m$$

Here, $\alpha_{j,k}$ and $\beta_{j,k}$ denote the (j,k)th element of the corresponding matrices.

V. Examples

In this section, we demonstrate the benefits of the proposed approach on two simulated examples.

A. Feasibility

We illustrate on a numerical example the case when the proposed conditions can be used to prove stability of a fuzzy system, which otherwise requires much more complicated analysis.

Consider the fuzzy system:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{2} w_i(\boldsymbol{z}) A_i \boldsymbol{x} \tag{41}$$

with $w_1(\boldsymbol{z}) \geq 0$, $w_2(\boldsymbol{z}) \geq 0$, $w_1(\boldsymbol{z}) + w_2(\boldsymbol{z}) = 1$, $\forall \boldsymbol{z}$.

The state matrices of the local linear models are given as:

$$A_1 = \begin{pmatrix} -0.7 & -1.0 & 0 & 0 & 0 \\ -1.0 & -2.8 & 0 & 0 & 0 \\ -0.1 & -1.8 & -1.4 & 0.6 & 0.0 \\ 0.1 & -0.7 & 0.6 & -3.1 & 0.4 \\ -1.8 & 1.3 & 0.0 & 0.4 & -1.9 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} -3.3 & -1.3 & 0 & 0 & 0 \\ -1.3 & -2.6 & 0 & 0 & 0 \\ 0 & 0 & -1.1 & 0.6 & -0.7 \\ 0 & 0 & 0.6 & -5.2 & 1.7 \\ 0 & 0 & -0.7 & 1.7 & -2.0 \end{pmatrix}.$$

The LMI problem

$$P > 0$$
$$A_1^T P + P A_1 < 0$$
$$A_2^T P + P A_2 < 0$$

is infeasible, so Theorems 2 and 4 cannot be applied. The stability of this system can be investigated using Theorem 5.

By examining the form of the system matrices, one can easily see that the system can be cascaded, with $\boldsymbol{x}_1 = [x_1 \;\; x_2]^T$ and $\boldsymbol{x}_2 = [x_3 \;\; x_4 \;\; x_5]^T$.

Based on Theorem 9, the system (41) is stable if the individual subsystems are stable. As such, in order to prove the stability of the system (41), it is sufficient that the LMI problems

$$P_1 > 0$$
$$A_{11}^T P_1 + P_1 A_{11} < 0$$
$$A_{12}^T P_1 + P_1 A_{12} < 0$$

and

$$P_2 > 0$$
$$A_{12}^T P_2 + P_2 A_{12} < 0$$
$$A_{22}^T P_2 + P_2 A_{22} < 0$$

are feasible. Using Yalmip's *solvesdp* [34] one can easily see that this is so.

This example illustrates the main benefit of the proposed stability conditions: while the conditions imposed by conventional methods lead to an infeasible LMI system, it is still possible to prove stability of the system under study, using the stability conditions proposed in this paper.

B. Observer Design

This example of a real-world system [35] illustrates the benefits of using the cascaded approach instead of centralized observer design.

Consider the three tanks connected in a cascade as shown in Figure 4. Water is pumped from a reservoir into the upper tank (3). From this tank, the water flows to the lower tanks and from the lowest tank back to the reservoir. The system has one control input u, which is the voltage applied to the motor of the pump and two measured outputs: the water levels h_3 in the upper tank and h_1 in the lowest tank. The flow rate F_{in}, provided by the pump, and the water level h_2 in the middle tank need to be estimated, and therefore, an observer has to be designed. The differential equations describing the dynamics of this system are the following:

$$\begin{aligned} \tau \dot{F}_{\text{in}} &= -F_{\text{in}} + Q_s \cdot u \\ \dot{h}_3 &= \frac{F_{\text{in}}}{A_3} - \frac{s_3\sqrt{2gh_3}}{A_3} \\ \dot{h}_2 &= \frac{s_3\sqrt{2gh_3}}{A_2} - \frac{s_2\sqrt{2gh_2}}{A_2} \\ \dot{h}_1 &= \frac{s_2\sqrt{2gh_2}}{A_1} - \frac{s_1\sqrt{2gh_1}}{A_1} \end{aligned} \tag{42}$$

The parameter values are presented in Table I.

TABLE I
PARAMETER VALUES USED.

Parameter	Symbol	Value	Units
Acceleration due to gravity	g	9.81	m/s^2
Cross-sectional area tank 1	A_1	10	m^2
Cross-sectional area tank 2	A_2	8	m^2
Cross-sectional area tank 3	A_3	9	m^2
Outlet area of tank 1	s_1	0.25	m^2
Outlet area of tank 2	s_2	0.2	m^2
Outlet area of tank 3	s_3	0.3	m^2
Input to flow gain	Q_s	0.336	m^3/s/V
Motor time constant	τ	3	s

It is assumed that the tanks have the same height, $h_{\max} = 2\,\text{m}$. Therefore, all levels are bounded, $h_i \in [0, h_{\max}]$.

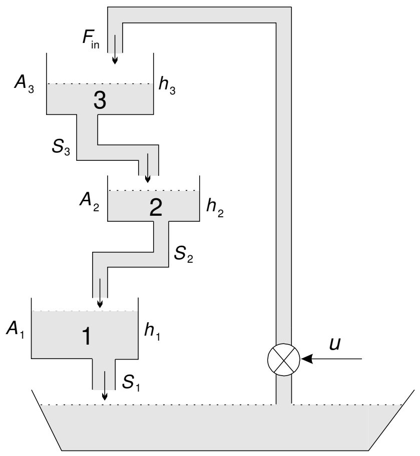

Fig. 4. Cascaded tanks system.

This system is highly nonlinear and a linear observer cannot be used. However, it is possible to design a fuzzy observer for this system. In order to use the proposed design, a TS fuzzy model of the system (42) is constructed. For each level h_i, four points $h_i \in \{0.1, 0.55, 1.05, 1.6\}$ are chosen, together with appropriate membership functions, as depicted in Figure 5. Note that the scheduling vector consists of the levels h_1, h_2 and h_3 which are the states to be estimated.

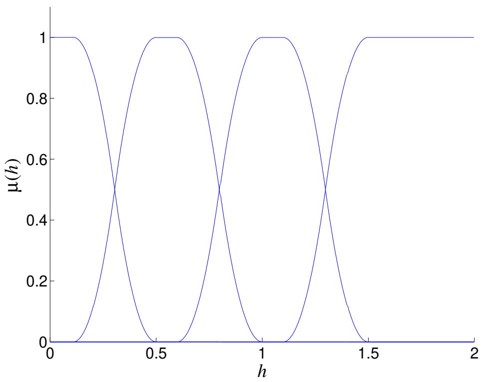

Fig. 5. Membership functions for the heights.

The system (42) is linearized for each combination of the chosen points. Since the linearization is not done in equilibria, the consequents are affine. For instance, the rule obtained by linearizing in $h_1 = 0.55$, $h_2 = 0.1$ and $h_3 = 0.55$ is:

If h_1 *is approximately* 0.55 *and* h_2 *is approximately* 0.1 *and* h_3 *is approximately* 0.55*, then* $\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu + a$, with

$$A = \begin{pmatrix} -0.3333 & 0 & 0 & 0 \\ 0.1111 & -0.0995 & 0 & 0 \\ 0 & 0.1120 & -0.1751 & 0 \\ 0 & 0 & 0.1401 & -0.0747 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.1120 & 0 & 0 & 0 \end{pmatrix}^T$$

$$a = (0 \quad -0.0547 \;\; 0.0441 \quad -0.0271)^T$$

where $\boldsymbol{x} = [F_{\text{in}} \;\; h_3 \;\; h_2 \;\; h_1]^T$. To compute the membership degree of the scheduling vector, the algebraic product operator is used. Note that, by using this operator, the membership functions obtained for the scheduling vectors are smooth. Other operators, such as MIN and MAX would render the membership functions non-smooth.

By examining the form of system (42) and the matrices of the fuzzy system, one can easily see that the system can be cascaded, with $\boldsymbol{x}_1 = [F_{\text{in}} \; h_3]^T$ and $\boldsymbol{x}_2 = [h_2 \; h_1]^T$. Therefore, observers can be designed separately for the individual subsystems. The observers are designed both for the whole system and the individual subsystems using the same pole-placement method and conditions. Both observers have the form (22).

To simulate the system, the differential equations were discretized with the Euler method, using a sampling period $T = 0.1$s. The input was randomly generated, and so were the "true" and "estimated" initial states. For the presented cases, the true initial conditions were $[1.7 \; 0.4 \; 0.1 \; 0.4]^T$, while the estimated ones were $[1.5 \; 0.2 \; 1.3 \; 0.8]^T$. The observers were designed using different LMI regions. The regions and the CPU time needed to solve the LMIs for these regions using the Yalmip toolbox [34] are presented in Table II, for the centralized and cascaded observers. As can be seen, the time needed to solve the LMIs for the centralized observer is in most cases more than 10 times larger than the time needed for the cascaded observer. This is due to the fact that for the centralized system 64 4-by-4 LMIs need to be solved, while for the cascaded approach this number is reduced to 2 times 4 LMIs of dimension 2.

TABLE II
LMI REGIONS AND THE CPU TIME.

Case	d_l	d_u	θ	Centralized [s]	Cascaded [s]
1	–	–	–	2.73	0.17
2	-10	-2	–	103.00	0.23
3	–	–	$\pi/4$	1.59	0.22
4	-10	-2	$\pi/4$	30.05	0.53
5	–	–	$\pi/36$	1.75	0.31
6	-10	-2	$\pi/36$	33.89	0.53

The estimation errors of F_{in} and h_2, when using centralized and cascaded observers, for the six cases are presented in Figures 6 and 7, respectively.

If the LMI region is the entire left half-plane, the cascaded observer converges much faster than the centralized (Figures 6(a) and 7(a)). If the closed-loop poles are restricted to the interval $[-10, -2]$, but there is no restriction on θ (case 2), the imaginary parts of the closed-loop poles of the centralized observer are of the order 10^6 – so the observer effectively becomes unusable. Therefore, only the results obtained by the cascaded setting are presented (Figures 6(b) and 7(b)). If θ is constrained, the performance of the observers is comparable. For no constraints on the poles' real part (i.e., no vertical strip in Figure 3), the estimation error on h_2 of the cascaded observer converges faster (Figures 7(c) and 7(e)). When both the real part and the damping are constrained, the overshoot of the cascaded observer is slightly larger than that of the centralized one (Figures 6(d) and 6(f)).

VI. CONCLUSIONS

In many real-life applications, a complex process model can be decomposed into simpler, cascaded subsystems. This partitioning of a process leads to increased modularity and reduced complexity of the problem, while also making the analysis easier. In this paper, we have studied the stability of a cascaded fuzzy system, based on its individual subsystems. We have proven that the stability of the individual subsystems implies the stability of the global fuzzy system. Furthermore, the proposed approach relaxes the conventional stability conditions and reduces the dimension of the LMI problem to be solved.

We have also extended the cascaded setting to state estimation. If a complex process model can be decomposed into a cascade of simpler subsystems, observers can be designed for these individual subsystems. This partitioning of a process and observer leads to increased modularity and reduced complexity of the problem, with reduced computational costs. The benefits of studying stability based on subsystems have been demonstrated on simulation examples.

In our future research, we will investigate the theoretical conditions under which fuzzy observers can be used for distributed, but not necessarily cascaded systems.

Acknowledgment

This research is financially supported by Senter, Ministry of Economic Affairs of the Netherlands within the project Interactive Collaborative Information Systems (grant no. BSIK03024).

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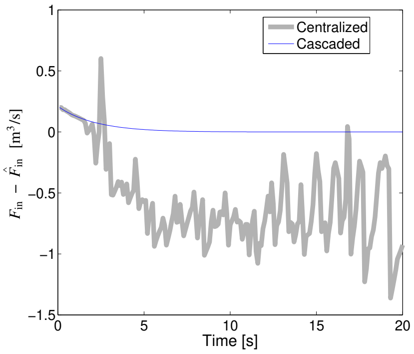
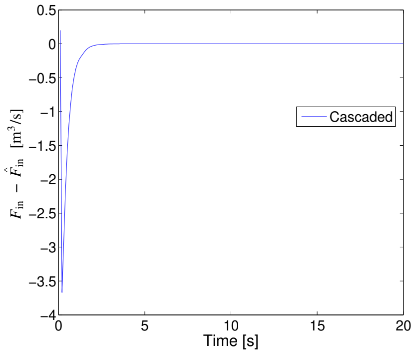
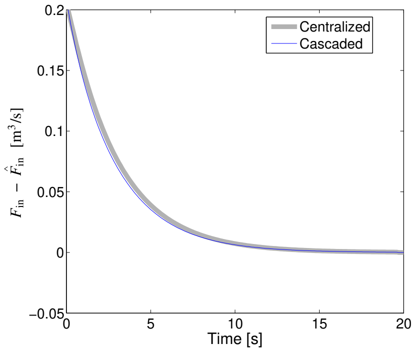


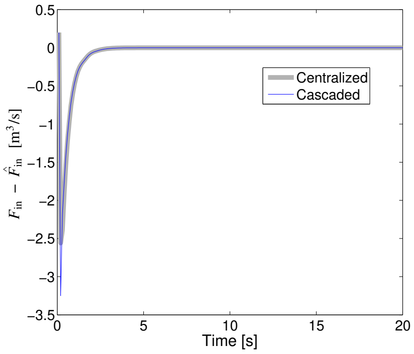
(a) Estimation error for F_{in}, case 1



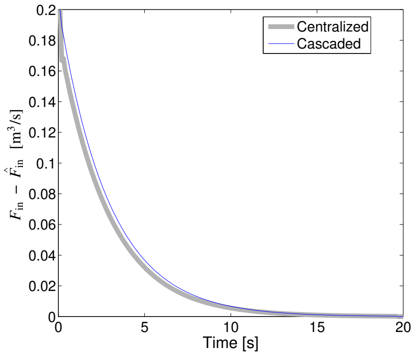
(b) Estimation error for F_{in}, case 2



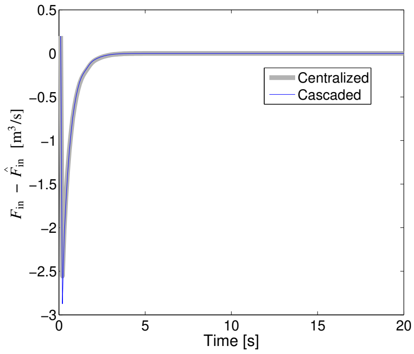
(c) Estimation error for F_{in}, case 3



(d) Estimation error for F_{in}, case 4

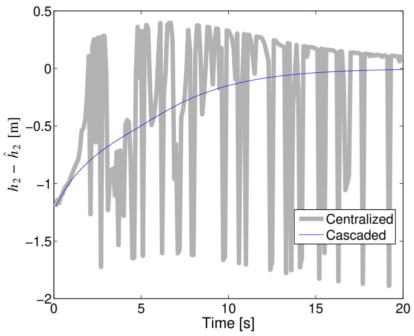


(e) Estimation error for F_{in}, case 5

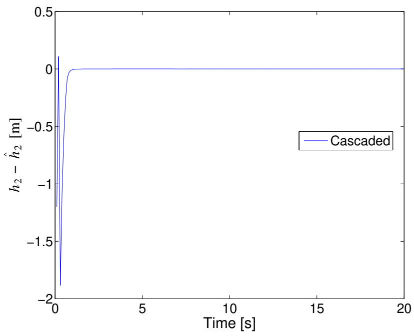


(f) Estimation error for F_{in}, case 6

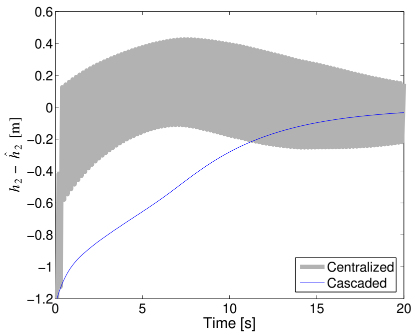
Fig. 6. Estimation errors for F_{in} centralized and cascaded observers.



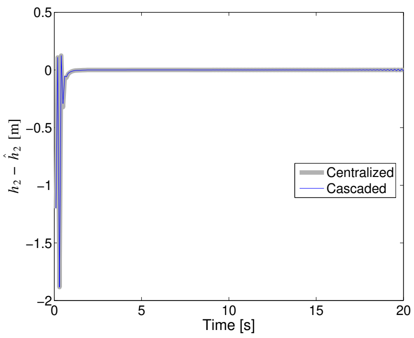
(a) Estimation error for h_2, case 1



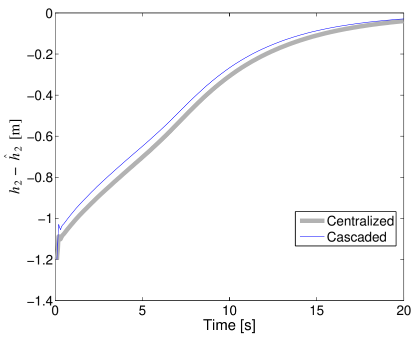
(b) Estimation error for h_2, case 2



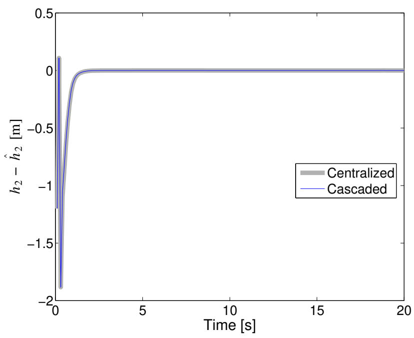
(c) Estimation error for h_2, case 3



(d) Estimation error for h_2, case 4



(e) Estimation error for h_2, case 5



(f) Estimation error for h_2, case 6

Fig. 7. Estimation errors for h_2 using centralized and cascaded observers.