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Optimal Steady-State Control for Isolated Traffic Intersections

Jack Haddad, Bart De Schutter, David Mahalel, Ilya Ioslovich, and Per-Olof Gutman

***Abstract*—A simplified isolated controlled vehicular traffic intersection with two movements is considered. A discrete-event max-plus model is proposed to formulate an optimization problem for the green-red switching sequence. In the case when the criterion is a strictly increasing, linear function of the queue lengths, the problem becomes a linear programming (LP) problem. Also, in this case, the steady-state control problem can be solved analytically. A sufficient and necessary condition for steady-state control is derived, and the structure of optimal steady-state traffic control is revealed. Our condition is the same as the necessary condition in [5] for both queue lengths to be non-increasing at an isolated intersection.**

***Index Terms*—Optimal control, steady-state, isolated traffic intersections**

I. Introduction

Webster [17] is probably the first researcher that investigated the undersaturated conditions for isolated traffic intersections, i.e. total flow entering the intersection can pass through during the cycle length. Webster derived an expression for the average delay per vehicle for a given movement, based on theoretical analysis and empirical results. The minimum and optimal fixed green split and cycle length formulas were derived by minimizing the total delay for all the approaches in the intersection. Other papers, e.g. [12], [15], aimed to improve the delay estimation formula for *over-saturated* conditions, while others proposed different models, methods, and strategies for controlling oversaturated isolated intersections [1], [2], [4]–[6], [9], [10], [13], [16] where the aim was to minimize delays or to maximize the intersection capacity.

With available sensors, it is easier to measure queue lengths than to estimate delays. Therefore, in this paper the criteria are functions of the queue lengths: inspired by Gazis *et al.* [5] the time integral over the sum of all queue lengths at the intersection, or the so-called "total delay", is minimized.

Under the assumptions of a continuous differential equation model for isolated intersection with only two movements, constant total throughput, constant cycle length, constant saturation flows, time varying arrival rates, and oversaturation (signifying that the initial queue lengths are larger than zero), Gazis *et al.* [5], found a necessary condition for not increasing any queue length. In [6] the same problem as in [5] is treated under the assumptions that the arrival rates are constant, and that, in addition, there are imposed maximum and minimum green duration constraints that enable both the queue lengths to be reduced to zero. As a consequence, the necessary and sufficient condition in [6] for decreasing both queue lengths to zero is different from the necessary condition in [5].

Under the assumption of a discrete-event model with constant arrival and departure rates, in this paper we give a necessary and sufficient condition for the existence of a constant cycle length steady-state solution. The green duration constraint is not imposed *a priori*, and steady-state queue lengths are not necessarily equal to zero at any time of the cycle which however turns out to be the case for the *optimal* steady-state solution. Our condition turns out to be the same as the necessary condition (19) in [5].

In the case when our optimization criterion is a strictly increasing, linear function of the queue lengths, the undersaturated problem can be solved by linear programming, and the optimal solution has a simple form that can also be found analytically. It is also shown how to bring initial oversaturated non-optimal queue lengths to optimum, thus enabling an N-stages control solution for the transient phase.

II. Problem Definition

A typical simplified isolated intersection is shown in Fig. 1. There are two movements (m_1 and m_2), defined as the sets of vehicles having reached but not passed the intersection. Each movement is governed by a traffic signal that each can be either green or red. Since the two movements cannot occupy the intersection area simultaneously, the traffic signals will be opposite, i.e. when movement m_1 has green light, movement m_2 sees red light, and vice versa. Each movement will encounter intertwined green and red periods. A cycle is defined as a pair of one green and one red period, whose durations may be time-varying.

The queue length $q_i(t)$ [veh] for movement $\mathrm{m_i}$ at time t, is defined as the number of vehicles belonging to $\mathrm{m_i}$ which is behind the stop line, i.e. the queue does not include the vehicles that are inside the intersection or have passed it. Let $a_i(t)$ [veh/s], and $d_i(t)$ [veh/s] be the arrival and departure rates for queue i, respectively. The following assumptions are made:

- A1: The arrival rates are known, non-negative constants for each green or red period, and the departure rates are known, non-negative constants within each green or red period.
- A2: $d_i(t) > a_i(t)$, and $d_i(t) = 0$, $i = 1, 2$ for green and red periods, respectively.

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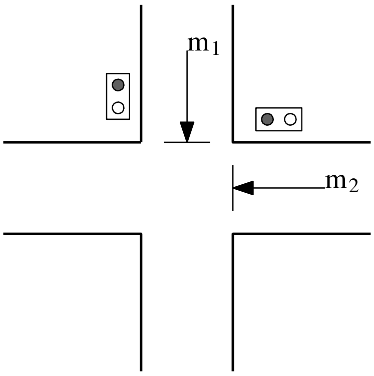

Fig. 1. Simplified isolated controlled intersection with two movements, each governed by a red-green traffic signal.

- A3: The queue lengths [veh] are approximated by non-negative real numbers.

For the isolated controlled intersection with constant traffic arrival and departure rates and constant cycle length T [s], we determine the steady-state traffic signal control solution that minimizes a given queue length dependent criterion. We also formulate a necessary and sufficient condition for steady-state control.

III. Discrete-Event Models for Isolated Controlled Intersections

A variety of models (cf. [14]) are based on the store-and-forward approach of modeling traffic networks that was first suggested in [4], [5]. This approach makes it possible to simplify the mathematical description of the traffic flow process without the use of switching variables. In this paper we consider the isolated controlled intersection as a switching system, as was done in [2], [3], [7], [8], and the optimization of the traffic signal switching sequences will be performed with a discrete-event max-plus model.

A. Basic model

Let k be the cycle index. By definition, in a cycle, each movement (m_1 or m_2) has only one green period. We want to determine two decision variables: the cycle length, $T(k)$ [s], and $g(k) \in [0,1]$ defined as the fraction of the green period time [s] of $T(k)$ for movement m_1.

The evolution of the system begins at time t_0. This implies that the state of the queue length i at time t is given by

$$q_i(t) = q_i(t_0) + \int_{t_0}^{t} \big(a_i(t) - d_i(t)\big) dt \tag{1}$$

There are two switching times for cycle k: t_{2k+1} and t_{2k+2} (see Fig. 2). Without loss of generality, let the green light for movement m_1 start at t_{2k}, which coincides with the start of cycle k. Hence, t_{2k+1} is the end of the green light for movement m_1, and the start of the green light for movement m_2, while t_{2k+2} is the end of the green light for movement m_2, and the start of the green light for movement m_1 in the next cycle. Note that $T(k) = t_{2k+2} - t_{2k}$. From Assumption A1 it follows that the arrival and departure rates are known and constant over each light period: for $t_{2k} \leq t < t_{2k+1}$ it holds that $a_i(t_{2k}) = a_i(t)$ and $d_i(t_{2k}) = d_i(t)$, and for $t_{2k+1} \leq t < t_{2k+2}$ it holds that $a_i(t_{2k+1}) = a_i(t)$ and $d_i(t_{2k+1}) = d_i(t)$, $i = 1, 2$. The relations between the time variables are as follows

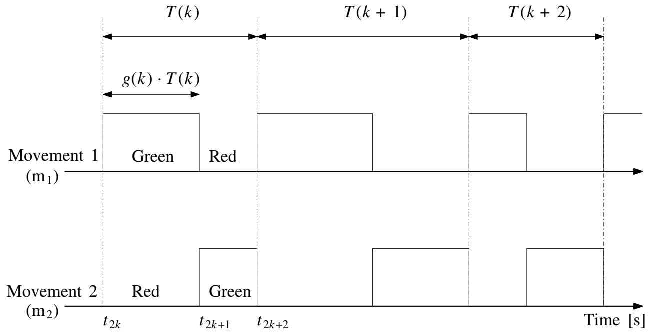

Fig. 2. Traffic signal switching sequences for movements m_1 and m_2

$$t_{2k+1} = t_{2k} + g(k) \cdot T(k) \tag{2}$$
$$t_{2k+2} = t_{2k} + T(k) \tag{3}$$

B. Formulation of an optimal discrete-event max-plus problem

The value of the queue length for movement m_1 in cycle k at the switching time instant t_{2k+1} is given by

$$q_1(t_{2k+1}) = \max\big(q_1(t_{2k}) + \big(a_1(t_{2k}) - d_1(t_{2k})\big) \cdot g(k) \cdot T(k), 0\big) \tag{4}$$

and at the switching time instant t_{2k+2}

$$q_1(t_{2k+2}) = q_1(t_{2k+1}) + a_1(t_{2k+1}) \cdot (1 - g(k)) \cdot T(k) \tag{5}$$

Recall that the signal light for movement m_2 is opposite to that of m_1; therefore the value of the queue lengths for movement m_2 in cycle k are given by

$$q_2(t_{2k+1}) = q_2(t_{2k}) + a_2(t_{2k}) \cdot g(k) \cdot T(k) \tag{6}$$

$$q_2(t_{2k+2}) = \max\big(q_2(t_{2k+1}) + \big(a_2(t_{2k+1}) - d_2(t_{2k+1})\big) \cdot (1 - g(k)) \cdot T(k), 0\big) \tag{7}$$

We now consider the following problem: for a given number of cycles N (not necessarily of equal lengths) and starting time t_0, compute an optimal switching time sequence $t_1, t_2, \ldots, t_{2N}$ that minimizes a given performance criterion J. There are a variety of criteria that can be chosen, e.g. average queue length, maximal queue length, and delay over all queues [2]. Two new variables are now defined, $T_1(k)$ [s] and $T_2(k)$ [s], where $T_1(k) = g(k) \cdot T(k)$ and $T_2(k) = (1 - g(k)) \cdot T(k)$. Substituting these variables into (4)–(7) leads to the following Discrete-event Max-Plus (DMP) problem:

$$\min_{T_1(0), T_2(0), T_1(1), T_2(1), \cdots, T_1(N-1), T_2(N-1)} J \tag{8}$$

subject to

$$q_1(t_{2k+1}) = \max\big(q_1(t_{2k}) + \big(a_1(t_{2k}) - d_1(t_{2k})\big) \cdot T_1(k), 0\big) \tag{9}$$
$$q_1(t_{2k+2}) = q_1(t_{2k+1}) + a_1(t_{2k+1}) \cdot T_2(k) \tag{10}$$
$$q_2(t_{2k+1}) = q_2(t_{2k}) + a_2(t_{2k}) \cdot T_1(k) \tag{11}$$
$$q_2(t_{2k+2}) = \max\big(q_2(t_{2k+1}) + \big(a_2(t_{2k+1}) - d_2(t_{2k+1})\big) \cdot T_2(k), 0\big) \tag{12}$$

for $k = 0, 1, 2, \ldots, N-1$. Hence, the number of variables to be determined is $2N$.

IV. Steady-state Control with Constant Cycle Length

Consider the steady-state control problem with constant cycle length. It is assumed that all cycles and their flow rates are identical, and hence only one cycle needs to be studied. The decision variables are T_1 [s] and T_2 [s], and the cycle duration is $T = T_1 + T_2$. Let the start time of the steady-state cycle be 0. The switching times are τ_1, and τ_2, respectively, whereby $T = \tau_2$, $T_1 = \tau_1$, and $T_2 = \tau_2 - \tau_1$. In steady-state it is postulated that the queue length for movement i at the start of the cycle will be equal to the queue length at the start of the next cycle:

$$q_1(0) = q_1(\tau_2) \tag{13}$$

$$q_2(0) = q_2(\tau_2) \tag{14}$$

Let the steady-state queue length vector q be defined as $[q_1(\tau_1), q_1(\tau_2), q_2(\tau_1), q_2(\tau_2)]^{\mathrm{T}}$. The criterion function J is said to be strictly increasing, if, for all queue length vectors $\hat{q}$, $\tilde{q}$ with $\hat{q} \leq \tilde{q}$ (elementwise) and $\hat{q}_i < \tilde{q}_i$ for at least one index i, we have $J(\hat{q}) < J(\tilde{q})$.

In the following, we consider the case when the criterion J is a strictly increasing function of the queue lengths, such as the average queue length, a positively weighted sum of queue lengths, or the average travel time. We show that for such a criterion, the optimal steady-state constant cycle length switching sequence problem and its necessary and sufficient condition can be formulated using a discrete-event max-plus model, and solved analytically for a strictly increasing and *linear* criterion.

A. Formulation of an optimal cyclic discrete-event max-plus problem

The formulation is based on the DMP problem (8)–(12). The cyclic queue lengths equations (13)–(14) are added to the DMP problem and then optimized over only one cycle, i.e. $N = 1$ and $k = 0$. The number of decision variables becomes two: $T_1(0)$ and $T_2(0)$, for simplicity written as T_1 and T_2, respectively. We also assume that a lower bound, $T_{\min} > 0$, for the cycle duration is a priori given, i.e. $T = T_1 + T_2 \geq T_{\min}$. The Cyclic Discrete-event Max-Plus (CDMP) problem is then defined as follows:

$$\min_{T_1, T_2} J \tag{15}$$

subject to

$$q_1(\tau_1) = \max\big(q_1(0) + (a_1(0) - d_1(0)) \cdot T_1, 0\big) \tag{16}$$

$$q_1(\tau_2) = q_1(\tau_1) + a_1(\tau_1) \cdot T_2 \tag{17}$$

$$q_2(\tau_1) = q_2(0) + a_2(0) \cdot T_1 \tag{18}$$

$$q_2(\tau_2) = \max\big(q_2(\tau_1) + (a_2(\tau_1) - d_2(\tau_1)) \cdot T_2, 0\big) \tag{19}$$

$$T_1 + T_2 \geq T_{\min} \tag{20}$$

$$\text{and } (13), (14)$$

Note that for scalars $a, b, c \in \mathbb{R}$ we have that $a = \max(b, c)$ implies $a \geq b$ and $a \geq c$. In a similar way the CDMP problem can be rewritten in such a way that the $\max$ equations are "relaxed" to linear inequality equations. But first, the cyclic queue lengths equations (13) and (14) are substituted into (16) and (18), respectively:

$$q_1(\tau_1) = \max\big(q_1(\tau_2) + (a_1(0) - d_1(0)) \cdot T_1, 0\big) \tag{21}$$

$$q_2(\tau_1) = q_2(\tau_2) + a_2(0) \cdot T_1 \tag{22}$$

The $\max$ equations (21) and (19) can then be relaxed into linear inequality equations as follows:

$$q_1(\tau_1) \geq q_1(\tau_2) + (a_1(0) - d_1(0)) \cdot T_1 \tag{23}$$

$$q_1(\tau_1) \geq 0 \tag{24}$$

$$q_2(\tau_2) \geq q_2(\tau_1) + (a_2(\tau_1) - d_2(\tau_1)) \cdot T_2 \tag{25}$$

$$q_2(\tau_2) \geq 0 \tag{26}$$

This leads to the "Relaxed" Cyclic Discrete-event Max-Plus (R-CDMP) problem:

$$\min_{T_1, T_2} J \tag{27}$$

subject to

$$(17), (20), (22), (23), (24), (25), (26)$$

Let $\tilde{q} = [\tilde{q}_1(\tau_1), \tilde{q}_1(\tau_2), \tilde{q}_2(\tau_1), \tilde{q}_2(\tau_2)]^{\mathrm{T}}$ and $\tilde{T} = [\tilde{T}_1, \tilde{T}_2]^{\mathrm{T}}$ be an optimal solution of the R-CDMP problem.

Proposition 1: If the criterion J is a strictly increasing function of the queue lengths, then any optimal solution of the R-CDMP problem has the following properties:

1) it holds that $\tilde{q}_1(\tau_1) = 0$,
2) it is also an optimal solution of the CDMP problem.

Proof: Let us denote the feasible set of the R-CDMP problem as R and the feasible set of the CDMP problem as C. Clearly it holds that $C \subset R$. Let us denote the optimal value of J over C as J_C and an optimal solution of the CDMP problem as $q_C = \operatorname*{argmin}_{q \in C} J$. Correspondingly we shall denote the optimal value of J over R as J_R, and an optimal solution of the R-CDMP problem as $q_R = \operatorname*{argmin}_{q \in R} J$. Clearly $J_R \leq J_C$. From Krotov's lemma [11] it follows that if $q_R \in C$, then $J_C = J_R$.

Statement 2 follows from the fact that an optimal solution of the R-CDMP problem with $\tilde{q}_1(\tau_1) = 0$ belongs to C, because in this case (16) and (17) are satisfied, whereby we note that (16) and (17) are the only equations where $\tilde{q}_1(\tau_1)$ is present. Thus, what remains is to prove statement 1. The analogous proposition that $\tilde{q}_2(\tau_2) = 0$ has a similar proof.

Statement 1 is proven by contradiction, where the idea of the proof is that the steady-state queues can be shifted down until $\tilde{q}_1(\tau_1)$ equals zero. Assume that $\tilde{q}_2(\tau_1)$, $\tilde{q}_2(\tau_2)$ satisfy (18) and (19), and suppose that for $\tilde{q}_1(\tau_1)$ (21) is not satisfied,

$$\tilde{q}_1(\tau_1) > \max(\tilde{q}_1(\tau_2) + (a_1(0) - d_1(0)) \cdot \tilde{T}_1, 0) \tag{28}$$

or equivalently

$$\tilde{q}_1(\tau_1) > \tilde{q}_1(\tau_2) + (a_1(0) - d_1(0)) \cdot \tilde{T}_1 \tag{29}$$

$$\tilde{q}_1(\tau_1) > 0 \tag{30}$$

Now we choose $\hat{q} = [\hat{q}_1(\tau_1), \hat{q}_1(\tau_2), \hat{q}_2(\tau_1), \hat{q}_2(\tau_2)]^{\mathrm{T}}$ and $\hat{T}$ as follows

$$\hat{q}_1(\tau_1) = 0 \tag{31}$$

$$\hat{q}_1(\tau_2) = \tilde{q}_1(\tau_2) - \tilde{q}_1(\tau_1) \tag{32}$$

$$\hat{q}_1(0) = \hat{q}_1(\tau_2) \tag{33}$$

The other variables stay the same, i.e. $\hat{q}_2(\tau_1)$, $\hat{q}_2(\tau_2)$, and $\hat{T}$ are equal to $\tilde{q}_2(\tau_1)$, $\tilde{q}_2(\tau_2)$, and $\tilde{T}$, respectively. We verify that $\hat{q}$, $\hat{T}$ is also a feasible solution of the R-CDMP problem by substitution in (29). Recall that the criterion J is a strictly increasing function of the queue lengths. Since $\hat{q} \leq \tilde{q}$ and $\hat{q}_i < \tilde{q}_i$ for some i due to (31) and (32) (i.e. $\hat{q}_1(\tau_1) < \tilde{q}_1(\tau_1)$ and $\hat{q}_1(\tau_2) < \tilde{q}_1(\tau_2)$), this implies $J(\hat{q}, \hat{T}) < J(\tilde{q}, \tilde{T})$, which is in contradiction with the fact that $(\tilde{q}, \tilde{T})$ is an optimal solution of the R-CDMP problem. So the assumption (28) is not satisfied, and (16) and in particular (31) hold.

Since the optimal solution of the R-CDMP problem satisfies (16), the optimal solution of the R-CDMP problem is also an optimal solution of the CDMP problem. Note that we consider the case (29) and (30), and that the proof for the other cases is similar. ■

B. Necessary and sufficient condition for steady-state control

In this section, a necessary condition for steady-state control is derived based on the R-CDMP problem. We can eliminate $q_1(\tau_2)$ and $q_2(\tau_1)$ from the constraint equations of the R-CDMP problem (i.e. (17), (20), (22), (23), (24), (25), and (26)) by substituting (17) and (22) into (23) and (25), respectively, resulting in

$$(d_1(0) - a_1(0)) \cdot T_1 \geq a_1(\tau_1) \cdot T_2 \tag{34}$$

$$(d_2(\tau_1) - a_2(\tau_1)) \cdot T_2 \geq a_2(0) \cdot T_1 \tag{35}$$

If $T_1 = 0$, then (34) and (35) imply that $T_2 = 0$, and vice versa. But this is a contradiction to (20) and the fact that T_1, T_2, and T are all positive. It follows from assumptions A1 and A2 that $d_1(0) - a_1(0) > 0$ and $d_2(\tau_1) - a_2(\tau_1) > 0$, and also $a_1(\tau_1) > 0$ and $a_2(0) > 0$. The inequalities (34) and (35) are divided by T_2, and the fraction T_1/T_2 is eliminated by substitution. Then the following necessary condition for a steady-state solution is obtained:

$$\frac{a_1(\tau_1)}{d_1(0) - a_1(0)} \leq \frac{d_2(\tau_1) - a_2(\tau_1)}{a_2(0)} \tag{36}$$

Conversely, assuming (36) it can be shown that the constraint equations of the R-CDMP problem can be satisfied with positive T_1 and T_2. Hence, the necessary condition (36) is also a sufficient condition for steady-state control.

C. Analytic solution for linear criterion

In this section, it is shown that if the criterion J is a strictly increasing *linear* function of the queue lengths, then the R-CDMP problem can be solved analytically.

We define a "zero-queue-length period" (ZQLP) as the time period (larger than zero) for which the queue length is equal to zero, see Fig. 3. Given the assumptions, a movement can encounter at most one ZQLP per cycle, and it may happen only before the end of the green light, i.e. between 0 and τ_1 for movement m_1, and between τ_1 and τ_2 for movement m_2. Let us denote the start of the ZQLP for movements m_1 and m_2 by τ_1^{e} and τ_2^{e}, respectively. Then the ZQLP for movement m_1 starts at time τ_1^{e} and ends at time τ_1, and the ZQLP for movement m_2 starts at time τ_2^{e} and ends at time τ_2.

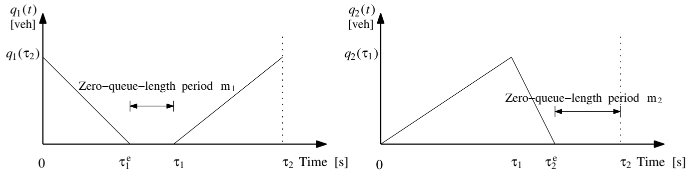

Fig. 3. Zero-queue-length period for movements m_1 and m_2.

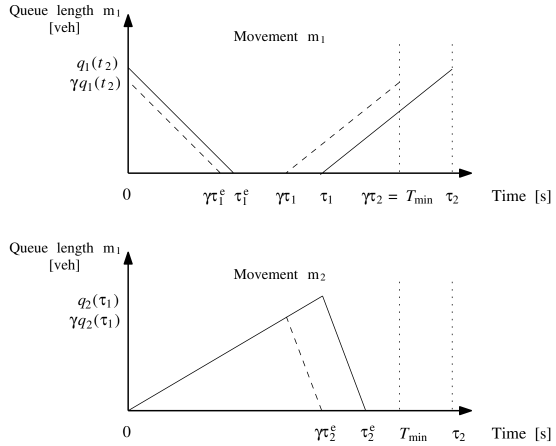

Fig. 4. Decreasing cycle time to the minimum by scaling multiplication.

Now we focus on a criterion J which is a strictly increasing *linear* function of the queue lengths. Let J be the weighted $(w_1, w_2 > 0)$ sum of the queue lengths,

$$J = w_1 q_1(\tau_1) + w_1 q_1(\tau_2) + w_2 q_2(\tau_1) + w_2 q_2(\tau_2) \tag{37}$$

Proposition 2: For the R-CDMP problem with a criterion J that is a strictly increasing linear function of the queue lengths, the optimal cycle time is equal to the given minimum cycle time.

Proof: In the optimum we have $q_1(\tau_1) = q_2(\tau_2) = 0$ by Proposition 1, which implies that in the optimum the criterion J only depends on the maximum queue lengths,

$$J = w_1 q_1(\tau_2) + w_2 q_2(\tau_1) \tag{38}$$

The general case when the cycle time is larger than the minimum cycle time and each movement has a ZQLP is shown in Fig. 4. The cycle time τ_2 can be decreased to $T_{\min}$ by multiplying all the values by the coefficient $\gamma = T_{\min}/\tau_2$ as shown in Fig. 4. Decreasing the cycle time decrease the maximum queue lengths from $q_1(\tau_2)$ and $q_2(\tau_1)$ to $\gamma q_1(\tau_2)$ and $\gamma q_2(\tau_1)$, respectively. The value of J is decreased, i.e. the maximum queue lengths decrease with the decrease of the cycle time which proves that the optimal cycle time will be equal to the minimum cycle time $T_{\min}$[1]. ■

[1]Recall that $T_{\min}$ is an endogenous and non-optimized design parameter.

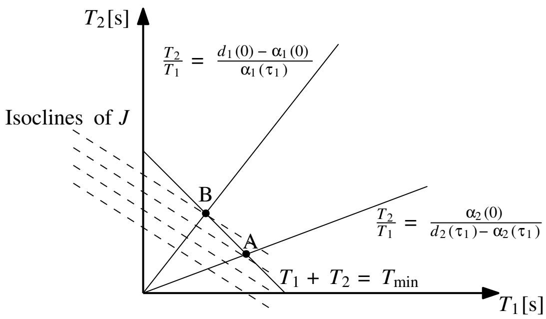

Fig. 5. Analytic solution for the linear programming problem.

According to Propositions 1 and 2, we obtain the following linear programming (LP) problem when J is given by (38),

$$\min_{T_1,T_2} J = w_2 a_2(0) \cdot T_1 + w_1 a_1(\tau_1) \cdot T_2 \tag{39}$$

subject to

$$(d_1(0) - a_1(0)) \cdot T_1 \geq a_1(\tau_1) \cdot T_2 \tag{40}$$

$$(d_2(\tau_1) - a_2(\tau_1)) \cdot T_2 \geq a_2(0) \cdot T_1 \tag{41}$$

$$T_1 + T_2 = T_{\min} \tag{42}$$

In case the necessary condition (36) is satisfied strictly, i.e. $a_1(\tau_1)/(d_1(0) - a_1(0)) < (d_2(\tau_1) - a_2(\tau_1))/a_2(0)$, the solution of the problem depends on the slope of the linear isoclines of J in the T_1, T_2-plane, see Fig. 5. If $w_2 a_2(0) < w_1 a_1(\tau_1)$ the optimal solution is found in point A, whereby the movement m_2 will not have a ZQLP. When $w_2 a_2(0) > w_1 a_1(\tau_1)$ the optimal solution is found in point B, and the movement m_1 will not have a ZQLP. Points A and B are given by

$$(T_1, T_2)_A = \left(\frac{T_{\min} \cdot (d_2(\tau_1) - a_2(\tau_1))}{a_2(0) + d_2(\tau_1) - a_2(\tau_1)}, \frac{T_{\min} \cdot a_2(0)}{a_2(0) + d_2(\tau_1) - a_2(\tau_1)} \right) \tag{43}$$

$$(T_1, T_2)_B = \left(\frac{-T_{\min} \cdot a_1(\tau_1)}{a_1(0) - d_1(0) - a_1(\tau_1)}, \frac{T_{\min} \cdot (a_1(0) - d_1(0))}{a_1(0) - d_1(0) - a_1(\tau_1)} \right) \tag{44}$$

If $w_2 a_2(0) = w_1 a_1(\tau_1)$ all points on the straight line between A and B (i.e. the convex combination of $\alpha\,(T_1, T_2)_A + (1 - \alpha)\,(T_1, T_2)_B$, $0 \leq \alpha \leq 1$) are optimal. The inner points will have two ZQLPs, one ZQLP for each movement. In the case when the necessary condition (36) is satisfied with equality, i.e. $a_1(\tau_1)/(d_1(0) - a_1(0)) = (d_2(\tau_1) - a_2(\tau_1))/a_2(0)$, the two points A and B are equal. In this case the optimal solution will not have any movement with ZQLP. Based on the above explanation the following proposition holds:

Proposition 3: There is always an optimal solution with at most one ZQLP.

Remark 1: The solution to (39)–(42) can also be found through direct substitution.

According to Proposition 1, the queue lengths $q_1(\tau_1) = 0$ and $q_2(\tau_2) = 0$ in the optimal cyclic solutions. Hence, the problem arises how to bring the queue lengths to their optimal values. N-stages control can be used to solve this problem, see Section V.

D. Numerical examples for the steady-state control

In this section two numerical examples are shown. The criterion J is the weighted sum of the queue lengths (39)

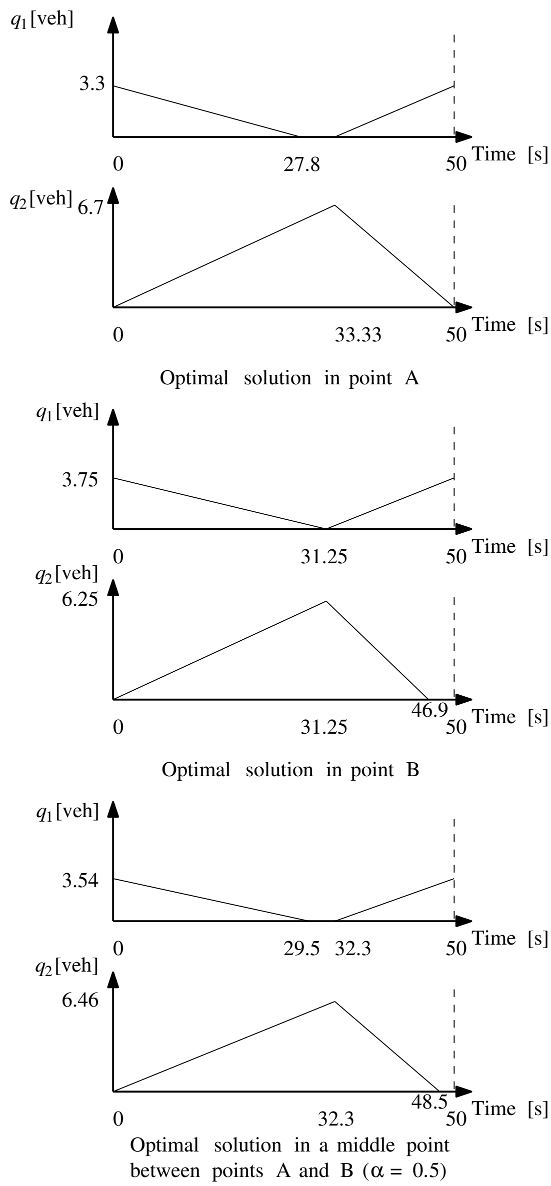

Fig. 6. Example 1. Multiple optimal solutions

with $w_1 = w_2 = 1$, and the minimum cycle time is $T_{\min} = 50$ [sec].

1) Example 1. - Multiple optimal solutions: Given that the arrival rates for movement m_1 and m_2, $a_1(t) = 0.2$ [veh/s] and $a_2(t) = 0.2$ [veh/s], respectively, and the departure rates for m_1 and m_2, $d_1(t) = 0.32$ [veh/s] and $d_2(t) = 0.6$ [veh/s], respectively. The necessary and sufficient condition for the steady-state control (36) is satisfied, $0.2/0.12 \leq 0.4/0.2$. Since $w_2 a_2(0) = w_1 a_1(\tau_1)$ is also satisfied, all points on the straight line between A (43), $(T_1, T_2)_A = (33.33, 16.67)$, and B (44), $(T_1, T_2)_B = (31.25, 18.75)$, are optimal. The queue lengths over one cycle for three optimal solutions are shown in Fig. 6.

2) Example 2. - Steady-state solution does not exist: Given that the arrival rates for movement m_1 and m_2, $a_1(t) = 0.3$ [veh/s] and $a_2(t) = 0.15$ [veh/s], respectively, and the departure rates for m_1 and m_2, $d_1(t) = 0.4$ [veh/s] and

$d_2(t) = 0.55$ [veh/s], respectively. The steady-state solution does not exist since the necessary condition (36) is not satisfied, $0.3/0.1 \not\leq 0.4/0.15$.

Remark 2: Congestion in isolated controlled intersections is defined as the situation when the queue lengths at the intersection are increasing over time. "Classic" congestion occurs when, for one of the movements, the arrival rate is larger than the departure rate during the green light period, i.e. when assumption A2 does not hold. In this example we have shown that another case of congestion will occur even if assumption A2 holds, but when the necessary condition for the steady-state control (36) is violated.

Note, however, that also for congested intersections, the DMP problem formulation (8)–(12) can be used. Since the queue lengths are increasing over time, the max equations (9) and (12) become linear equations, and for a given N, given starting time t_0 with initial queue lengths, and linear criterion the DMP problem can be solved by linear programming. The final queue lengths will be at least as large as the initial ones.

V. N-stages Control

In the N-stages control problem we consider a finite number of switchings in the optimization procedure. Let us specifically consider the following problem: for a given integer N and a given starting time t_0 with initial queue lengths $q_{1,\text{init}}, q_{2,\text{init}}$ we want to compute an optimal switching sequence consisting of N cycles[2]. For the simplified isolated controlled intersection the problem is formulated for the case when the criterion is a strictly increasing function of the queue lengths. The DMP problem (8)–(12) is used to solve the optimal problem for N-stages control when the criterion J is a strictly increasing function of the queue lengths. In this case, each max equation can be relaxed to two inequality equations, which leads to the "Relaxed" Discrete-event Max-Plus (R-DMP) problem

$$\min_{\substack{T_1(0),T_2(0),T_1(1),T_2(1),\cdots,T_1(N-1),T_2(N-1),\\ q_1(t_1),q_2(t_2),q_1(t_3),q_2(t_4),\cdots,q_1(t_{2N-1}),q_2(t_{2N})}} J \tag{45}$$

subject to

$$q_1(t_{2k+1}) \geq q_1(t_{2k}) + (a_1(t_{2k}) - d_1(t_{2k})) \cdot T_1(k) \tag{46}$$

$$q_2(t_{2k+2}) \geq q_2(t_{2k+1}) + (a_2(t_{2k+1}) - d_2(t_{2k+1})) \cdot T_2(k) \tag{47}$$

$$q_1(t_{2k+1}) \geq 0 \text{ and } q_2(t_{2k+2}) \geq 0 \tag{48}$$

$$q_1(t_0) = q_{1,\text{init}} \text{ and } q_2(t_0) = q_{2,\text{init}} \tag{49}$$

$$q_1(t_{2N}) = T_2^* \cdot a_1(\tau_1) \text{ and } q_2(t_{2N}) = 0 \tag{50}$$

$$T_1(k) + T_2(k) \geq T_{\min} \text{ and } T_1(k) + T_2(k) \leq T_{\max} \tag{51}$$

$$\text{and } (10), (11)$$

for $k = 0, 1, 2, \ldots, N-1$, where $T_{\max}$ is an upper bound for the cycle duration. Note the endpoint constraints (50) where the endpoint queue lengths $q_1(t_{2N}), q_2(t_{2N})$ are equal to the optimal queue lengths of the cyclic solution, $T_2^* \cdot a_1(\tau_1)$, and 0, respectively, where T_2^* is the optimal cyclic solution for T_2.

Proposition 4: If the criterion J is a strictly increasing function of the queue lengths, then any optimal solution of the R-DMP problem is also an optimal solution of the DMP problem.

Proof: See the proof of Proposition 3.3 in [2] which also applies here. ■

Hence, in order to bring initial queue lengths to the optimal cyclic queue lengths, N-stages control can be solved by linear programming. The endpoint queue lengths $q_1(t_{2N}), q_2(t_{2N})$ are equal to the optimal cyclic solutions and the condition (36) has to be satisfied.

VI. Conclusions

For the simplified isolated controlled intersection, in the case when the criterion J is a strictly increasing linear function of the queue lengths, we can compute the optimal switching sequence for the steady-state problem with constant cycle length by solving a linear programming problem analytically. A necessary and sufficient condition for the steady-state control with constant cycle length was also derived. The N-stages control problem was formulated. It is shown that the N-stages control problem can be solved by linear programming if the criterion J is linear and strictly increasing. Furthermore, N-stages control can be used to bring the queue lengths to optimum.

[2] In the N-stages control, the queue length vector q is defined as: $[q_1(t_1), q_1(t_2), q_2(t_1), q_2(t_2), \cdots, q_1(t_{2N-1}), q_1(t_{2N}), q_2(t_{2N-1}), q_2(t_{2N})]^{\mathrm{T}}$.

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