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Optimal Steady-State Traffic Control for Isolated Intersections ⋆

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Abstract: In this paper a simplified isolated controlled intersection is introduced. A discrete-event max-plus model is proposed to formulate the optimization problem for the switching sequences. The formulated max-plus problem is converted to be solved by linear programming (LP). In the special case when the criterion is a strictly increasing and linear function of the queue lengths, the steady-state control problem can be solved analytically. In addition, necessary condition for the steady-state control is derived.

Keywords: Traffic control, Steady-state control, Discrete-event max-plus model.

1. INTRODUCTION

Different models, methods, and strategies have been proposed and applied for controlling urban isolated signalized intersections Allsop (1971, 1976); Gartner et al. (1976); Improta and Cantarella (1984); Kashani and Saridis (1983); Lim et al. (1981); Little (1966); Roberston (1969); Talmor and Mahalel (2007). These researches aim to minimize delays or to maximize the intersection capacity. Some recent research considers the isolated intersection in the urban traffic network as a hybrid system Di Febbraro et al. (2004); Di Febbraro and Sacco (2004); Dotoli and Pia Fanti (2006); Lei and Ozguner (2001) and others propose the game theory approach Villalobos et al. (2008) to model signalized intersections.

In De Schutter (2002), the optimal acyclic (or N-stages) control was dealt with, where the Extended Linear Complementary Problem (ELCP), which is a mathematical programming problem, was used.

In practice it is difficult to measure delays. Therefore, in this paper we propose to minimize queue lengths which are easily measured with available sensors. We give necessary condition for the existence of a constant cycle length steady-state solution. The solution is given as an LP-problem, and the analytic solution has as a simple form. We also show how to bring initial non-optimal queue lengths to optimum thus enabling an N-stages control solution for the transient phase.

The paper is organized as follows. After describing the problem definition in Section 2, the discrete-event models of an isolated intersection and the formulation of the optimal problems are given in Section 3. The cyclic steady-state control problem is considered in Section 4. The transient N-stages control problem is dealt with in Section 5.

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2. PROBLEM DEFINITION

In this paper, a typical simplified isolated intersection will be dealt with. As shown in Fig. 1, there are two movements (m_1 and m_2), where each movement has a traffic signal that can be green or red. There is a traffic conflict in the intersection area between the two movements, therefore they cannot travel simultaneously and the traffic signal will be opposite, i.e. when movement m_1 has a green light, movement m_2 has a red light, and vice versa.

A given movement will encounter intertwined green and red periods. A cycle is defined as a pair of one green and one red period, whose durations may be time-varying.

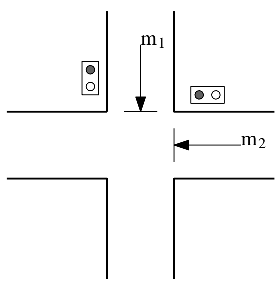

Fig. 1. Simplified isolated controlled intersection

The evolution of the queue lengths will be considered for the two movements m_1 and m_2. The length of the queue for movement m_i at time t, which is the number of vehicles stopping behind the stop line in the intersection, is denoted by $q_i(t)$ [veh]. Let $f_{\text{arr},i}(t)$ [veh/s], $f_{\text{dep},i}(t)$ [veh/s] be, respectively, the arrival rate and the departure rate for queue i at time t. The queue length growth rate $\alpha_i(t)$ [veh/s] for queue i at time t is given by $\alpha_i(t) = f_{\text{arr},i}(t) - f_{\text{dep},i}(t)$.

The following assumptions are made:

- A1: The arrival and departure rates in the isolated intersection are known. The arrival rate is constant within each cycle, and the departure rate is constant within each green or red period.
- A2: When the traffic signal is green, the departure rate is bigger than the arrival rate, i.e. $f_{\text{dep},i}(t) > f_{\text{arr},i}(t)$, and when the traffic signal is red, the departure rate is equal to zero, $f_{\text{dep},i}(t) = 0$, and the arrival rate is non-negative, $f_{\text{arr},i}(t) \geq 0$.
- A3: The queue lengths [veh] are approximated by real non-negative numbers.

For the isolated controlled intersection with a constant traffic arrival and departure rates, we determine the constant-cycle steady-state traffic signal control solution that minimizes a given queue length dependent criterion. We also formulate the necessary condition for the steady-state control.

3. DISCRETE-EVENT MODELS FOR ISOLATED CONTROLLED INTERSECTIONS

A variety of models Diakaki et al. (2002); Papageorgiou et al. (2003) are based on the store-and-forward approach of modeling traffic networks that was first suggested in Gazis (1964); Gazis and Potts (1963). This approach enables the simplification of the mathematical description of the traffic flow process without the use of switching variables. In this paper we consider the isolated controlled intersection as switching systems, as was done in De Schutter (2000, 2002); De Schutter and De Moor (1998). Here, the optimization of traffic signal switching sequences will be performed with a discrete-event max-plus model.

3.1 Basic model

Let k be the index of the cycle. By definition, in a cycle, each movement (m_1 or m_2) will have only one green signal. We want to determine two decision variables: the cycle time, $T_{\text{cyc},k}$ [s], and $g_k \in [0,1]$ the fraction of the green time of movement m_1 of $T_{\text{cyc},k}$.

The evolution of the system begins at time t_0. This implies that the state of the queue length i at time t is given by

$$q_i(t) = q_i(t_0) + \int_{t_0}^{t} \alpha_i(t)dt \tag{1}$$

There are two switching times for cycle k: t_{2k+1} and t_{2k+2} (see Fig. 2). Without loss of generality, let the green light for movement m_1 start at t_{2k}, which coincides with the start of cycle k. Hence, t_{2k+1} is the end of the green light for movement m_1, and the start of the green light for movement m_2, while t_{2k+2} is the end of the green light for movement m_2, and the start of the green light for movement m_1 in the next cycle. We note that $T_{\text{cyc},k} = t_{2k+2} - t_{2k}$. By assumption A1 the arrival rate of queue i in phase $2k$ (i.e. the time period between t_{2k} and t_{2k+1}), $f_{\text{arr},i}(t_{2k})$, and the departure

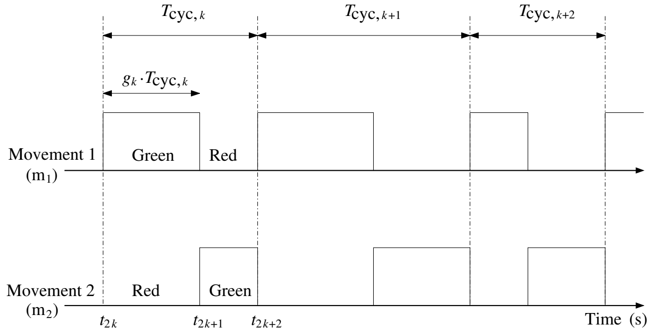

Fig. 2. Traffic signal switching sequences for movements m_1 and m_2

rate of queue i in phase $2k$, $f_{\text{dep},i}(t_{2k})$, are known and constant. The same holds for phase $2k+1$: the arrival rate of queue i, $f_{\text{arr},i}(t_{2k+1})$, and the departure rate of queue i, $f_{\text{dep},i}(t_{2k+1})$, are known and constant. This means that the growth rate $\alpha_i(t_{2k}) = f_{\text{arr},i}(t_{2k}) - f_{\text{dep},i}(t_{2k})$ is constant between the two switching times t_{2k} and t_{2k+1}. The relations between the time sequences are as follows

$$t_{2k+1} = t_{2k} + g_k \cdot T_{\text{cyc},k} \tag{2}$$

$$t_{2k+2} = t_{2k} + T_{\text{cyc},k} \tag{3}$$

3.2 Formulation of an optimal discrete-event max-plus problem

The value of the queue length for movement m_1 in cycle k at the switching time instant t_{2k+1} is given by

$$q_1(t_{2k+1}) = \max(q_1(t_{2k}) + \alpha_1(t_{2k}) \cdot g_k \cdot T_{\text{cyc},k}, 0) \tag{4}$$

and at the switching time instant t_{2k+2}

$$q_1(t_{2k+2}) = q_1(t_{2k+1}) + \alpha_1(t_{2k+1}) \cdot (1 - g_k) \cdot T_{\text{cyc},k} \tag{5}$$

Recall that the signal light for movement m_2 is opposite to that of m_1; therefore the value of the queue lengths for movement m_2 in cycle k are given by

$$q_2(t_{2k+1}) = q_2(t_{2k}) + \alpha_2(t_{2k}) \cdot g_k \cdot T_{\text{cyc},k} \tag{6}$$

$$q_2(t_{2k+2}) = \max(q_2(t_{2k+1}) + \alpha_2(t_{2k+1}) \cdot (1 - g_k) \cdot T_{\text{cyc},k}, 0) \tag{7}$$

We now consider the following problem: for a given number of cycles N and starting time t_0, we compute an optimal switching time sequence $t_1, t_2, \ldots, t_{2N}$ that minimizes a given performance criterion J. There are a variety of criteria that can be chosen, e.g. average queue length, maximal queue length, and delay over all queues De Schutter (2002).

Two new variables are now defined, $T_1(k)$ [s] and $T_2(k)$ [s], where $T_1(k) = g_k \cdot T_{\text{cyc},k}$ and $T_2(k) = (1 - g_k) \cdot T_{\text{cyc},k}$. Substituting these variables into (4) - (7) leads to the following Discrete-event Max-Plus (DMP) problem:

$$\min_{T_1(0),T_2(0),T_1(1),T_2(1),\cdots,T_1(N-1),T_2(N-1)} J \tag{8}$$

subject to

$$q_1(t_{2k+1}) = \max(q_1(t_{2k}) + \alpha_1(t_{2k}) \cdot T_1(k), 0) \tag{9}$$

$$q_1(t_{2k+2}) = q_1(t_{2k+1}) + \alpha_1(t_{2k+1}) \cdot T_2(k) \tag{10}$$

$$q_2(t_{2k+1}) = q_2(t_{2k}) + \alpha_2(t_{2k}) \cdot T_1(k) \tag{11}$$

$$q_2(t_{2k+2}) = \max(q_2(t_{2k+1}) + \alpha_2(t_{2k+1}) \cdot T_2(k), 0) \tag{12}$$

for $k = 0, 1, 2, \ldots, N-1$. The optimization problem is formulated by minimization of the criterion J over N

cycles. Hence, the number of variables to be determined is $2N$.

4. STEADY-STATE CONTROL WITH CONSTANT CYCLE LENGTH

We start by finding a solution to the steady-state control problem with constant cycle length. It is assumed that all cycles, and their flow rates are identical, and hence only one cycle needs to be considered. Hence, the decision variables are T_1 [s] and T_2 [s] whereby the cycle duration $T_{\text{cyc}} = T_1 + T_2$. Let the start time of the steady-state cycle be τ_0. The switching times are τ_1, and τ_2, respectively, whereby $T_1 = \tau_1 - \tau_0$, and $T_2 = \tau_2 - \tau_1$.

The queue length for movement i at the start of the cycle will be equal to the queue length at the start of the next cycle:

$$q_1(\tau_0) = q_1(\tau_2) \tag{13}$$
$$q_2(\tau_0) = q_2(\tau_2) \tag{14}$$

Let the steady-state queue length vector q be defined as $[q_1(\tau_1), q_2(\tau_1), q_1(\tau_2), q_2(\tau_2)]^T$. The criterion function J is said to be strictly increasing, if, for all queue length vectors $\hat{q}$, $\tilde{q}$ with $\hat{q} \leq \tilde{q}$ (elementwise) and $\hat{q}_i < \tilde{q}_i$ for at least one index i, we have $J(\hat{q}) < J(\tilde{q})$.

In the following, we consider the case when the criterion J is a strictly increasing function of the queue lengths, such as the average queue length, a positively weighted sum of queue lengths, or the average travel time. We show that for such a criterion, the optimal steady-state constant cycle length switching sequence problem, and its necessary condition can be formulated using a discrete-event max-plus model, and solved analytically for a strictly increasing and linear criterion.

4.1 Formulation of an optimal cyclic discrete-event max-plus problem

The formulation is based on the DMP problem (8) - (12). The cyclic queue lengths equations (13) - (14) are added to the DMP problem and then we optimize it over only one cycle time ($N = 1$ and $k = 0$). Therefore, the number of decision variables will decrease to two: $T_1(0)$ and $T_2(0)$. For simplicity we write $T_1(0)$ and $T_2(0)$ as T_1 and T_2, respectively. We also assume that a lower bound $T_{\min}$ (with $T_{\min} > 0$) for the sum of T_1 and T_2 is given, i.e. $T_1 + T_2 \geq T_{\min}$.

The Cyclic Discrete-event Max-Plus (CDMP) problem is then defined as follows:

$$\min_{T_1,T_2} J \tag{15}$$

subject to

$$q_1(\tau_1) = \max(q_1(\tau_0) + \alpha_1(\tau_0) \cdot T_1, 0) \tag{16}$$
$$q_1(\tau_2) = q_1(\tau_1) + \alpha_1(\tau_1) \cdot T_2 \tag{17}$$
$$q_2(\tau_1) = q_2(\tau_0) + \alpha_2(\tau_0) \cdot T_1 \tag{18}$$
$$q_2(\tau_2) = \max(q_2(\tau_1) + \alpha_2(\tau_1) \cdot T_2, 0) \tag{19}$$
$$T_1 + T_2 \geq T_{\min} \tag{20}$$
$$\text{and } (13), (14)$$

Note that for scalars $a, b, c \in \mathbb{R}$ we have that $a = \max(b, c)$ implies $a \geq b$ and $a \geq c$. In a similar way the CDMP problem can be rewritten in such a way that the max equations are "relaxed" to linear inequality equations. But first, the cyclic queue lengths equations (13) and (14) are substituted into (16) and (18) respectively:

$$q_1(\tau_1) = \max(q_1(\tau_2) + \alpha_1(\tau_0) \cdot T_1, 0) \tag{21}$$
$$q_2(\tau_1) = q_2(\tau_2) + \alpha_2(\tau_0) \cdot T_1 \tag{22}$$

The max equations (21) and (19) can then be relaxed into linear inequality equations as follows:

$$q_1(\tau_1) \geq q_1(\tau_2) + \alpha_1(\tau_0) \cdot T_1 \tag{23}$$
$$q_1(\tau_1) \geq 0 \tag{24}$$
$$q_2(\tau_2) \geq q_2(\tau_1) + \alpha_2(\tau_1) \cdot T_2 \tag{25}$$
$$q_2(\tau_2) \geq 0 \tag{26}$$

This leads to the "Relaxed" Cyclic Discrete-event Max-Plus (R-CDMP) problem:

$$\min_{T_1,T_2} J \tag{27}$$

subject to

$$(17), (20), (22), (23), (24), (25), (26)$$

Proposition 1. If the criterion J is a strictly increasing function of the queue lengths, then any optimal solution of the R-CDMP problem is also an optimal solution of the CDMP problem.

Proof. The proof is done by contradiction.
Let $\tilde{q} = (\tilde{q}_1(\tau_1), \tilde{q}_1(\tau_2), \tilde{q}_2(\tau_1), \tilde{q}_2(\tau_2))^T$ and $\tilde{T} = (\tilde{T}_1, \tilde{T}_2)^T$ be an optimal solution of the R-CDMP problem such that (16) is not satisfied, i.e.

$$\tilde{q}_1(\tau_1) > \max(\tilde{q}_1(\tau_2) + \alpha_1(\tau_0) \cdot \tilde{T}_1, 0) \tag{28}$$

or equivalently

$$\tilde{q}_1(\tau_1) > \tilde{q}_1(\tau_2) + \alpha_1(\tau_0) \cdot \tilde{T}_1 \tag{29}$$
$$\tilde{q}_1(\tau_1) > 0 \tag{30}$$

and such that $\tilde{q}_2(\tau_1)$, $\tilde{q}_2(\tau_2)$ satisfy (19) (note that we consider the case (29) and (30), but the proof for other cases is similar).
Now we replace $\tilde{q}_1(\tau_1)$ and $\tilde{q}_1(\tau_2)$ by

$$\hat{q}_1(\tau_1) = \tilde{q}_1(\tau_1) - \varepsilon \tag{31}$$
$$\hat{q}_1(\tau_2) = \tilde{q}_1(\tau_2) - \varepsilon \tag{32}$$

where $\varepsilon > 0$. The other variables stay the same, i.e. $\hat{q}_2(\tau_1)$, $\hat{q}_2(\tau_2)$, and $\hat{T}$ are equal to $\tilde{q}_2(\tau_1)$, $\tilde{q}_2(\tau_2)$, and $\tilde{T}$, respectively.
In the following we verify that $(\hat{q}, \hat{T})$ is also a feasible solution of the R-CDMP problem as long as $\hat{q}_1(\tau_1) \geq 0$ (i.e. (24)) is satisfied. We fill out $\hat{q}_1(\tau_1)$ and $\hat{q}_1(\tau_2)$ into (29) and obtain $\tilde{q}_1(\tau_1) - \varepsilon > \tilde{q}_1(\tau_2) - \varepsilon + \alpha_1(\tau_0) \cdot \tilde{T}_1$ for any ε, which implies $\hat{q}_1(\tau_1) \geq \hat{q}_1(\tau_2) + \alpha_1(\tau_0) \cdot \hat{T}_1$ (i.e. (23) holds).
Since the variables $\hat{q}_2(\tau_1)$, $\hat{q}_2(\tau_2)$, and $\hat{T}$ are assumed to be unchanged, they imply (20), (22), (25), and (26).
Equation (17) implies $\tilde{q}_1(\tau_2) - \varepsilon = \tilde{q}_1(\tau_1) - \varepsilon + \alpha_1(\tau_1) \cdot \tilde{T}_2$ or equivalently $\hat{q}_1(\tau_2) = \hat{q}_1(\tau_1) + \alpha_1(\tau_1) \cdot \hat{T}_2$.
Now we select ε such that $\hat{q}_1(\tau_1) = 0$. Then $(\hat{q}, \hat{T})$ is a feasible solution of the R-CDMP problem. Recall that the criterion J is a strictly increasing function of the queue lengths. Since $\hat{q} \leq \tilde{q}$ and $\hat{q}_i < \tilde{q}_i$ for some i due to (31) and (32), this implies $J(\hat{q}, \hat{T}) < J(\tilde{q}, \tilde{T})$, which is in contradiction with the fact that $(\tilde{q}, \tilde{T})$ is an optimal solution of the R-CDMP problem.

Hence, the optimal solution of the R-CDMP problem should satisfy (16) and as a consequence the optimal solution of the R-CDMP problem is also an optimal solution of the CDMP problem.

4.2 Necessary condition for the steady-state control

In this section, the necessary condition for the steady-state control is derived based on the R-CDMP problem. We can eliminate $q_1(\tau_2)$ and $q_2(\tau_1)$ from the constraints of the R-CDMP problem by substituting (17) and (22) into (23) and (25) respectively, resulting in

$$-\alpha_1(\tau_0)\cdot T_1 \geq \alpha_1(\tau_1)\cdot T_2 \quad (33)$$
$$-\alpha_2(\tau_1)\cdot T_2 \geq \alpha_2(\tau_0)\cdot T_1 \quad (34)$$

If $T_1 = 0$, then (33) and (34) imply that $T_2 = 0$, and vice versa. But this is a contradiction to (20) and the fact that $T > 0$. Hence, we will have $T_1 > 0$ and $T_2 > 0$. However, it follows from assumptions A1 and A2 that $\alpha_1(\tau_0) < 0$ and $\alpha_2(\tau_1) < 0$, whereby $\alpha_1(\tau_1) > 0$ and $\alpha_2(\tau_0) > 0$. We divide the two equations (33) and (34) by T_2 and eliminate the fraction $\frac{T_1}{T_2}$ by comparing between the two equations. Then we obtain the following necessary condition:

$$\frac{\alpha_1(\tau_1)}{-\alpha_1(\tau_0)} \leq \frac{-\alpha_2(\tau_1)}{\alpha_2(\tau_0)} \quad (35)$$

or by symmetry

$$\frac{-\alpha_1(\tau_0)}{\alpha_1(\tau_1)} \geq \frac{\alpha_2(\tau_0)}{-\alpha_2(\tau_1)} \quad (36)$$

4.3 Analytic solution for linear criterion

Now we show that if the criterion J is a strictly increasing *linear* function of the queue lengths, then the R-CDMP problem can be solved analytically. But first we prove the following proposition.

Proposition 2. The optimal solutions of the queue lengths $q_1(\tau_1)$ and $q_2(\tau_2)$ in the R-CDMP problem with a strictly increasing function of the queue lengths must be equal to zero in order to minimize J, i.e.

$$q_1^*(\tau_1) = 0 \quad (37)$$
$$q_2^*(\tau_2) = 0 \quad (38)$$

Proof. The proof is done by contradiction. We first assume that the optimal queue lengths $q_1(\tau_1), q_2(\tau_2)$ are non-zeros, i.e. $q_1(\tau_1) > 0$ and $q_2(\tau_2) > 0$. Then in this case, the max equations (21) and (19) can be written as follows,

$$q_1(\tau_1) = q_1(\tau_2) + \alpha_1(\tau_0)\cdot T_1 \quad (39)$$
$$q_2(\tau_2) = q_2(\tau_1) + \alpha_2(\tau_1)\cdot T_2 \quad (40)$$

This leads to slight changes in the equations (33) and (34), respectively

$$-\alpha_1(\tau_0)\cdot T_1 = \alpha_1(\tau_1)\cdot T_2 \quad (41)$$
$$-\alpha_2(\tau_1)\cdot T_2 = \alpha_2(\tau_0)\cdot T_1 \quad (42)$$

and the R-CDMP problem will be as follows,

$$\min_{T_1,T_2} J \quad (43)$$

subject to

$$(17), (20), (22), (39), (40)$$

Equations (41), (42) and (20) give a solution for T_1 and T_2. Then (39), (40) imply that $q_1(\tau_1)$ and $q_2(\tau_2)$ are constant, not depending on T_1 and T_2. Hence, we can reduce the criterion J by decreasing the queue lengths $q_1(\tau_1), q_2(\tau_2)$ to zeros. This is a contradiction with the assumption that the optimal queue lengths $q_1(\tau_1), q_2(\tau_2)$ are non-zeros.

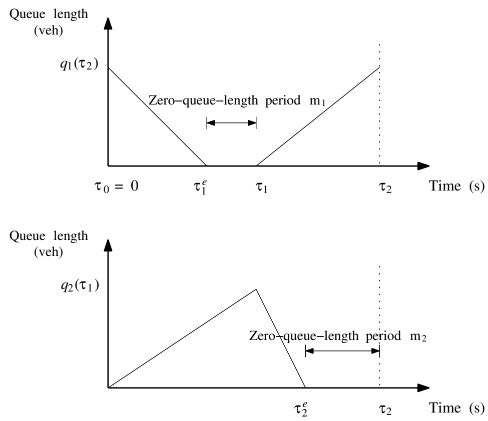

Fig. 3. Zero-queue-length period for movement m_1 and m_2

We define a "zero-queue-length period" (ZQLP) as the time period (larger than zero) for which the queue length is equal to zero (see Fig. 3). Given the assumptions, a movement can encounter only one ZQLP per cycle, and it may happen only before the end of the green light, i.e. between τ_0 and τ_1 for movement m_1, and between τ_1 and τ_2 for movement m_2. Let us denote the start of the ZQLP for movements m_1 and m_2 by τ_1^e and τ_2^e, respectively. Then the ZQLP for movement m_1 starts at time τ_1^e and ends at time τ_1, and the ZQLP for movement m_2 starts at time τ_2^e and ends at time τ_2. Without loss of generality let $\tau_0 = 0$, and the cycle time $T = \tau_2$.

As an example for a criterion J which is a strictly increasing *linear* function of the queue lengths, let J be the weighted sum of the maximum queue lengths,

$$J = w_1 q_1(\tau_2) + w_2 q_2(\tau_1) \quad (44)$$

where $w_1, w_2 > 0$.

Proposition 3. For the R-CDMP problem with a criterion J that is a strictly increasing "linear" function of the queue lengths, the optimal cycle time is equal to the minimum cycle time $T_{\min}$.

Proof. The general case where the cycle time is bigger than the minimum cycle time and each movement has a ZQLP is shown in Fig. 4. The cycle time τ_2 can be decreased to $T_{\min}$ by multiplying all the values by the coefficient $\gamma = \frac{T_{\min}}{\tau_2}$ as shown in Fig. 4. Decreasing the cycle time decrease the maximum queue lengths from $q_1(\tau_2)$ and $q_2(\tau_1)$ to $\gamma q_1(\tau_2)$ and $\gamma q_2(\tau_1)$, respectively, and decreases the value of the criterion J, i.e. the maximum queue lengths decreases as we decrease the cycle time, which proves that the optimal cycle time will be equal to the minimum cycle time $T_{\min}$.

According to the Propositions 2 and 3, we obtain the following linear programming (LP) problem when J is given by (44),

$$\min_{T_1,T_2} J = w_2\alpha_2(\tau_0)\cdot T_1 + w_1\alpha_1(\tau_1)\cdot T_2 \quad (45)$$

subject to

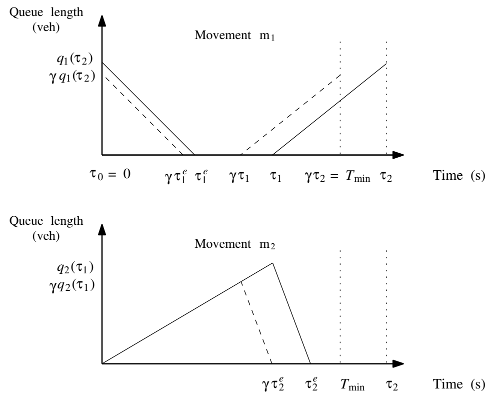

Fig. 4. Decreasing cycle time to the minimum by scaling multiplication

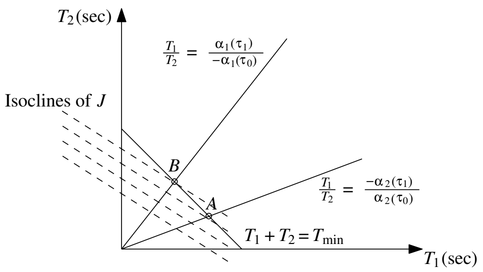

Fig. 5. Analytic solution for the linear programming problem

$$-\alpha_1(\tau_0) \cdot T_1 \geq \alpha_1(\tau_1) \cdot T_2 \quad (46)$$
$$-\alpha_2(\tau_1) \cdot T_2 \geq \alpha_2(\tau_0) \cdot T_1 \quad (47)$$
$$T_1 + T_2 = T_{\min} \quad (48)$$

In the case when the necessary condition (35) is satisfied by the strict inequality constraint, i.e. $\frac{\alpha_1(\tau_1)}{-\alpha_1(\tau_0)} < \frac{-\alpha_2(\tau_1)}{\alpha_2(\tau_0)}$, the solution of the problem depends on the slope of the linear objective function (see Fig. 5). If $w_2\alpha_2(\tau_0) < w_1\alpha_1(\tau_1)$ the optimal solution will be point A, where in this point the movement m_2 will not have a ZQLP. When $w_2\alpha_2(\tau_0) > w_1\alpha_1(\tau_1)$ the optimal solution will be point B, and the movement m_1 will not have a ZQLP. Points A and B are equal to

$$(T_1, T_2)_A = \left(\frac{-T_{\min}\alpha_2(\tau_1)}{\alpha_2(\tau_0) - \alpha_2(\tau_1)}, \frac{T_{\min}\alpha_2(\tau_0)}{\alpha_2(\tau_0) - \alpha_2(\tau_1)} \right) \quad (49)$$
$$(T_1, T_2)_B = \left(\frac{-T_{\min}\alpha_1(\tau_1)}{\alpha_1(\tau_0) - \alpha_1(\tau_1)}, \frac{T_{\min}\alpha_1(\tau_0)}{\alpha_1(\tau_0) - \alpha_1(\tau_1)} \right) \quad (50)$$

If $w_2\alpha_2(\tau_0) = w_1\alpha_1(\tau_1)$ all points between A and B (i.e. the convex combination of $\alpha\,(T_1, T_2)_A + (1-\alpha)\,(T_1, T_2)_B$ where $0 \leq \alpha \leq 1$) are optimal solutions for the problem. The inner points will have two zero queue length periods, one zero queue period for each movement. In the case when the necessary condition (35) is satisfied with equality, i.e. $\frac{\alpha_1(\tau_1)}{-\alpha_1(\tau_0)} = \frac{-\alpha_2(\tau_1)}{\alpha_2(\tau_0)}$, the two points A and B are equal. In this case the optimal solution will not have any movement with ZQLP. Based on the above explanation the following proposition holds:

Proposition 4. There is always an optimal solution with at most one ZQLP.

Remark 1: The solution to (45)–(48) can also be found through direct substitution.

According to the Proposition 2, the queue lengths $q_1(\tau_1) = 0$ and $q_2(\tau_2) = 0$ in the optimal cyclic solutions. Hence the problem arises how to bring the queue lengths to their optimal values. N-stages control can be used to solve this problem.

5. N-STAGES CONTROL

In the N-stages control problem we consider a finite number of switchings in the optimization procedure. Now we specifically consider the following problem: for a given integer N and a given starting time t_0 we want to compute an optimal switching sequence consisting of N cycles[1].

For the simplified isolated controlled intersection we formulate the problem for the case when the criterion is a strictly increasing function of the queue lengths. We use the DMP problem (8) - (12) to solve the optimal problem for N-stages control when the criterion J is a strictly increasing function of the queue lengths. In this case, each max equation can be relaxed to two inequality equations, which leads to the "Relaxed" Discrete-event Max-Plus (R-DMP) problem

$$\min_{T_1(0),T_2(0),T_1(1),T_2(1),\cdots,T_1(N-1),T_2(N-1)} J \quad (51)$$

subject to

$$q_1(t_{2k+1}) \geq q_1(t_{2k}) + \alpha_1(t_{2k}) \cdot T_1(k) \quad (52)$$
$$q_1(t_{2k+1}) \geq 0 \quad (53)$$
$$q_2(t_{2k+2}) \geq q_2(t_{2k+1}) + \alpha_2(t_{2k+1}) \cdot T_2(k) \quad (54)$$
$$q_2(t_{2k+2}) \geq 0 \quad (55)$$
$$q_1(t_{2N}) = T_2^* \cdot \alpha_1(\tau_1) \quad (56)$$
$$q_2(t_{2N}) = 0 \quad (57)$$
$$\text{and } (10), (11)$$

for $k = 0, 1, 2, \ldots, N-1$.
Notice that we also impose the endpoint constraints (56) and (57). The endpoint queue lengths $q_1(t_{2N}), q_2(t_{2N})$ are equal to the optimal queue lengths cyclic solutions $T_2^* \cdot \alpha_1(\tau_1), 0$, respectively, where T_2^* is the optimal cyclic solution of T_2.

Proposition 5. If the criterion J is a strictly increasing function of the queue lengths, then any optimal solution of the R-DMP problem is also an optimal solution of the DMP problem.

Proof. See the proof of Proposition 3.3 in De Schutter (2002) which also applies here.

So the R-DMP problem can be solved by linear programming when the criterion J is a strictly increasing linear function of the queue lengths.

Hence, in order to bring initial queue lengths to the optimal cyclic queue lengths the N-stages control can be used. The endpoint queue lengths $q_1(t_{2N}), q_2(t_{2N})$ are equal to the optimal cyclic solutions and the condition (35) or (36) has to be satisfied.

[1] In the N-stages control, the queue length vector q is defined as: $\left(q_1(t_1), q_2(t_1), q_1(t_2), q_2(t_2), \cdots, q_1(t_{2N-1}), q_2(t_{2N-1}), q_1(t_{2N}), q_2(t_{2N})\right)^T$.

6. CONCLUSIONS

For the simplified isolated controlled intersection we can compute the optimal switching sequences for the steady-state problem with constant cycle length by solving a linear programming problem. It is shown that if the criterion J is a strictly increasing function and linear of the queue lengths the steady-state control problem can be solved analytically.

A necessary condition for the steady-state control with constant cycle length has been derived. The N-stages control problem has been formulated. It is shown that the N-stages control problem can be solved by linear programming if the criterion J is linear and strictly increasing. Furthermore, the N-stages control can be used to bring the queue lengths to optimum.

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