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Technical report 12-001

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If you want to cite this report, please use the following reference instead:
Y. Li and B. De Schutter, "Control of a string of identical pools using non-identical feedback controllers," *IEEE Transactions on Control Systems Technology*, vol. 20, no. 6, pp. 1638–1646, Nov. 2012.

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Control of a String of Identical Pools Using Non-Identical Feedback Controllers

Yuping Li and Bart De Schutter

***Abstract*—In the distant-downstream control of irrigation channels, the interactions between pools and the internal time-delay for water to travel from upstream to downstream, impose limitations on global performance, i.e. there exists propagation of water-level errors and amplification of flows over gates in the upstream direction. This paper analyzes these coupling properties for a string of identical pools, both with identical feedback controllers and with non-identical feedback controllers. A definition of string stability in terms of bounded water-level errors and bounded flows is given. It is shown that for a string of an infinite number of pools, string stability cannot be achieved by decentralized distant-downstream feedback control mainly due to the internal time-delay for water to travel from upstream to downstream. Applying the analysis results on string stability to a string of a finite number of pools, i.e. using non-identical feedback controllers in a distant-downstream control structure such that the closed-loop bandwidths of the subsystems increase from downstream to upstream, a much better global performance can be achieved than in the case of using identical feedback controllers.**

***Index Terms*—Decentralized control, string stability, irrigation channels, performance trade-off.**

I. Introduction

In large-scale irrigation networks, water is often distributed via open water channels under the power of gravity (i.e. there is no pumping). The flow of water through the network is then regulated by automated gates positioned along the channels [3], [8], [22]. The stretch of a channel between two gates is commonly called a pool. Hence the open water channels in an irrigation network can each be thought of in terms of a string of pools linked by gates. Water offtake points to farms and secondary channels are distributed along the pools. Typically, most farms are situated at the downstream end of each pool. As such, an important control objective is setpoint regulation of the water-levels immediately upstream of each gate, which enables flow demand at the (often gravity-powered) offtake points to be met without over-supplying. When the number of pools to be controlled is large and the gates are widely dispersed, it is natural to employ a decentralized control structure. In practice, a distant-downstream control structure (i.e. using the upstream gate to control the downstream water-level of a pool) is implemented for a good management of water service [11]. Such a control strategy does not demand water storage at the end of the channels and hence attains an efficient water distribution.

Fig. 1 shows the sideview of a channel under decentralized distant-downstream control. For such a control structure, when water offtakes to farms occur in the downstream pools, the interactions between pools put requirements on managing the water-level error propagation and attenuating the amplification of flows over gates in the upstream direction. This is due to the fact that the flow into one pool equals to the flow supplied by its upstream pool, and the internal time-delay for the transportation of water from upstream to downstream.[1] The coupling effects between pools, and several ways of reducing them have already been studied in the past, see [3], [4], [13], [16], [18]. In particular, when a distant-downstream control structure is chosen, the decoupling measures proposed in [18] are actually a feedforward compensation. In fact, a decentralized feedback control with such a feedforward compensation is a special realization of the distributed control scheme proposed in [3], [10]. It is shown in [3] that there exists a trade-off between local performance of water-level setpoint regulation in each pool and global performance of decoupling the interactions between pools. However, when designing purely decentralized feedback controllers for irrigation networks,[2] one usually only takes local performance into account, i.e. regulating the water-level in a pool at its setpoint while rejecting offtake disturbances (i.e. water fed to farms). Such a design might present very bad global performance, e.g. in response to offtake disturbances in the downstream pools, the gates in the upstream pools may go beyond saturation or the water-levels in the upstream pools may drop too low to satisfy the water demands.

Therefore, this paper studies decentralized distant-downstream feedback control of a string of identical pools from the perspective of global performance, for which we suggest a control strategy involving the use of non-identical feedback controllers. A definition of string stability in terms of bounded water-level errors and bounded flows is given.

This work has been supported in part by the European 7th framework STREP project "Hierarchical and distributed model predictive control of large-scale systems (HD-MPC)", contract number INFSO-ICT-223854, the Delft Research Center NGI, and the BSIK project "Next Generation Infrastructures (NGI)", and the European 7th Framework Network of Excellence "Highly-complex and networked control systems (HYCON2)".

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[1]The geometry of canal pools is rather simple compared with those of rivers and other natural water infrastructures. Based on both the simulation results on St. Venant equations and field test results [15], one can anticipate that the movements of gates and the control actions involved will dominate the interconnections between pools. Hence, this paper will not focus on the effects of the geometry of the pools on the flows in the pools.

[2]Although a centralized scheme is popular in higher-level supervisory control, many irrigation networks still implement purely a decentralized scheme to guarantee robustness of the lower-level control (e.g. the Murray Irrigation's 2 000 kilometers of open channel system in Australia, the Transvase Tajo-Segura irrigation system in southeastern Spain, etc.).

It is shown that string stability cannot be achieved for a string of an infinite number of pools with decentralized distant-downstream feedback control. We also show that for a string of a finite number of pools (which is true in practice), by designing the non-identical feedback controllers such that the closed-loop bandwidths of the subsystems increase from downstream to upstream, a much better global performance than that with identical feedback controllers can be achieved. Furthermore, we extend the analysis result to a string of heterogeneous pools and give guidelines for designing feedback controllers based on global performance. This paper builds upon the results in [9]. It extends the sufficient conditions for bounded water-level errors and bounded flows (i.e. Lemmas 2.4 and 2.5 in [9] respectively) to necessary and sufficient conditions (see Lemmas 2.7 and 2.8 of Section II). Discussions on other decoupling strategies for decentralized feedback controller design are added in Section II-C. Moreover, the case study results given in this paper are based on an identified third-order *nonlinear* simulation model that gives very accurate predictions of water-levels [21].

In particular, this paper considers (a) *temporal stability* for a string of a finite number of pools, for which the related test is based on the stability concept that has been standing in the literature for long, see [20]; and (b) *string stability* for a string of an infinite number of pools. In reality, the number of pools in a channel is always finite. It is worthwhile to highlight that the reason we propose a string stability analysis is to improve global performance of a string of a finite number of pools with decentralized distant-downstream feedback control in terms of decoupling between subsystems. This paper has the following contributions: 1. A novel definition of string stability in terms of bounded water-level errors and bounded flows is given. 2. We show that for a string of an infinite number of pools with decentralized distant-downstream feedback control, the closed-loop bandwidth limitation of each subsystem, imposed by the internal time-delay, makes it impossible to achieve string stability. 3. Based on string stability analysis, we show that for a string of a finite number of pools, by selecting non-identical feedback controllers such that the closed-loop bandwidths of the subsystems increase from downstream to upstream, a much better global performance than that with identical feedback controllers can be achieved. 4. A case study is supplied that demonstrates the proposed guidelines to design decentralized distant-downstream feedback controllers from the perspective of global performance.

The paper is organized as follows. Section II gives the definition of string stability in terms of bounded water-level errors and bounded flows. Both the case of a string of identical pools (represented by an integrator with time-delay model) with identical feedback controllers and the case with non-identical feedback controllers are discussed. Section III presents simulation results on an identified (3rd-order non-linear) model that captures the dominant wave dynamics. A summary is finally given in Section IV.

II. Bounded water-level errors and bounded flows

Consider $n+1$ pools. Denote the first downstream pool by G_0, the second downstream pool by G_1, and so on, till the most upstream pool, G_n. The sideview of the interconnected closed-loop system is shown in Fig. 1, where y_i is the water-level in pool$_i$ and h_i is the head over gate$_i$. Note that the head over a gate is closely connected to the flow over the gate. In the figure, d_i represents the water offtakes from pool$_i$ to farms, which are typically load disturbances.

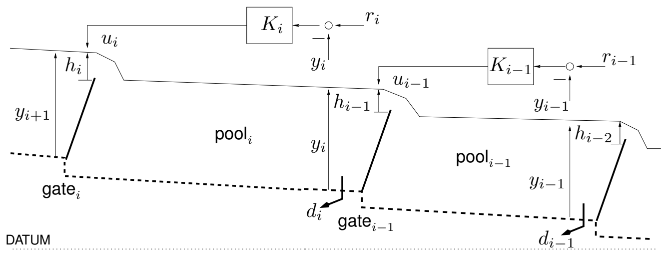

Fig. 1. Decentralized control of an open water channel

Based on mass balance, a simple model of the water-level in pool$_i$ that captures the dynamics at low frequencies is obtained (see [21]):

$$G_i:\; y_i(s) = \frac{c_i e^{-\tau_i s}}{s} u_i(s) - \frac{c_{i-1}}{s} u_{i-1}(s) + \frac{1}{s} d_i(s), \quad (1)$$

where the scalars c_i and c_{i-1} are discharge coefficients that depend on the pool surface area and the width of the upstream and the downstream gates respectively, and τ_i is the internal time-delay that the water takes to travel from the upstream end to the downstream end of the pool, $u_i(t) := h_i^{3/2}(t)$ is proportional to the flow over gate$_i$.[3] Denote the water-level error as $e_i(t) := r_i(t) - y_i(t)$, where $r_i(t)$ is the water-level setpoint. Essentially, for setpoint regulation, a decentralized controller K_i is selected as a PI compensator:

$$K_i : u_i(s) = \left(\kappa_i + \frac{\phi_i}{s}\right) e_i(s), \quad (2)$$

with $\kappa_i > 0$ and $\phi_i > 0$; the integrator is included for zero steady-state water-level error in rejection to load disturbance $d_i(t)$, while the phase-lead term helps for closed-loop stability.

As mentioned previously, the interaction between pools (i.e. the flow out of pool$_i$ equals to the flow into pool$_{i-1}$) influences the global performance of the closed-loop system. This is represented by the propagation of water-level errors and the amplification of control actions in the upstream direction. To analyze the above coupling properties between pools, we study a string of identical pools with decentralized feedback control.

[3]We consider here open water channels with overshot gates, for which the flow over gate$_i$ can be approximated by $c_i h_i^{3/2}(t)$ [2]. Note that the discussions on string stability in this paper also hold for channels with undershot gates. With an overshot gate, the water flow or level is controlled by lowering the top of the gate so water flows over the structure (head). With an undershot gate, the water flow or level is controlled by lifting the gate vertically in such a way that the water flows through the opening at the bottom of the structure.

In this case, in (1), $c_i = c$ and $\tau_i = \tau$ for $i = 0, \ldots, n$. One then has the following plant model:

$$G_i: \ y_i(s) = \frac{ce^{-\tau s}}{s} u_i(s) - \frac{c}{s}\left(u_{i-1} + d_i\right)(s), \tag{3}$$

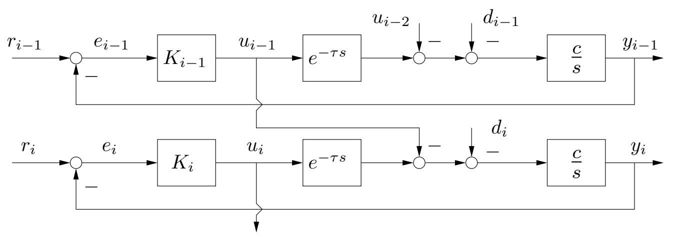

Fig. 2. Identical pools with decentralized feedback control

Fig. 2 shows the configuration of a string of identical pools with decentralized feedback control. From (3) and (2), the coupling transfer function from one closed-loop subsystem to the next one can be obtained as follows:

$$T_{ee,i}(s) := \frac{e_i(s)}{e_{i-1}(s)} = \frac{\frac{c(\kappa_{i-1}s + \phi_{i-1})}{s^2}}{1 + \frac{c(\kappa_i s + \phi_i)}{s^2} e^{-s\tau}} \tag{4}$$

$$T_{uu,i}(s) := \frac{u_i(s)}{u_{i-1}(s)} = \frac{\frac{c(\kappa_i s + \phi_i)}{s^2}}{1 + \frac{c(\kappa_i s + \phi_i)}{s^2} e^{-s\tau}} \tag{5}$$

Denote the coupling transfer functions from the first downstream pool to the most upstream pool as $E_n(s) := \frac{e_n(s)}{e_0(s)}$ and $F_n(s) := \frac{u_n(s)}{u_0(s)}$.

Definition 2.1: Given a string of $n+1$ pools under centralized or decentralized control, if $\lim_{n\to\infty} |E_n(j\omega)| < \infty$ and $\lim_{n\to\infty} |F_n(j\omega)| < \infty$ for all $\omega \geq 0$, the system is said to be string-stable in terms of bounded water-level errors and bounded flows. ∎

For a string of pools with decentralized control one has

$$E_n(s) = \prod_{i=1}^{n} T_{ee,i}(s) = \prod_{i=1}^{n} \frac{c\left(\kappa_{i-1}s + \phi_{i-1}\right)}{s^2 + c\left(\kappa_i s + \phi_i\right)e^{-s\tau}} \tag{6}$$

$$F_n(s) = \prod_{i=1}^{n} T_{uu,i}(s) = \prod_{i=1}^{n} \frac{c\left(\kappa_i s + \phi_i\right)}{s^2 + c\left(\kappa_i s + \phi_i\right)e^{-s\tau}}. \tag{7}$$

A. Coupling of pools with identical feedback controllers

If one designs the decentralized controller based on local performance, in particular, if one takes the interaction between pools as an unknown disturbance, then for identical pools, it is natural to select K_i in (2) the same for $i = 0, \ldots, n$, i.e.

$$u_i(s) = \left(\kappa_0 + \frac{\phi_0}{s}\right) e_i(s), \tag{8}$$

where κ_0 and ϕ_0 are selected by just considering the local closed-loop system: regulating the water-level in a pool to its setpoint while rejecting the offtake disturbances in the pool. Then the couplings between neighboring pools are:

$$T_{ee}(s) = T_{uu}(s) = \frac{\frac{c(\kappa_0 s + \phi_0)}{s^2}}{1 + \frac{c(\kappa_0 s + \phi_0)}{s^2} e^{-s\tau}} \tag{9}$$

Similar as Lemma 1 of [3], we have the following result.

Lemma 2.2: For a string of identical pools with identical feedback controllers, there exists an $\omega > 0$ such that $|T_{ee}(j\omega)| > 1$ and $|T_{uu}(j\omega)| > 1$. ∎

Proof: The proof follows the lines of the proof for Lemma 9.3 of [6].

We first prove that $\int_0^\infty \ln |T_{ee}(j\omega)| \frac{d\omega}{\omega^2} \geq 0$. Denote $L(s) := \frac{c(\kappa_0 s + \phi)}{s^2}$, then $T_{ee}(s) = \frac{L(s)e^{-s\tau}}{1+L(s)e^{-s\tau}} e^{s\tau}$. Correspondingly,

$$|T_{ee}(j\omega)| = \left| \frac{L(j\omega)\exp(-j\tau\omega)}{1 + L(j\omega)\exp(-j\tau\omega)} \right| \tag{10}$$

for all $\omega \in \mathbb{R}$. Applying Cauchy's Theorem to the integral of the function $F(s) := \frac{1}{s^2} \ln\left(\frac{L(s)\exp(-\tau s)}{1+L(s)\exp(-\tau s)}\right)$ along the standard Nyquist contour C with infinitesimal indentation C_ϵ around the origin, we have

$$\oint_C F(s)ds = 0 = \int_{C_{i-}} F(s)ds + \int_{C_\epsilon} F(s)ds + \int_{C_\infty} F(s)ds,$$

where C_{i-} is the imaginary axis minus the indentation C_ϵ and C_∞ the semicircle arc with radius $\to \infty$ that starts at $0 + j\infty$ and travels clock-wise to $0 - j\infty$. Since $L(s)$ has two poles at the origin, the integral along C_ϵ is 0. By straightforward calculation, the integral along C_∞ is equal to $j\pi\tau$. Using the conjugate symmetry of the integrand and rearranging terms, yields

$$\int_0^\infty \ln \left| \frac{L(j\omega)\exp(-j\tau\omega)}{1 + L(j\omega)\exp(-j\tau\omega)} \right| \frac{d\omega}{\omega^2} = \frac{\pi\tau}{2} > 0. \tag{11}$$

Indeed, $L(s)$ is strictly proper; hence $\ln |T_{ee}(j\omega)| < 0$ at high frequencies. It follows from (11) that there exists an $\omega_0 \in (0, \infty)$, such that $|T_{ee}(j\omega_0)| > 1$. From (9), $|T_{uu}(j\omega_0)| > 1$. ∎

Note that for the string of pools with identical feedback controllers, $E_n(s) = \left(T_{ee}(s)\right)^n$. Hence there exists an $\omega > 0$ such that $\lim_{n\to\infty} |E_n(j\omega)|$ is unbounded. Similarly, there exists an $\omega > 0$ such that $\lim_{n\to\infty} |F_n(j\omega)|$ is unbounded. Following Definition 2.1, we have

Theorem 2.3: For a string of infinite number of pools (3) controlled by identical decentralized feedback controllers (8), the closed-loop system is not string-stable. ∎

Let us consider a numerical example for a string of 101 identical pools. The model of the pools is given in (3) with the coefficient $c = 0.68 \frac{\mathrm{m}^{3/2}}{\mathrm{min}}$ and the transportation time delay $\tau = 20$ min. For local performance, select $\kappa_0 = 0.31$ and $\phi_0 = 8.2 \times 10^{-4}$ for the feedback controller (8). The magnitudes of the coupling transfer functions $T_{ee}(s)$ and $T_{uu}(s)$ are shown in Fig. 3. It is observed that $\max_\omega |T_{ee}(j\omega)| \approx 2.28$. The maximum occurs around 0.14 rad/min. Hence $\max_\omega \frac{|e_{100}|}{|e_0|} = 2.28^{100}$ (and $\max_\omega \frac{|u_{100}|}{|u_0|} = 2.28^{100}$), which is intolerable in practice.

B. Coupling of pools with non-identical feedback controllers

In fact, a string of $n+1$ identical pools with identical feedback controllers involves the strongest coupling between

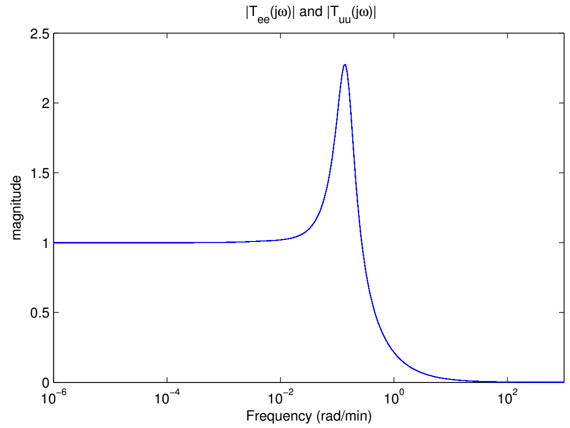

Fig. 3. Closed-loop coupling (with identical feedback controllers applied)

pools, e.g. $\max_\omega |T_{ee,i}(j\omega)|(>1)$ occurs at the same ω for all i, which makes bounded water-level errors impossible. To decouple the interaction and hence for a better global closed-loop performance, we consider non-identical feedback controllers as follows:

$$K_0: u_0(s) = \left(\kappa_0 + \frac{\phi_0}{s}\right) e_0(s), \tag{12}$$

$$K_i: u_i(s) = \left((\kappa_0+\alpha i) + \frac{\phi_0}{s}\right) e_i(s) \text{ for } i = 1,\dots,n \tag{13}$$

with $\alpha > 0$. Using a first-order Padé approximation [1] to represent the transportation time-delay τ and substituting (12-13) into (6-7) results in

$$\begin{aligned} |E_n(j\omega)|^2 &= \prod_{i=1}^{n} |T_{ee,i}(j\omega)|^2 \\ &\approx \prod_{i=1}^{n} \frac{i^2 + A_e i + B_e}{i^2 + Ci + D} =: |E_n^a(j\omega)|^2 \end{aligned} \tag{14}$$

$$\begin{aligned} |F_n(j\omega)|^2 &= \prod_{i=1}^{n} |T_{uu,i}(j\omega)|^2 \\ &\approx \prod_{i=1}^{n} \frac{i^2 + A_f i + B_f}{i^2 + Ci + D} =: |F_n^a(j\omega)|^2 \end{aligned} \tag{15}$$

for all $\omega > 0$, where

$$A_e = \frac{2\alpha c^2(\kappa_0-\alpha)\omega^4 + \frac{8\alpha c^2(\kappa_0-\alpha)}{\tau^2}\omega^2}{\alpha^2c^2\omega^4 + \frac{4\alpha^2c^2}{\tau^2}\omega^2}$$

$$B_e = \frac{(\kappa_0-\alpha)^2c^2\omega^4 + \left(\phi_0^2 + \frac{4(\kappa_0-\alpha)^2}{\tau^2}\right)c^2\omega^2 + \frac{4\phi_0^2c^2}{\tau^2}}{\alpha^2c^2\omega^4 + \frac{4\alpha^2c^2}{\tau^2}\omega^2}$$

$$A_f = \frac{2\kappa_0\alpha c^2\omega^4 + \frac{8\kappa_0\alpha c^2}{\tau^2}\omega^2}{\alpha^2c^2\omega^4 + \frac{4\alpha^2c^2}{\tau^2}\omega^2}$$

$$B_f = \frac{\kappa_0^2c^2\omega^4 + \left(\phi_0^2 + \frac{4\kappa_0^2}{\tau^2}\right)c^2\omega^2 + \frac{4\phi_0^2c^2}{\tau^2}}{\alpha^2c^2\omega^4 + \frac{4\alpha^2c^2}{\tau^2}\omega^2}$$

$$C = \frac{\left(2\kappa_0\alpha c^2 - \frac{8\alpha c}{\tau}\right)\omega^4 + \frac{8\kappa_0\alpha c^2}{\tau^2}\omega^2}{\alpha^2c^2\omega^4 + \frac{4\alpha^2c^2}{\tau^2}\omega^2}$$

$$D = \frac{\omega^6 + \left(\kappa_0^2c^2 - \frac{8\kappa_0 c}{\tau} + \frac{4}{\tau^2} + 2\phi_0 c\right)\omega^4}{\alpha^2c^2\omega^4 + \frac{4\alpha^2c^2}{\tau^2}\omega^2} + \frac{\left(\frac{4\kappa_0^2c^2 - 8\phi_0 c}{\tau^2} + \phi_0^2c^2\right)\omega^2 + \frac{4\phi_0^2 c2}{\tau^2}}{\alpha^2c^2\omega^4 + \frac{4\alpha^2c^2}{\tau^2}\omega^2}.$$

Remark 1: In practice, such an approximation does not change the analysis result for the delayed system given that the offtake disturbance that induces e_0 is significant in the low-frequency range, while the high-frequency resonances caused by time-delay are dampened by the feedback controller with an extra low-pass filter, as illustrated by the simulation in Section III. ◦

The following conditions for bounded water-level errors (in Lemma 2.7) and bounded flows (in Lemma 2.8) use properties 2.4 – 2.6 of the Gamma function defined in (16) (see Chapter 5 of [14]).

Property 2.4: If the real part of the complex number z is positive (i.e. Re$[z] > 0$), then the integral

$$\Gamma(z) := \int_0^\infty e^{-t}t^{z-1}dt \tag{16}$$

converges absolutely.

Property 2.5: For $x > 0$, $\Gamma(x) > 0$.

Property 2.6: $|\Gamma(x+yj)| \geq \left(\cosh^{-1}(\pi y)\right)^{1/2}\Gamma(x)$ for $x \geq \frac{1}{2}$.

Lemma 2.7: Assume that for a fixed $\omega > 0$, $A_e(\omega) > 0$, $C(\omega) > 0$, $D(\omega) > 0$. Then $\lim_{n\to\infty} |E_n^a(j\omega)|$ exists *if and only if* $A_e(\omega) \leq C(\omega)$. ■

Proof: We consider three cases: 1) $A_e(\omega) = C(\omega)$, 2) $A_e(\omega) < C(\omega)$, and 3) $A_e(\omega) > C(\omega)$.

For the case of $A_e(\omega) = C(\omega)$, one has

$$|E_n^a(j\omega)|^2 = \prod_{i=1}^{n}\left[1 + \frac{B_e - D}{(i-z_1)(i-z_2)}\right], \tag{17}$$

where z_1, z_2 are the roots of $z^2+Cz+D=0$. When $n\to\infty$, expression (17) corresponds to equation (89.5.7) of [7], which gives

$$\lim_{n\to\infty} |E_n^a(j\omega)|^2 = \frac{\Gamma(1-z_1)\Gamma(1-z_2)}{\Gamma(1-z_3)\Gamma(1-z_4)}, \tag{18}$$

where z_3, z_4 are the roots of $z^2+Cz+B_e=0$. Note that $B_e(\omega) > 0$ for all $\omega > 0$ and by assumption, for the ω under consideration, $A_e(\omega) > 0$ (equivalently $C(\omega) > 0$) and $D(\omega) > 0$. It follows that Re$[z_1] < 0$, Re$[z_2] < 0$, Re$[z_3] < 0$, Re$[z_4] < 0$. Based on the Properties 2.4 – 2.6 of the Gamma function, one has that $\Gamma(1-z_1)\Gamma(1-z_2)$ and $\Gamma(1-z_3)\Gamma(1-z_4)$ are finite and $\Gamma(1-z_3)\Gamma(1-z_4) \neq 0$. Therefore, $\frac{\Gamma(1-z_1)\Gamma(1-z_2)}{\Gamma(1-z_3)\Gamma(1-z_4)}$ is finite and $\lim_{n\to\infty}|E_n^a(j\omega)|$ exists.

For $A_e(\omega) < C(\omega)$, one has

$$\prod_{i=1}^{n}\frac{i^2 + A_e(\omega)i + B_e(\omega)}{i^2 + C(\omega)i + D(\omega)} < \prod_{i=1}^{n}\frac{i^2 + C(\omega)i + B_e(\omega)}{i^2 + C(\omega)i + D(\omega)} \tag{19}$$

for $A_e(\omega) > 0$, $C(\omega) > 0$ and $D(\omega) > 0$.[4] Taking the limit of both sides of (19),

$$\lim_{n\to\infty} |E_n^a(j\omega)|^2 \leq \lim_{n\to\infty} \prod_{i=1}^{n} \frac{i^2 + C(\omega)i + B_e(\omega)}{i^2 + C(\omega)i + D(\omega)}. \qquad (20)$$

As proved previously, the right-hand side of (20) is finite (i.e. $= \frac{\Gamma(1-z_1)\Gamma(1-z_2)}{\Gamma(1-z_3)\Gamma(1-z_4)}$). Therefore, $\lim_{n\to\infty} |E_n^a(j\omega)|$ exists.

For $A_e(\omega) > C(\omega)$, since $B_e(\omega) > 0$ for all $\omega > 0$ and by assumption, $A_e(\omega) > 0$, $C(\omega) > 0$ and $D(\omega) > 0$ for a fixed $\omega > 0$, it is easy to verify that $\prod_{i=1}^{n} \frac{i^2 + A_e(\omega)i + B_e(\omega)}{i^2 + C(\omega)i + D(\omega)}$ increases monotonously with n when $n > \frac{D(\omega)-B_e(\omega)}{A_e(\omega)-C(\omega)}$. Hence, $\lim_{n\to\infty} |E_n^a(j\omega)|$ is unbounded. The lemma is thus proved. ■

Remark 2: a) For all $\omega > 0$, the condition $A_e(\omega) > 0$ holds *if and only if* $\kappa_0 > \alpha$.

b) For all $\omega > 0$, the condition $A_e(\omega) \leq C(\omega)$ holds *if and only if* $-2\alpha^2 c^2 \omega^4 - \frac{8\alpha^2 c^2}{\tau^2}\omega^2 \leq -\frac{8\alpha c}{\tau}\omega^4$, which is equivalent to

$$\alpha c - \frac{4}{\tau} \geq -\frac{4\alpha c}{\tau^2}\omega^{-2}. \qquad (21)$$

Since $\alpha > 0$, $c > 0$ and $\omega > 0$, (21) thus holds if $\alpha c - \frac{4}{\tau} \geq 0$.

c) Note that the denominator of $D(\omega)$ is positive for $\omega > 0$. The numerator of $D(\omega)$ can be written as

$$\left(\omega^3 - \frac{2\kappa_0 c}{\tau}\omega\right)^2 + \left(\frac{2}{\tau}\omega^2 - \frac{2\phi_0 c}{\tau}\right)^2 + \left(2\phi_0 c\omega^2 + \phi_0^2 c^2\right)\omega^2 + \left(\kappa_0^2 c^2 - \frac{4\kappa_0 c}{\tau}\right)\omega^4.$$

For all $\omega > 0$, the condition $D(\omega) > 0$ holds if $\kappa_0 \geq \frac{4}{c\tau}$.

From the above points a), b) and c), if $\kappa_0 > \alpha \geq \frac{4}{c\tau}$, then the conditions for the existence of $\lim_{n\to\infty} |E_n^a(j\omega)|$ hold for all $\omega > 0$. ○

Similarly, one has the following result for bounded flows.

Lemma 2.8: Assume that for a fixed $\omega > 0$, $C(\omega) > 0$, $D(\omega) > 0$. Then $\lim_{n\to\infty} |F_n^a(j\omega)|$ exists if and only if $A_f(\omega) \leq C(\omega)$. ■

Proof: The proof follows the same lines as the proof of Lemma 2.7. ■

Remark 3: For $\omega > 0$, the condition $A_f(\omega) \leq C(\omega)$ holds *if and only if* $0 \leq -\frac{8\alpha c}{\tau}\omega^4$, which is impossible given the assumption that $\alpha > 0$. So $|F_n^a(j\omega)|$ always grows unbounded as $n \to \infty$. ○

In fact, under distant-downstream control, to compensate the influence of the internal time-delay, the amplification of the control actions in the upstream direction is unavoidable. This is shown in Fig. 4. Initially, the system is at steady-state. At time t_s, the flow out of pool$_i$ increases, see the change of u_{i-1} (the dashed line in Fig. 4(a)). To compensate for the influence of u_{i-1} on y_i, the flow into the pool, u_i, also increases (the solid line in Fig. 4(a)). However, the influence of u_i on the downstream water-level y_i will be τ_i later than that of u_{i-1} on y_i (see Fig. 4(b)). For a zero steady-state error of y_i

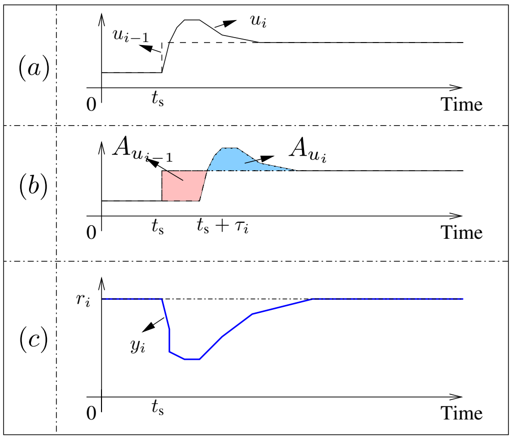

Fig. 4. Control actions for zero steady-state water-level error

with respect to r_i (see Fig. 4(c))), u_i should be greater than u_{i-1} for some time such that the area of A_{u_i} (as indicated in Fig. 4) is equal to the area of $A_{u_{i-1}}$. Correspondingly in the frequency domain, we have that for a small enough $\omega > 0$, $|T_{uu,i}(j\omega)| > 1$.[5] Hence, there exists an $\omega > 0$ such that $\lim_{n\to\infty} |F_n(j\omega)|$ is unbounded. Then to have bounded water-level errors for an infinite number of identical pools with decentralized control, the energy of the control actions in the upstream pools goes to infinity, which is impossible in practice. Indeed, for robust stability of the closed-loop, one has the condition on the closed-loop bandwidth that $\omega_b < \frac{1}{\tau}$ (see [19]). However, with the condition that $\alpha \geq \frac{4}{c\tau}$, the bandwidths of the string of pools increase from downstream to upstream. Hence, for a string of an infinite number of pools, there exists an $N < \infty$, such that the temporal stability condition for the subsystems $i > N$ is not satisfied.

From the above discussions and Definition 2.1, the following conclusion is obtained.

Theorem 2.9: For a string of infinite number of pools (3) controlled by the decentralized feedback controller (12-13), the closed-loop system is not string-stable. ■

Consider the numerical example given in Section II-A for a string of 101 identical pools. Select $\kappa_0 = 0.31$, $\phi_0 = 8.2\times10^{-4}$ and $\alpha = 0.29$ for the feedback controller in (12-13). The magnitudes of the coupling transfer functions $T_{ee,i}(s)$ and $T_{uu,i}(s)$, for $i = 1, \ldots, 100$, are shown in Fig. 5. The decoupling function of applying non-identical feedback controller is observed. Indeed, for all $i = 1, \ldots, 100$, $|T_{ee,i}(j\omega)| \leq 1$ for all $\omega \geq 0$. Hence, we can expect a decreasing propagation of the water-level errors in the upstream direction, which is confirmed by the top graph of $|E_n(j\omega)|$ in Fig. 6. Furthermore, an attenuation of the amplification of the control action (i.e. flows over gates) is also achieved, see in the bottom graph in Fig. 6 that $\max_\omega \frac{|u_{100}|}{|u_0|} = 17.3$, while as analyzed in Section II-A, $\max_\omega \frac{|u_{100}|}{|u_0|} = 2.28^{100}$ for the case with identical feedback controllers.

[4]Again, note that $B_e(\omega) > 0$ for all $\omega > 0$.

[5]Note that $|T_{uu,i}(j0)| = 1$ for all $i = 1, \ldots, n$.

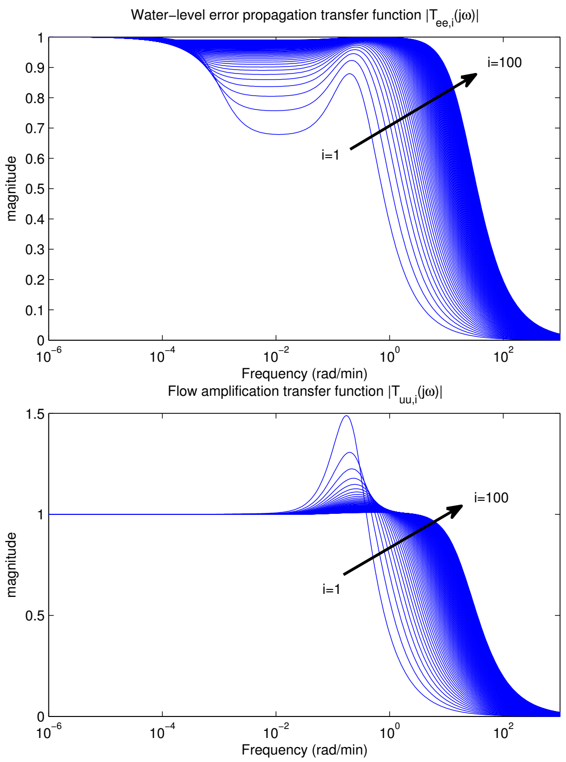

Fig. 5. Closed-loop coupling (with non-identical feedback controllers applied)

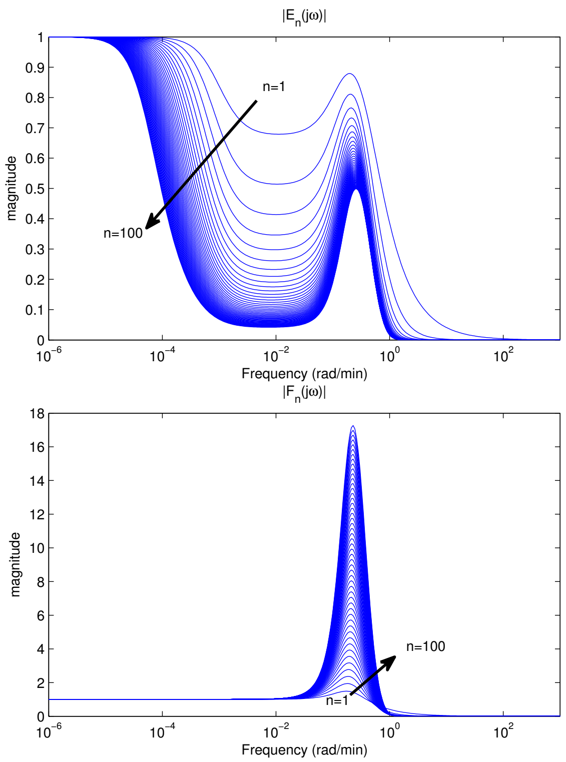

Fig. 6. $|E_n(j\omega)|$ and $|F_n(j\omega)|$ for $n = 1, \ldots, 100$ with non-identical feedback controllers

C. Further Remarks

In reality, the number of the pools in a channel is finite. When designing decentralized feedback control for a string of $n + 1$ ($< \infty$) similar pools (in terms of e.g. pool lengths, gate properties, etc.), one can first select κ_0, ϕ_0 such that the local performance in terms of setpoint regulation is guaranteed. Then we select an $\alpha > 0$ such that the tradeoff between local and global performance is managed, i.e. with the selected α, the bandwidth of the most upstream closed-loop $\omega_{b,n}$ should satisfy $\omega_{b,n} < \frac{1}{\tau}$.[6] Indeed, by including an $\alpha > 0$ in the non-identical feedback controllers, the bandwidths of the closed-loop subsystems increase in the upstream direction; hence, one can expect a faster response of the interconnected system to the offtake disturbance in the downstream pools than the case with identical feedback controllers. Note that in the distributed control strategy discussed in [3], [10], such a speed-up of the closed-loop response is achieved by involving the known interaction between neighboring pools in the input signals to be rejected and by solving an optimization problem to manage the tradeoff between local and global performance.

One can extend the above analysis results to channels with heterogeneous pools: For a channel with distant-downstream control, given that the temporal stability is ensured for each subsystem, one can guarantee good global performance, i.e. management of the water-level error propagation and attenuation of the amplification of flows over gates in the upstream direction, by ensuring that the closed-loop bandwidths increase from downstream to upstream.

Remark 4: For a channel in which the pool lengths increase from upstream to downstream, the above condition that the closed-loop bandwidths increase from downstream to upstream can be satisfied even by simply designing the decentralized feedback controllers just based on local performance. In reality however, based on the consideration of storing water to satisfy demands from farms, civil engineers design irrigation networks such that the pool lengths, in general, tend to decrease from upstream to downstream. However, the previous guidelines for decentralized feedback control design (i.e. 1) the temporal stability of each subsystem should be attained; 2) the bandwidths of the closed-loop subsystems should increase in the upstream direction) should still be kept in mind for a good tradeoff between local and global performance. ◦

Designing non-identical feedback controllers for a string of identical pools based on other decoupling strategies might work in improving global performance. For example, based on the analysis of the interactions between subsystems with identical feedback controllers (see Section II-A), one might directly think of setting the closed-loop bandwidths of the subsystems such that the string of the controlled pools alternate between a fast subsystem and a slow subsystem. In this way, the maximum of the magnitudes of the coupling transfer functions $T_{ee,i}(s)$ will not occur at the same frequency for different i. However, it is not easy to design such non-identical feedback controllers. For example, one simple design scheme

[6] As discussed in [20], a string of finite number of pools with decentralized feedback control is stable *if and only if* each closed-loop (i.e. each single pool with its feedback controller) is stable, which is guaranteed by $\omega_{b,i} < \frac{1}{\tau_i}$ for all $i = 0, \ldots, n$ [19].

could be:

$$K_i : (e_i \mapsto u_i) = \begin{cases} \kappa_0 + \frac{\phi_0}{s} & \text{for } i = 0, 2, \ldots, n, \\ \kappa_1 + \frac{\phi_0}{s} & \text{for } i = 1, 3, \ldots, n-1; \end{cases}$$

where $\kappa_1 > \kappa_0 > 0$ and without loss of generality, we assume n to be even. However, it can be proved that with such a design scheme, the water-level error goes unbounded when n approximates infinity. Indeed, from (4) one has

$$T_{ee,i}(s) = \begin{cases} \frac{\frac{c(\kappa_0 s+\phi_0)}{s^2}}{1+\frac{c(\kappa_1 s+\phi_0)}{s^2}e^{-s\tau}} & \text{for } i = 1, 3, \ldots, n-1, \\ \frac{\frac{c(\kappa_1 s+\phi_0)}{s^2}}{1+\frac{c(\kappa_0 s+\phi_0)}{s^2}e^{-s\tau}} & \text{for } i = 2, 4, \ldots, n. \end{cases}$$

Then from (6),

$$\begin{aligned} E_n(s) &= \left(\frac{\frac{c(\kappa_0 s+\phi_0)}{s^2}}{1+\frac{c(\kappa_1 s+\phi_0)}{s^2}e^{-s\tau}} \right)^{\frac{n}{2}} \left(\frac{\frac{c(\kappa_1 s+\phi_0)}{s^2}}{1+\frac{c(\kappa_0 s+\phi_0)}{s^2}e^{-s\tau}} \right)^{\frac{n}{2}} \\ &= \left(\frac{\frac{c(\kappa_0 s+\phi_0)}{s^2}}{1+\frac{c(\kappa_0 s+\phi_0)}{s^2}e^{-s\tau}} \right)^{\frac{n}{2}} \left(\frac{\frac{c(\kappa_1 s+\phi_0)}{s^2}}{1+\frac{c(\kappa_1 s+\phi_0)}{s^2}e^{-s\tau}} \right)^{\frac{n}{2}}. \end{aligned}$$

Similar as the proof of Lemma 2.2, using Cauchy's integral formula, one can prove that there exists an $\omega > 0$ such that $E_n(j\omega) > 1$. Hence $\lim_{n\to\infty} |E_n(j\omega)|$ is unbounded.

III. Simulation Results

In this section, simulation results are shown for the case of a string of five identical pools with identical feedback controllers and for the case with non-identical feedback controllers. The following third-order nonlinear model that captures the dominant wave-frequency dynamics in the pools is used as the simulation model (see [21]):

$$\begin{aligned} y_i(t+1) =& b_1 h_i^{\frac{3}{2}}(t-\tau) + b_3 h_i^{\frac{3}{2}}(t-\tau-1) + b_5 h_i^{\frac{3}{2}}(t-\tau-2) \\ &+ b_2 \left(y_i - p_{i-1}\right)^{\frac{3}{2}}(t) + b_4 \left(y_i - p_{i-1}\right)^{\frac{3}{2}}(t-1) \\ &+ b_6 \left(y_i - p_{i-1}\right)^{\frac{3}{2}}(t-2) + y_i(t) \\ &+ (1-a_1)\left(y_i(t) - 2y_i(t-1) + y_i(t-2)\right) \\ &+ (1-a_2)\left(y_i(t) - y_i(t-1)\right), \end{aligned}$$

where p_{i-1} is the position of gate$_{i-1}$. The parameters of the pool are given in Table I. Saturations are set for gate positions and flows over gates to simulate the operation of the real water infrastructure system.[7] The parameters of the pool are the same as those identified for pool$_{10}$ of the Haughton Main Channel. As shown in Section 7.2 of [21], such a third-order nonlinear model gives very accurate predictions.

Different from the feedback controller (2) given in Section II, to guarantee no excitement of the dominant waves (which are captured by the above third-order nonlinear model), here the feedback controllers involve an extra low-pass filter $\frac{1}{s+0.125}$, i.e. to make sure that we have a very low loop-gain around the wave frequency.[8] Hence, a) the identical feedback controllers are set as

$$K_i(s) = \left(0.1050 + \frac{0.0008}{s}\right)\frac{1}{s+0.125} \text{ for } i = 0, 1, \ldots, 4;$$

while b) the non-identical feedback controllers are set as

$$\begin{aligned} K_0(s) &= \left(0.1050 + \frac{0.0008}{s}\right)\frac{1}{s+0.125}, \text{ and} \\ K_i(s) &= \left((0.1050 + 0.1i) + \frac{0.0008}{s}\right)\frac{1}{s+0.125} \end{aligned}$$

for $i = 1, \ldots, 4$. Note that in the above design of non-identical feedback controllers $\alpha = 0.1$ ($\mathrm{m}^{-3/2}$) $> \frac{4}{c\tau}(= 0.0904)$.[9] Fig. 7 and 8 give the closed-loop responses to an offtake disturbance in the downstream pool. Clearly, a much better decoupling performance is obtained by the strategy with the non-identical feedback controllers.

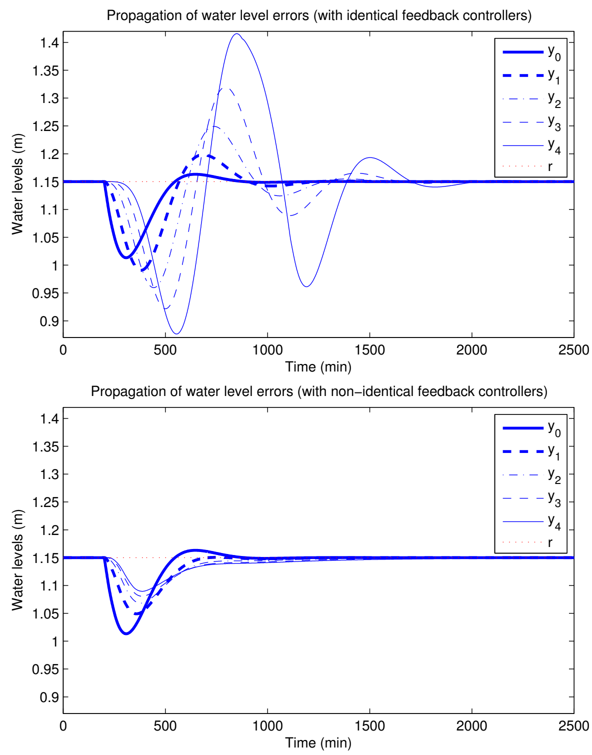

Fig. 7. Water-level error propagation, with identical feedback controllers (top graph) and with non-identical feedback controllers (bottom graph)

[9]Increasing α will improve the global performance. However, as discussed in Section II-C, considering (robust) stability of the upstream subsystems, a value of $\alpha \gg \frac{4}{c\tau}$ is undesirable.

TABLE I
Parameters of the Pool and Saturation Values Set

Pool length (m)		Wave frequency (rad/min)		τ (min)		c $\left(\frac{\mathrm{m}^{3/2}}{\mathrm{min}}\right)$	
3129		0.20		16		2.767	
($\mathrm{m}^{-1/2}$)						(p.u.)	
b_1	b_2	b_3	b_4	b_5	b_6	a_1	a_2
0.13	−0.13	−0.24	0.24	0.11	−0.11	0.70	0.19
Saturations of gate positions				Saturations of flows			
max (m)		1.487		max (Ml/day)		300	
min (m)		0		min (Ml/day)		0	

[7]In the real-life running of an irrigation channel, saturation limits of gate positions are always hard criteria for water-level (or flow) control. Further, the minimum flow is always 0 (i.e. it cannot be negative); while the maximum flow is set by the water management authorities based on the calculation of water capacity.

[8]With such a low-pass filter, the influence of the velocity of gate movement on the interconnections between pools, i.e. the resonance in the flows over gates and the water-levels around the dominant wave-frequency, is dampened, see [17], [22]. Hence the analysis results in Section II still work.

Fig. 7 shows the water-level errors in the five pools when an offtake of 75 Ml/day at the downstream pool begins at time 200 min. The water-level setpoints for the pools are set the same: $r = 1.15$ m. Note that the local water-level error in the first downstream pool (i.e. $r - y_0$) is the same for identical and non-identical feedback controllers. With identical feedback controllers (the top graph), the water-level errors in the pools increase in the upstream direction. In the upstream pool, the maximum water-level error caused by the offtake is 0.28 m. In contrast, with the non-identical feedback controllers (the bottom graph), the water-level errors in the pools decrease in the upstream direction. In the upstream pool, the maximum water-level error caused by the offtake is then 0.06 m.

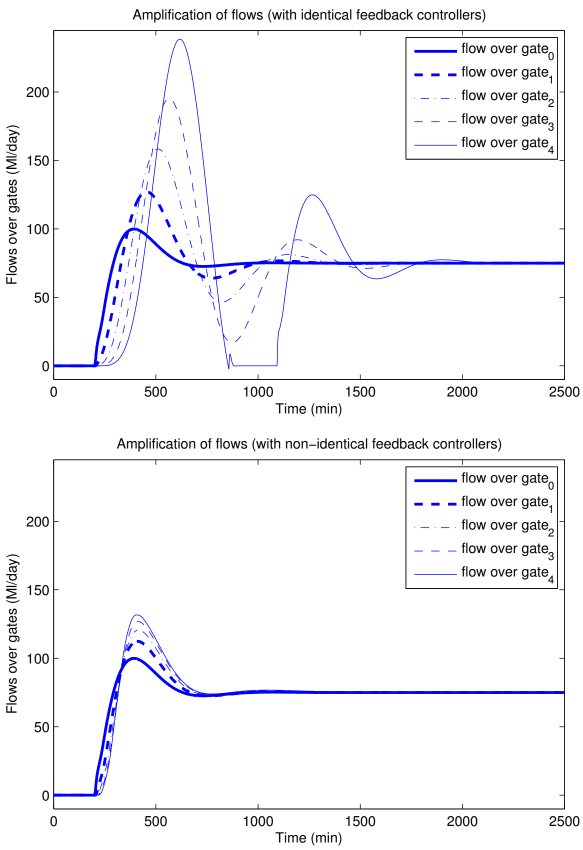

Fig. 8. Flow amplification, with identical feedback controllers (top graph) and with non-identical feedback controllers (bottom graph)

Fig. 8 shows the amplification of flows to compensate the influence of the offtake of 75 Ml/day at the downstream pool begins at time 200 min. With identical feedback controllers (top graph), the amplification of flows is significant, e.g. the maximum flow over the most upstream gate is 240 Ml/day around 600 min; more seriously, the flow over the most upstream gate goes below the minimum limit from 870 min to 1170 min.[10] In the simulation, On the other hand, with non-identical feedback controllers (bottom graph), the amplification of flows over gates is well attenuated, e.g. the maximum flow over the most upstream gate is 130 Ml/day around 450 min. Note that, as expected, the control actions in the upstream pools, i.e. flow over gate$_i$ for $i = 1, \ldots, 4$, in response to the offtake disturbance are faster than those in the case with identical controllers.

[10]No anti-windup augmentation of the linear control system is considered in the simulation. To mitigate the degradation in performance when saturation limits are reached, an anti-windup compensation [12] can be included.

IV. Conclusions

This paper has discussed the design of decentralized feedback controllers for a string of identical pools based on the global performance of managing water-level error propagation and attenuating the amplification of flows over gates in the upstream direction. A definition of string stability in terms of bounded water-level errors and bounded flows has been given. It has been shown that for a string of an infinite number of pools with decentralized distant-downstream feedback control, the closed-loop bandwidth limitation of each subsystem, imposed by the internal time-delay, makes it impossible to achieve string stability. For a string of a finite number of pools, temporal stability is guaranteed if and only if each closed-loop (i.e. each single pool with its feedback controller) is stable. Moreover, by selecting non-identical feedback controllers such that the closed-loop bandwidths of the subsystems increase from downstream to upstream, a better global performance than that with identical feedback controllers is achieved.

Further research will compare the management of the trade-off between the global and local performance attained by the decentralized feedback control strategy given in this paper and that by the distributed control approach introduced in [3], [10], which involves the interaction between pools as a known disturbance of the closed-loop system in the synthesis of the distributed controller. Moreover, it is also of interest to explore the coupling properties of other interconnection topologies of a water network, e.g. the influence of the discharge through side gates on the interactions between pools.

V. ACKNOWLEDGMENTS

The authors would like to thank Prof. Michael Cantoni of the University of Melbourne for his valuable comments regarding this work.

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