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Technical report 10-044

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If you want to cite this report, please use the following reference instead:
T. Luspay, B. Kulcsár, I. Varga, S.K. Zegeye, B. De Schutter, and M. Verhaegen, "On acceleration of traffic flow," *Proceedings of the 13th International IEEE Conference on Intelligent Transportation Systems (ITSC 2010)*, Madeira Island, Portugal, pp. 741–746, Sept. 2010.

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On Acceleration of Traffic Flow

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Abstract— **The paper contributes to the derivation and analysis of accelerations in freeway traffic flow models. First, a solution based on fluid dynamics and on pure mathematical manipulations is given to express accelerations. The continuous-time acceleration is then approximated by a discrete-time equivalent. By applying continues time microscopic and macroscopic traffic flow velocity definitions, spatial and material derivatives are used to describe the continuous-time and exact changes in the velocity vector field. A forward-difference Euler method is proposed to discretize the acceleration both in time and space. For applicability purposes the use of average quantities is proposed. The finite-difference approximation by space-mean speed is shown to be consistent, and its solution is convergent to the original continuous-time form. As an alternative, the acceleration obtained from a second-order macroscopic freeway model by means of physical interpretation [1] is analyzed and found to be an appropriate discrete approximation. Comparative remarks as well as future research questions conclude the paper.**

I. Introduction

Traffic models play an important role in both today's traffic research and in many traffic applications such as traffic flow prediction, monitoring, incident detection, and traffic control. Traffic models can be categorized along several dimensions, one of which is the distinction between microscopic, mesoscopic, and macroscopic traffic models [2], [3], [4]. In microscopic traffic models each individual vehicle is described separately. In a mesoscopic model individual vehicles with the same characteristics are grouped into a package. So, each vehicle within a package has the same origin and destination, the same route, the same driver characteristics, and so on. In that way the computation time needed for the simulation is reduced compared to microscopic models. In a macroscopic models the individual vehicles are aggregated and described as fluid flows, which are then characterized by average space-mean speeds, densities, and flows (or throughput rates). In that way the computation time needed for the simulation is reduced even further (at the cost of accuracy). In general, there is thus a trade-off between the accuracy of the model and the computation time required to simulate the model. In this context, many on-line model-based traffic prediction and traffic control approaches [5], [6] that require the model to be simulated repeatedly and at a high rate (much faster than real-time), use macroscopic traffic flow models.

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However, in some traffic prediction and control applications, we also need to have information about the acceleration of the vehicles. A prime example of this is the estimation of traffic emissions and fuel consumption. Many macroscopic emission models that are based on the average speed of vehicles only provide coarse estimates of emissions and fuel consumption of the traffic flow due to two reasons. First, the emissions and fuel consumptions are estimated based on trip-based average speeds of *all* vehicles (i.e., the models do not consider the emissions and fuel consumption of each vehicle at a time). Second, macroscopic models neglect the second-by-second dynamics of the traffic flow, while emissions and fuel consumption are sensitive to the acceleration of vehicles. This implies that for accurate estimation of emissions and fuel consumption of the traffic flow, dynamic emission and fuel consumption models are required. However, such models in general require the acceleration and speed of each vehicle on a second-by-second basis, while in general macroscopic traffic flow models do not provide the acceleration of the vehicles. In fact, macroscopic traffic flow models (such as, e.g., the METANET model used in [5], [6]) only provide macroscopic variables such as space-mean speed, density, and flow. Therefore, one needs to derive the acceleration of the vehicles from these macroscopic variables. However, this is not a trivial task. On the one hand macroscopic variables are average quantities that describe the characteristics of vehicles. Nevertheless, our goal is to appropriately estimate the acceleration of individual vehicles and try to derive it from macroscopic quantities if possible.

One way to simplify this task is to compute the average acceleration of a number of vehicles that have the same average speed. This can be done in two ways: using the discrete-time macroscopic traffic flow model, or using the continuous-time continuous-space traffic flow model. In the first approach, the average accelerations and the numbers of vehicles subject to these accelerations are computed from the discrete traffic flow model variables. Hence, the resulting accelerations are already discrete. In the second approach, first the accelerations are derived from the continuous-time continuous-space speed field. Next, the accelerations have to discretized.

The first approach has already been developed and applied in the estimation and prediction of emissions and fuel consumption in [1]. The second approach is far from being a trivial task and has not been yet investigated. In this paper we therefore first derive the continuous-time domain acceleration of macroscopic traffic flow models. Next, we show how to discretize and prove consistency and convergence of the discretized accelerations to the continuous values. Our paper was partially motivated by the derivation of higher-order macroscopic models in [7].

Therefore, the contribution of the paper is twofold. After a rigorous mathematical derivation of the exact and continuous time acceleration (continuous equation (CE)) its finite-difference approximate (FDA) is formulated, resulting in a discrete acceleration. Asymptotical mathematical properties such as consistency and convergence of the FDA are proposed to validate any type of approximation for traffic flow acceleration.

Finally, throughout the above conditions, not only the novel FDA of original CE will been validated but also the results obtained in [1].

The layout of the paper is as follows. After motivating the paper and the importance of the accelerations in the Introduction, the continuous-time and exact acceleration field is derived in Section II. Section III presents the rigorous discretization of the continuous-time acceleration both in time and in space and its links to the discrete-time equivalent forms to average macroscopic quantities. Physical interpretation of the acceleration is briefly analyzed in Section IV. Comparison and conclusion is given at the end of the paper in Section V.

II. Continuous time velocities and accelerations

A. Traffic variables

Let us consider (for sake of simplicity) a one-lane freeway, with cars moving in the same direction. Our aim is to determine the acceleration of the flow.

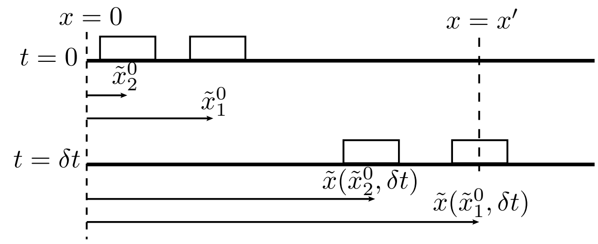

Fig. 1. Illustration of spatial and material variables

The following notations will be used along the paper, see also the illustration of Figure 1:

- t denotes the continuous time,
- x denotes the spatially fixed coordinate (spatial variable), independent from time t,
- $\tilde{x}^0$ denotes the initial position of a vehicle at time $t = 0$ (material variable), independent from time t,
- $\tilde{x}(\tilde{x}^0, t)$ denotes the position of a vehicle as a function of its initial position and the elapsed time t.

Some terminology of fluid dynamics are adopted to discuss traffic behavior. The basic notions are the Lagrangian and Eulerian coordinate frames [8]. The motion of a streaming fluid in the Lagrangian coordinate system is described by the motion of individual particles. It is known as microscopic traffic modeling in the transportation literature [3]. The Eulerian description investigates the flow at fixed spatial points. This approach corresponds to macroscopic traffic modeling. Every physical quantity (such as speed, acceleration or density) can be described in both Lagrangian and Eulerian coordinate frames, as detailed in the sequel.

B. Micro- and macroscopic description of the velocity

It is known that the speed of individual vehicles can be calculated as the first time derivative of their positions, therefore one can write:

$$\tilde{v}(\tilde{x}^0, t) = \left.\frac{\partial \tilde{x}(\tilde{x}^0, t)}{\partial t}\right|_{\tilde{x}^0}. \tag{1}$$

Note, $\tilde{x}^0$ is a variable in the Lagrangian coordinate system (see Figure 1), hence fixed $\tilde{x}^0$ indicates the tracking of one vehicle's motion [1]. Derivation with a fixed material variable is called material (or substantial) derivative and denoted as follows [9]:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \left.\frac{\partial}{\partial t}\right|_{\tilde{x}^0}. \tag{2}$$

The material derivative can be considered as the rate of change at which the property varies when measured by an observer traveling together with a group of particles.

At the same time, the velocity in the Eulerian coordinate frame should represent the velocity of the streaming continuum at every fixed spatial point. For this propose a velocity field ($v(x,t)$) is used, which returns the measured speed at the fixed spatial position x and time t. If the fixed spatial coordinate x coincides with the position of a vehicle $\tilde{x}(\tilde{x}^0, t)$, then:

$$v(x,t)\big|_{x=\tilde{x}(\tilde{x}^0,t)} = \tilde{v}(\tilde{x}^0, t). \tag{3}$$

This equation relates the velocity field to the speed of moving vehicles and gives the connection between Lagrangian (microscopic) and Eulerian (macroscopic) modeling methodologies.

C. Continuous time accelerations

Let us investigate the motion in Eulerian and Lagrangian coordinate systems. We analyze the rate of changes of the velocity.

The acceleration of a single vehicle can be calculated in Lagrangian coordinates system by using the material derivative, eq. (2) of the vehicle's speed:

$$\frac{\mathrm{D}\tilde{v}(\tilde{x}^0, t)}{\mathrm{D}t} = \left.\frac{\partial \tilde{v}(\tilde{x}^0, t)}{\partial t}\right|_{\tilde{x}^0}. \tag{4}$$

The material derivative of the individual vehicle speed is called microscopic acceleration.

[1] $\tilde{x}_i^0$ denotes the initial position of i-th vehicle and $\left.\frac{\partial \tilde{x}(\tilde{x}^0,t)}{\partial t}\right|_{\tilde{x}_i^0}$ denotes the speed of vehicle initially positioned at $\tilde{x}_i^0$.

To analyze the acceleration in the Eulerian coordinates, we introduce the local (or spatial) time derivative as the time rate of change at a given fixed point in the space:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \left.\frac{\partial}{\partial t}\right|_x . \tag{5}$$

Consequently, the macroscopic acceleration field is defined as the local time derivative of the velocity field given by:

$$\frac{\mathrm{d}v(x,t)}{\mathrm{d}t} = \left.\frac{\partial v(x,t)}{\partial t}\right|_x . \tag{6}$$

Due to eq. (3) one can write:

$$v(x,t)|_{x=\tilde{x}(\tilde{x}^0,t)} = v(\tilde{x}(\tilde{x}^0,t),t), \tag{7}$$

i.e., the velocity field depends on material variables, hence its material derivative can be calculated by using the chain rule:

$$\left.\frac{\mathrm{D}v(\tilde{x}(\tilde{x}^0,t),t)}{\mathrm{D}t}\right|_{\tilde{x}^0} = \left.\frac{\partial v(\tilde{x}(\tilde{x}^0,t),t)}{\partial t}\right|_x + \\ + \left.\frac{\partial v(\tilde{x}(\tilde{x}^0,t),t)}{\partial x}\right|_t \left.\frac{\partial \tilde{x}(\tilde{x}^0,t)}{\partial t}\right|_{\tilde{x}^0=\tilde{x}^{-1}(x,t)} .$$

By using the definitions of material derivative eq. (2) and local derivative eq. (5) one gets the following form:

$$\frac{\mathrm{D}v(x,t)}{\mathrm{D}t} = \frac{\partial v(x,t)}{\partial t} + \frac{\partial v(x,t)}{\partial x} v(x,t). \tag{8}$$

Note, the derivative in eq. (8) is not equal to the acceleration of the fluid at a fixed point in space, neither the acceleration of the particle. It is the rate of change in the velocity field observed by the particle as it moves in space. Eq. (8) states that the variation in the velocity field is a sum of two effects: the velocity change in time at a given spatial point and the change in velocity due to the movement in space, also known as advection acceleration [9].

Remark 1: Using the same derivation, the material derivative of the density field $\rho(x,t)$ would take the following form:

$$\frac{\mathrm{D}\rho(x,t)}{\mathrm{D}t} = \frac{\partial \rho(x,t)}{\partial t} + \frac{\partial \rho(x,t)}{\partial x} v(x,t), \tag{9}$$

which is the well-known vehicle conservation law [2].

The continuous-time and space representation of the acceleration in eq. (8) has only a theoretical importance. Due to its continuous dependence eq. (8) is not suitable for real applications with limited temporal and spatial measurements. To overcome this difficulty two major issues are addressed in the following sequels. Firstly, the Euler discretization of eq. (8) is introduced. Secondly, the use of space-mean speeds is suggested and validated.

Remark 2: Consequently, the discretized acceleration has to be a "good" approximation of the continuous equation, attention has to be paid on the selection of the step size. Furthermore, difference approximation using average speed terms has to fulfill several conditions to appropriately characterize the acceleration vector field.

III. Discretization of acceleration

The section provides the Euler discretization of eq. (8) in space and time respectively [10].

A. The discrete form of acceleration

Firstly, let us investigate the motion of a single particle in a time span δt. Obviously, its position will be changed by δx. The velocity can be described with the velocity field by: $v(x+\delta x, t+\delta t)$. In case of small δt, one can use a first-order Taylor approximation to express this velocity:

$$v(x+\delta x, t+\delta t) \approx v(x,t) + \frac{\partial v(x,t)}{\partial t}\delta t + \frac{\partial v(x,t)}{\partial x}\delta x. \tag{10}$$

Furthermore the variation in the velocity during this small time can be expressed as:

$$\begin{aligned} \delta v &= v(x+\delta x, t+\delta t) - v(x,t) && (11)\\ &\approx \frac{\partial v(x,t)}{\partial t}\delta t + \frac{\partial v(x,t)}{\partial x}\delta x, && (12)\end{aligned}$$

and accordingly the rate of change as:

$$\begin{aligned} \frac{\delta v}{\delta t} &= v(x+\delta x, t+\delta t) - v(x,t) = \\ &\approx \frac{\partial v(x,t)}{\partial t} + \frac{\partial v(x,t)}{\partial x}\frac{\delta x}{\delta t}. && (13)\end{aligned}$$

If one takes the limit $\delta t \to 0$ then we return to the initial condition:

$$\lim_{\delta t \to 0} \frac{\delta x}{\delta t} = v(x,t), \tag{14}$$

and the following equivalence is straightforward:

$$\frac{\mathrm{D}v(x,t)}{\mathrm{D}t} = \lim_{\delta t \to 0} \frac{v(x+\delta x, t+\delta t) - v(x,t)}{\delta t}. \tag{15}$$

At the same time, one can discretize the right-hand side of eq. (8) using first order forward-difference Euler method with finite time step T and space step Δ:

$$\begin{aligned} \frac{\partial v(x,t)}{\partial t} &= \lim_{T \to 0} \frac{v(x,t+T) - v(x,t)}{T}, && (16)\\ \frac{\partial v(x,t)}{\partial x} &= \lim_{\Delta \to 0} \frac{v(x+\Delta,t) - v(x,t)}{\Delta}. && (17)\end{aligned}$$

It is important to denote that, there are two different finite temporal (δt and T) and spatial steps (δx and Δ) introduced. T and Δ are chosen for the Euler discretization method as the discrete step sizes, while δt can be chosen optionally, while δx represents the distance traveled by the particle with speed $v(x,t)$ during a δt time. To connect these notations one can chose $\delta t = T$ and as a consequence δx will represent the distance traveled by a moving particle during a single time-step T. In case of several moving particles one has to choose different δx values for different vehicles according to their actual speed. To overcome this difficulty one could select a single and uniform value besides. This uniform bound should be chosen carefully, taking the fact, that no vehicles could be created or disappeared in cells (i.e., all vehicles must cross all segments), into consideration. This condition can be fulfilled by the following selection:

$$\max\{\delta x\} = \max\{v(x,t)\}\, T \le \Delta. \tag{18}$$

Note, that a similar reasoning is used in the traffic literature [11]. Moreover eq. (18) is in accordance with the Courant-Friedrichs-Lewy condition known from the theory of partial differential equations [12], [13].

B. Approximation of velocity field

The discrete form necessitates the knowledge of the velocity field at given spatial and temporal coordinates, which is usually not available in real traffic applications due to the limited number of detector locations. In order to calculate the acceleration with limited measurements the velocity field is approximated by space-mean speed terms.

The average speed of vehicles in a given space segment (with Δ length) is defined as follows:

$$v_i(t) = \frac{\int_{x_i}^{x_i+\Delta} v(x,t)\mathrm{d}x}{\Delta}, \tag{19}$$

where subscript indices the ith discrete cell (segment) on the freeway with origin x_i. Consequently, in case of the next segment the space-mean speed:

$$v_{i+1}(t) = \frac{\int_{x_{i+1}}^{x_{i+1}+\Delta} v(x,t)\mathrm{d}x}{\Delta} = \frac{\int_{x_i+\Delta}^{x_i+2\Delta} v(x,t)\mathrm{d}x}{\Delta}. \tag{20}$$

By using averaged speed values the finite-difference approximation of eq. (15) reads as follows:

$$\frac{\mathrm{D}v(x,t)}{\mathrm{D}t} \approx \frac{v_i(k+1)-v_i(k)}{T} + \frac{v_{i+1}(k)-v_i(k)}{\Delta}v_i(k). \tag{21}$$

Indeed, the above approximation might be very conservative but is required to connect the discrete approximations to the mean speed components. Consequently, it allows us to use space mean speed in the sequel. Before proceeding, the following section shows the analytical properties of the above FDA with mean speed terms.

C. Numerical analysis of the finite difference approximation

This section gives the validation of the discretization in eq. (21) using the notions of consistency and convergence [12], [14]:

1) Consistency: the finite difference approximation (FDA) converges to the original continuous equation (CE) as the discretization steps approach 0:

$$\mathrm{CE}(x,t) = \lim_{\Delta,T\to 0} \mathrm{FDA}(i,k)$$

2) Convergence: the solution of the finite difference approximation (FDA) converges to the solution of the original continuous equation (CE) as the discretization steps approach 0:

$$\int_{t_1}^{t_2} \mathrm{CE}(x,t)\,\mathrm{dt} = \lim_{\Delta,T\to 0}\int_{t_1}^{t_2} \mathrm{FDA}(i,k)\,\mathrm{dt}. \tag{22}$$

Both properties describe the validity of a finite-difference approximation. Theorem 1 proves the validity of the finite difference approximation:

Theorem 1: The finite-difference approximations (FDAs) in eq. (21) of the acceleration eq. (8) are consistent and convergent.

In the proof of the Theorem 1, the following Lemmas are used:

Lemma 1: Lebesque's dominated convergence theorem [15]: Let $\{f_n\}$ denote a sequence of real-valued measurable functions on a measure space described by the σ-algebra Σ over the set S and measure μ: (S,Σ,μ). Assume that the sequence converges point-wise to a function f and is dominated by some integrable function g. Then the limiting function f is integrable and:

$$\lim_{n\to\infty}\int_S f_n d\mu = \int_S f d\mu.$$

Lemma 2: L'Hospital's rule [16]: Let f and g be two real-valued functions defined and differentiable on $E : \{x|\,|x_0-\delta|<\epsilon\}$, $\epsilon,\delta>0$, moreover: $g'(x)\neq 0$ if $x\in E$. If:

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0,$$

and the:

$$L' = \lim_{x\to x_0}\frac{f'(x)}{g'(x)},$$

limit exists, then:

$$L = \lim_{x\to x_0}\frac{f(x)}{g(x)},$$

also exists and $L = L'$.

Lemma 3: If the real valued scalar function f is continuous on the closed interval $[a,b]$, then the function:

$$I:\ I(y) = \int_a^y f(z)dz,$$

is defined on the closed interval $[a,b]$. Moreover; I is differentiable on the open interval (a,b), and:

$$I'(y) = f(y),\quad \text{if } y\in(a,b),$$

i.e., the function I is a primitive function of f on (a,b) [17].

Now we are at the point of proving one of the most important contribution of the paper.

Proof of Theorem 1: Now we are at the position of giving the proof of the Theorem 1. Assumptions of Lemma 1 are fulfilled for the sequence of finite difference approximations. As a consequence the limit process and the integration are commutable in eq. (22) and hence the finite difference approximations are convergent if they are consistent.

The proof of consistency requires the following equality:

$$\lim_{T,\Delta\to 0}\mathrm{FDA}(i,k) = \mathrm{CE}(x,t). \tag{23}$$

From the definition of space-mean speed eq. (19), it follows that $v_i(k)$ is the average speed of vehicles in segment i at time step k or equivalently the sample of $v_i(t)$ at time-step $t=kT$, i.e.: the order of averaging and time sampling can be interchanged. This remark leads us to:

$$\lim_{\Delta\to 0}\mathrm{FDA}(i,t) = \mathrm{CE}(x,t). \tag{24}$$

which can be proved by the verification of:

$$\lim_{\Delta \to 0} v_i(t) = v(x_i, t). \tag{25}$$

Let us investigate the limit of the space-mean speed as the segment length tends to zero:

$$\lim_{\Delta \to 0} v_i(t) = \lim_{\Delta \to 0} \frac{\int_{x_i}^{x_i+\Delta} v(x,t)\,\mathrm{dx}}{\Delta}. \tag{26}$$

As $\Delta \to 0$ the upper integration limit tends to the lower one and hence the integral tends to zero, i.e.: the limit in eq. (26) is indeterminate (zero over zero). Lemma 2 can be applied to calculate the limit, by forming the derivative of the numerator and denominator in (26). Since the integral is the function of its upper limit, Lemma 3 is used to calculate its derivative:

$$\frac{\mathrm{d} \int_{x_i}^{x_i+\Delta} v(x,t)\,\mathrm{dx}}{\mathrm{d}\Delta} = v(x_i + \Delta, t), \tag{27}$$

$$\frac{\mathrm{d}\Delta}{\mathrm{d}\Delta} = 1, \tag{28}$$

i.e.:

$$\lim_{\Delta \to 0} v_i(t) = v(x_i, t), \tag{29}$$

which completes the proof of Theorem 1. ■

Accordingly, we have shown that the most important properties of the FDA by space mean speed satisfy the analytical condition and we can suggest to apply them as valid approximators. As mention in the Introduction, acceleration variables can also be obtained directly out from the discrete-time second-order macroscopic freeway models by means of physical interpretations.

IV. Physical interpretation

The sequel gives the physical interpretation of the derived discrete acceleration eq. (21) using the previously published notions of "temporal" and "spatiotemporal" accelerations [1].

Consider a segment of a link with length Δ as in Fig. 2, illustrating the traffic flow at time step k and $k+1$. At the time step k the number of vehicles in segment i in λ lanes is equal to $\lambda \cdot \Delta \cdot \rho_i(k)$ and the number of vehicles going from segment i to segment $i+1$ in the time period $[Tk, T(k+1)]$ is $Tq_i(k)$. Therefore, the number of vehicles that stayed in segment i in the time period $[kT,\ (k+1)T]$ is equal to $\lambda \cdot \Delta \cdot \rho_i(k) - Tq_i(k)$. From time step k to $k+1$ the acceleration is not only due to the change in speed of the vehicles within the segment i, but also there is an acceleration for the vehicles flowing from segment i to segment $i+1$.

The acceleration of the vehicles that stay within a segment in the time period $[Tk, T(k+1)]$ is called the "temporal" acceleration, and it is given by:

$$a_i^{\mathrm{tmp}}(k) = \frac{v_i(k+1) - v_i(k)}{T}. \tag{30}$$

The corresponding number of vehicles subject to this acceleration is:

$$n_i^{\mathrm{tmp}}(k) = \lambda \cdot \Delta \cdot \rho_i(k) - Tq_i(k). \tag{31}$$

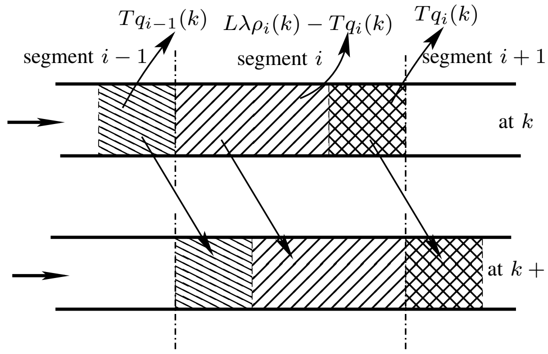

Fig. 2. Illustration of traffic flow in METANET

Moreover, the acceleration of the vehicles moving from segment i to segment $i+1$ in the time period $[Tk, T(k+1)]$ is named as "spatiotemporal" and it is given by:

$$a_i^{\mathrm{spatmp}}(k) = \frac{v_{i+1}(k+1) - v_i(k)}{T} \tag{32}$$

and the number of vehicles that are subject to this acceleration is:

$$n_i^{\mathrm{spatmp}}(k) = Tq_i(k). \tag{33}$$

It could be deduced from the above concept that the "temporal" acceleration equals with the macroscopic acceleration field in the discrete framework, i.e., the discretized local time derivative of the velocity field.

At the same time, the "spatiotemporal" acceleration turns out to be equal with the material derivative of the velocity field in the discrete representation (left-hand side of eq. (21)).

Moreover eq. (21) also gives the connection between "temporal" and "spatiotemporal" accelerations. "Temporal" acceleration can be considered as the special case of "spatiotemporal" acceleration, when the transition from segment i to $i+1$ is zero, hence the "spatial" term is zero in the right-hand side of eq. (21).

Finally one can compute the average acceleration of vehicles that are in segment i at time step k. Since this acceleration is connected with the number of vehicles in segment i, it is denoted by $a_i^n(k)$ and defined as:

$$a_i^n(k) = \frac{n_i^{\mathrm{tmp}}(k) a_i^{\mathrm{tmp}}(k) + n_i^{\mathrm{spatmp}}(k) a_i^{\mathrm{spatmp}}(k)}{n_i^{\mathrm{tmp}}(k) + n_i^{\mathrm{spatmp}}(k)},$$

using the definitions of $n_i^{\mathrm{tmp}}, n_i^{\mathrm{spatmp}}$ and $a_i^{\mathrm{tmp}}, a_i^{\mathrm{spatmp}}$:

$$\begin{aligned} a_i^n(k) &= \frac{(\lambda \cdot \Delta \cdot \rho_i(k) - Tq_i(k)) \frac{v_i(k+1) - v_i(k)}{T}}{\lambda \cdot \Delta \cdot \rho_i(k) - Tq_i(k) + Tq_i(k)} + \\ &+ \frac{Tq_i(k) \dfrac{v_{i+1}(k+1) - v_i(k)}{T}}{\lambda \cdot \Delta \cdot \rho_i(k) - Tq_i(k) + Tq_i(k)}. \end{aligned}$$

Using the continuity relation $q_i(k) = \lambda \rho_i(k) v_i(k)$ and rearranging terms:

$$\begin{aligned} a_i^n(k) &= \left(1 - \frac{T}{\Delta} v_i(k)\right) \frac{v_i(k+1) - v_i(k)}{T} + \\ &+ \frac{T}{\Delta} v_i(k) \frac{v_{i+1}(k+1) - v_i(k)}{T} = \\ &= \frac{v_i(k+1) - v_i(k)}{T} + v_i(k) \frac{v_{i+1}(k+1) - v_i(k+1)}{\Delta}. \end{aligned} \tag{34}$$

The FDA in eq. (21) and the interpretation in eq. (34) are not identical but both of them are valid, consistent, and convergent approximations of the original CE.

V. Conclusion

This paper has been motivated by the derivation and analysis of accelerations used in macroscopic freeway environments. Two different approaches have been investigated and found to be identical in terms of consistent and convergent approximations of the continuous-time freeway acceleration.

One of the major contributions of the paper is to present (approximative) acceleration terms expressed purely in terms of macroscopic and mean quantities. Consequently, by using average (measurable) speed components not only the continuous-time velocity vector field can be approximated but also its derivative.

Acceleration has a more and more larger impact on the performance of traffic control systems since reliable emission models depend on it. Average emission minimization as well as more complex performance requirements can be formulated with the help of accelerations, computable based on mean-speed components.

Comparative analysis of the approximation errors is require further research in this framework.

VI. Acknowledgment

The authors gratefully acknowledge the support by the Hungarian National Scientific Research Fund (OTKA CNK78168).

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