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N. Groot, B. De Schutter, and H. Hellendoorn

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Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
phone: +31-15-278.51.19 (secretary)
fax: +31-15-278.66.79
URL: http://www.dcsc.tudelft.nl

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Reverse Stackelberg Games, Part I: Basic Framework

Noortje Groot, Bart De Schutter, and Hans Hellendoorn

Abstract— **The class of reverse Stackelberg games, also known as incentives, embodies a structure for sequential decision making that has been recognized as a suitable approach for hierarchical control problems like road tolling and electricity pricing. In this game, a leader player announces a mapping of the follower's decision space into the leader's decision space, after which a follower player determines his optimal decision variables. Compared to the original Stackelberg game, the reverse Stackelberg approach has several advantages that will be emphasized in this survey. Since the reverse Stackelberg game has been studied in different research areas, first a comprehensive overview is provided of the definition of the game. Further, several areas of application are stated. In the companion paper entitled 'Reverse Stackelberg Games, Part II: Results and Open Issues', main contributions are subsequently summarized along with several characteristics of the game and open issues that are relevant for further research, are presented.**

I. Introduction

Since the introduction of the Stackelberg game in 1934 [1], [2] this hierarchical leader-follower game has broadened its form and application areas to show its diversity in fields from game theory to optimization. While originally introduced in the economic context of duopolies in which one firm has the power to act before the other firm, in the 1970s the Stackelberg game has been recognized as a tool for dealing with large-scale optimization and control problems [3], [4]. In particular, a hierarchical approach can be adopted to distribute a complex problem into layers of sequential problems in order to ease the problem solving, as well as to deal with systems that contain a natural hierarchy. An overview highlighting the relevance of hierarchical control as an alternative to decentralized and distributed approaches can be found in [5].

An interesting related problem is the reverse Stackelberg game, at which the current survey is aimed. Compared to the original Stackelberg game, in the reverse Stackelberg game, the type of leader action is generalized from making a direct decision to determining a *function* that maps the follower's decision space into the leader's decision space. Thus, although the leader remains the first to act by proposing a leader function, her actual decision variables will not be determined until the follower acts and proposes his decisions or control inputs [6]. Most of the literature mentioned in this survey will be on the reverse Stackelberg game as perceived from what may be called a control-theoretic perspective [7]. There, the reverse Stackelberg game is often referred to as an incentive problem.

N. Groot, B. De Schutter, and H. Hellendoorn are with the Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands. E-mail: n.b.groot@tudelft.nl

The reverse Stackelberg game can serve as a structure for multi-level control, where in comparison to the original Stackelberg game that poses a truly top-down hierarchical scheme, there is more communication from the lower levels to the higher levels [6]. In other words, instead of solely responding to a leader input by the choice of a follower decision variable, in the reverse Stackelberg game the follower chooses a *combination* of optimal follower inputs and the associated leader decision variables. Moreover, the reverse Stackelberg game structure is capable of:

- dealing with a nonunique follower response. Whereas in the Stackelberg game the leader can only determine her input and therefore she has no complete control over the (nonunique) response of the follower to this input, in the reverse case, the leader can construct a more advanced leader function inducing the follower to behave as desired [6].
- inferring information on the state in case of incomplete information, by observing the follower's response to the leader function [6]. Similarly, by a sudden deviation from a certain response pattern by the follower, the leader could detect a change in the conditions of the follower. However, it should be noted that in multi-stage games with incomplete information, such derivations are complex.
- inducing multiple followers to behave cooperatively by means of the leader function, while the followers themselves behave noncooperatively [6], [8].

In the following, first the original Stackelberg game will be defined in Section II, after which an overview is provided of different terminology that is used to describe the reverse Stackelberg game. In Section III the computational complexity of the game is considered and possible algorithms to derive numerical solutions are briefly discussed. Several areas of application are finally considered in Section IV, followed by an intermediate conclusion of the current overview in Section V.

II. Background

The reverse Stackelberg game falls within the branch of noncooperative game theory, which deals with players that act individually according their own objectives. Although cooperative game theory is sometimes adopted in control applications [9], most often the controllers or players are perceived as individual and rational operators that do not voluntarily exchange information. When players act simultaneously, the well-known Nash equilibrium is adopted as a solution concept; in such an equilibrium no player can improve his/her situation by unilaterally deviating from the decisions associated with the Nash equilibrium [10]. An equivalent solution concept in the case of sequentially operating players is the Stackelberg equilibrium. Information on general game theory can be found in several classical books [11], [12].

A. Stackelberg Games

1) Static: The basic single-leader single-follower single-stage or static Stackelberg game may be described as follows. Leader and follower decision variables are denoted by $u_{\mathrm{L}} \in \Omega_{\mathrm{L}} \subseteq \mathbb{R}^{n_{\mathrm{L}}}$ and $u_{\mathrm{F}} \in \Omega_{\mathrm{F}} \subseteq \mathbb{R}^{n_{\mathrm{F}}}$, with as cost functions: $\mathcal{J}_p : \Omega_{\mathrm{L}} \times \Omega_{\mathrm{F}} \to \mathbb{R}, p \in \{\mathrm{L}, \mathrm{F}\}$.

In order to determine the leader's optimal input u_{L}^*, she takes into consideration the follower's reaction curve $l_{\mathrm{F}} : \Omega_{\mathrm{L}} \to \Omega_{\mathrm{F}}$, i.e.,

$$u_{\mathrm{F}} = l_{\mathrm{F}}(u_{\mathrm{L}}) \qquad \text{s.t.} \min_{u_{\mathrm{F}} \in \Omega_{\mathrm{F}}} \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}, u_{\mathrm{F}}) = \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}, l_{\mathrm{F}}(u_{\mathrm{L}})).$$

Based on this knowledge, the leader can announce

$$u_{\mathrm{L}}^* \in \arg \min_{u_{\mathrm{L}} \in \Omega_{\mathrm{L}}} \mathcal{J}_{\mathrm{L}}(u_{\mathrm{L}}, l_{\mathrm{F}}(u_{\mathrm{L}})).$$

2) Dynamic: In case of a multi-stage discrete-time game an additional state variable $x(k) \in \mathcal{X} \subseteq \mathbb{R}^{n_x}$ and associated state update equation $x(k+1) = f(x(k), u_{\mathrm{L}}(k), u_{\mathrm{F}}(k), k)$ with initial state $x_0 = x(0)$ should be added that define the global system, with $k \in \{1, 2, \ldots, K\}$, with K the number of stages. The continuous-time equivalent, i.e., the differential game, will be described in Section II-B of the companion paper. One can also distinguish between the state space of leader and follower, i.e., $\mathcal{X} = \mathcal{X}_{\mathrm{L}} \times \mathcal{X}_{\mathrm{F}}$. The general game can now be denoted by a tuple

$$\langle \mathcal{P}, K, \{\mathcal{J}_i\}_{i \in \mathcal{N}}, \{(\Omega_i)^K\}_{i \in \mathcal{N}}, \mathcal{X}^K, f, x_0, \\ \{(\mathcal{I}_i)^K\}_{i \in \mathcal{N}}, \{(\Gamma_i)^K\}_{i \in \mathcal{N}} \rangle,$$

where

- $\mathcal{P} : \mathcal{N} \to \{\mathrm{L}, \mathrm{F}\}$ indicates the type of player associated with each of the N players $i \in \mathcal{N} = \{1, ..., N\}$.
- $\mathcal{I}_i \subseteq \mathbb{R}^{n_i \times N}$ denotes the information space of player i.
- $\Gamma_i \subseteq \{\gamma_i | \gamma_i : \mathcal{I}_i \to \Omega_i\}$ denotes the strategy space.

To elaborate, $\iota_i(k) \in \mathcal{I}_i$ captures the knowledge that is available to player i at stage k: it can contain information of the other players' decision space and objective function, state, etc. Finally, the strategy space refers to the permissible mappings $\gamma_i(k) : \mathcal{I}_i \to \Omega_i$ that leads to a decision $u_i(k)$: both the reverse and original Stackelberg thus fit in this general description, depending on how one defines the interaction between the players. Strategies will thus be dependent on previous actions of some (other) player(s), where:

- *Open-loop* decisions are a function of time and of the initial conditions only, where the leader function in the reverse Stackelberg game is in addition dependent on the follower's decision: $\gamma_{\mathrm{L}}(k) = g_{\mathrm{ol}}(x_0, u_{\mathrm{F}}(k), k)$.
- *Closed-loop* decisions are defined to be a function of time and of the state and decision variables $x(k-\alpha), ..., x(k), u_{\mathrm{F}}(k-\alpha), ..., u_{\mathrm{F}}(k)$ of some previous – and the current – $\alpha + 1$ stages in discrete time.
- State *feedback* strategies can be seen as a special type of closed-loop decision with one-step memory, i.e., the strategies depend on the previous state and decision variables, with $\gamma_{\mathrm{L}}(k) = g_{\mathrm{fb}}(x(k-1), u_{\mathrm{F}}(k-1), k)$.

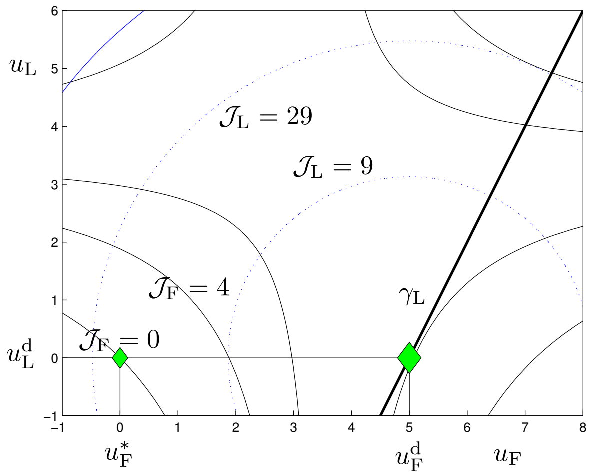

Fig. 1. Graphical representation of Example 1.

Further, also cases with several players operating on a single level can be considered, as well as cases in which the leader has incomplete, partial information on the follower, or on an uncertain state. For an overview of results on the original Stackelberg game, please refer to [13].

B. Reverse Stackelberg Games in Several Fields

Briefly stated, in a reverse Stackelberg game the leader first proposes instead of the decision u_{L}, a function $\gamma_{\mathrm{L}} : \Omega_{\mathrm{F}} \to \Omega_{\mathrm{L}}$. Thus, the associated leader decision u_{L} is determined when the follower chooses his input u_{F}. This strategy mapping is based on knowledge of the follower's reaction curve to this function, i.e., the follower will determine his decision u_{F}^*, where the superscript $*$ denotes optimality:

$$u_{\mathrm{F}}^* \in \arg \min_{u_{\mathrm{F}} \in \Omega_{\mathrm{F}}} \mathcal{J}_{\mathrm{F}}(\gamma_{\mathrm{L}}^*(u_{\mathrm{F}}), u_{\mathrm{F}}).$$

Keeping this information into account, the leader then determines an optimal leader function:

$$\gamma_{\mathrm{L}}^*(\cdot) \in \arg \min_{\gamma_{\mathrm{L}}(\cdot) \in \Gamma_{\mathrm{L}}} \mathcal{J}_{\mathrm{L}}(\gamma_{\mathrm{L}}(u_{\mathrm{F}}^*(\gamma_{\mathrm{L}}(\cdot))), u_{\mathrm{F}}^*(\gamma_{\mathrm{L}}(\cdot))). \quad (1)$$

Example 1: (Adopted from [14]) In order to illustrate the Stackelberg and reverse Stackelberg games, consider the following simple example also depicted in Fig. 1. Consider a static, single-leader single-follower reverse Stackelberg game. Let the objective functions be:

$$\mathcal{J}_{\mathrm{L}}(u_{\mathrm{L}}, u_{\mathrm{F}}) = (u_{\mathrm{F}} - 5)^2 + u_{\mathrm{L}}^2, \\ \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}, u_{\mathrm{F}}) = u_{\mathrm{L}}^2 + u_{\mathrm{F}}^2 - u_{\mathrm{L}} u_{\mathrm{F}},$$

with decision variables $u_{\mathrm{L}} \in \mathbb{R}, u_{\mathrm{F}} \in \mathbb{R}$. The leader's global optimum is $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}) = (0, 5)$. In the original Stackelberg formulation, the follower's response to the desired variable $u_{\mathrm{L}}^{\mathrm{d}} = 0$ would be the suboptimal $u_{\mathrm{F}}^* = 1/2 u_{\mathrm{L}} = 0$.

However, under the leader function

$$u_{\mathrm{L}} = \gamma_{\mathrm{L}}(u_{\mathrm{F}}) = 2u_{\mathrm{F}} - 10,$$

the follower's response will indeed be:

$$\arg\min_{u_{\mathrm{F}}\in\Omega_{\mathrm{F}}}\mathcal{J}_{\mathrm{F}}(u_{\mathrm{F}})=\arg\min_{u_{\mathrm{F}}\in\Omega_{\mathrm{F}}}(2u_{\mathrm{F}}-10)^2+u_{\mathrm{F}}^2+(2u_{\mathrm{F}}-10)u_{\mathrm{F}}=5.$$

This situation is depicted in Fig. 1 where several level curves are plotted; the leader's optimum $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$ is in the center of the dotted level curves for $\mathcal{J}_{\mathrm{L}}$. The contours centered around the four corners of the plotted decision space represent the level curves of $\mathcal{J}_{\mathrm{F}}$. The follower's optimal response to $u_{\mathrm{L}}^{\mathrm{d}}$ and $\gamma_{\mathrm{L}}(u_{\mathrm{F}})$ are respectively u_{F}^{*} and $u_{\mathrm{F}}^{\mathrm{d}}$ for the original versus the reverse Stackelberg game.

1) Generalized and Inverse Stackelberg Games: The first step towards the reverse Stackelberg game formulation may be found in [15] where a generalized strategy is introduced that leads to the best solution the leader can achieve in case the follower's response to the original Stackelberg decision u_{L} is nonunique. In the original Stackelberg game, uniqueness of the follower response is usually assumed at a loss of generality, in order to simplify the problem. Formally, if for every $u_{\mathrm{L}} \in \Omega_{\mathrm{L}}$, the follower's reaction set

$$\Omega_{\mathrm{F}}^{*}(u_{\mathrm{L}})=\{u_{\mathrm{F}}^{*}\in\Omega_{\mathrm{F}}|\mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}},u_{\mathrm{F}}^{*})\leq\mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}},u_{\mathrm{F}})\forall u_{\mathrm{F}}\in\Omega_{\mathrm{F}}\}$$

is not empty, and if $\exists u_{\mathrm{Lgen}}^{*} \in \Omega_{\mathrm{F}}$ s.t.

$$\sup_{u_{\mathrm{F}}\in\Omega_{\mathrm{F}}^{*}(u_{\mathrm{Lgen}}^{*})}\mathcal{J}_{\mathrm{L}}(u_{\mathrm{Lgen}}^{*},u_{\mathrm{F}})=\min_{u_{\mathrm{L}}\in\Omega_{\mathrm{L}}}\sup_{u_{\mathrm{F}}\in\Omega_{\mathrm{F}}^{*}(u_{\mathrm{L}})}\mathcal{J}_{\mathrm{L}}(u_{\mathrm{L}},u_{\mathrm{F}})=\mathcal{J}_{\mathrm{L}}^{*},$$

then u_{Lgen}^{*} is called a generalized Stackelberg strategy, i.e., it leads to the least upper bound on $\mathcal{J}_{\mathrm{L}}$ [15].

It should be noted that the generalized strategy basically results in a reduced set of possible Stackelberg solutions for the leader that constitute an upper bound to her objective function value. The generalized strategy thus deals with the problem of a nonunique follower response by accepting a solution that leads to a reduced performance for the leader. In contrast, the reverse Stackelberg game deals with a nonunique response by substituting the leader strategy u_{L} with a more complex function γ_{L}.

The term reversed Stackelberg game first appeared in [6] where it was chosen to explain the order of first announcing the leader strategy γ_{L} (rather than her action u_{F} as in the original Stackelberg formulation), followed by the follower's actual action or decision u_{F}, from which u_{L} follows. Instead of approaching only an upper bound on $\mathcal{J}_{\mathrm{L}}$ by using the generalized strategy, the leader may in fact be able to reach exactly her desired equilibrium.

As additional reasons for adopting a reverse structure, it is mentioned [6] that (1) the leader may infer information on the state from knowing the follower's decision first, especially in a stochastic setting in which the leader does not possess the follower's full information, and that (2) the follower's decision may directly affect the leader's objective function value. However, to the latter argument it should be added that also in the original Stackelberg game $\mathcal{J}_{\mathrm{L}}$ is dependent on u_{F}. The reverse structure however provides more power to the leader to enforce her desired solution, as compared to providing only the decision u_{L}.

Most recently, the game in which the leader announces a strategy as a mapping $\Omega_{\mathrm{F}} \to \Omega_{\mathrm{L}}$ has been studied as the 'inverse Stackelberg game' [14], [16], [17].

2) Theory of Incentives: Theory of incentives, also known as contract theory, involves principal-agent problems in which some quantity produced by the agent or follower is exchanged for a (monetary) transfer by the principal or leader. A new element of information is considered, i.e., the so-called type of an agent that refers to e.g., skills or opportunity cost. The agent may not reveal his type to the principal or he may even provide false characteristics. Therefore, an aspect of paramount importance within this area is uncertainty due to a lack of information. The three main types of principle-agent problems can be divided into moral hazard, adverse selection, and signaling. Here, the agent has either (1) private information concerning actions that occur after the signing of a contract, (2) private information concerning his type before the composition of the contract, or (3) the ability to send information to the principal during the game [18].

Although controller agents are usually assumed to provide their available information truthfully, results from incentives theory concerning incomplete information can provide useful insight to the reverse Stackelberg game formulation in control settings [19].

Another important part of the problem definition in incentives theory is the participation constraint or bail-out option of the follower, which allows him to withdraw from participating in the game in case the leader proposes a contract that leaves the follower with insufficient performance. This constraint does not directly appear in the reverse Stackelberg game formulations mentioned in Sections II-B.1 and II-B.3.

3) Incentive Strategies: From a control-theoretic rather than economic perspective, the leader strategy is often called incentive [7], [19]; as in Section II-B.2, the term is chosen to indicate the problem of how the leader can incentivize the follower to perform as desired. Different from the leader function as described in Section II-B.1, the incentive strategy is not always a mapping $\Omega_{\mathrm{F}} \to \Omega_{\mathrm{L}}$; some authors define the incentive strategy more generally as a function of the available information [19], or solely of the state variables x [7], [20], [21]. In fact, in [21] the use of state feedback is motivated by argument that it is unrealistic to have access to the follower's decision variables in a real-life dynamic setting. At the same time however, some authors consider such state-dependent leader function as a regular (feedback or closed-loop) Stackelberg strategy without mentioning the concept of incentives [3], [4], [22].

The incentives information structure has also been considered as a fourth alternative along with the open-loop, closed-loop, and feedback information structure in a multi-stage context [23]. Although the last three patterns are indeed only relevant in a dynamic framework, the reverse Stackelberg game or incentives structure with $u_{\mathrm{L}} = \gamma_{\mathrm{L}}(u_{\mathrm{F}})$ can however very well occur in a single-stage context without the introduction of a state variable.

A link has also been made between incentives and social choice theory [7]. In social choice theory, agents need to propose an ordering of preferences (e.g., in the voting for elections) based on which a final listing (the solution or election outcome) is developed, depending on a predetermined choice rule [24]. In order to make people reveal their true preferences, the choice rule should be strategy-proof. In [7] the equivalence is stated between a leader function γ_{L} of a reverse Stackelberg game and the social choice rule that allocates a final ordering (solution) to a preference ordering (decision variables) of the agents in a strategy-proof manner. However, there is no desired outcome (election order) that the leader strives after in social choice theory, as opposed to in the reverse Stackelberg game where the leader optimizes $\mathcal{J}_{\mathrm{L}}$, which is directly dependent on u_{F}. Therefore the proposed resemblance with a reverse Stackelberg game does not completely fit.

In order to put the different terms in perspective, the incentive problem of determining the leader function γ_{L} to induce the follower to behave as desired can be seen as a part of the overall reverse Stackelberg game described at the start of Section II-B.

III. Computational Complexity and Solution Approach

The general reverse Stackelberg or incentive problem is difficult to solve analytically [14], [16], [25]. Reasons mentioned are the occurrence of composed functions, i.e., intertwined expressions as in (1), and the existence of multiple solutions and nonunique follower responses. The latter problem is eased by taking the determination of the desired solution of the leader as a separate problem that should be solved a priori [4], [6], [7]. In the original Stackelberg game it is often assumed that additionally, the follower has a unique response or that the leader is indifferent between the possible follower responses. While the former assumption can be made without loss of generality, the latter assumption is exactly what we like to circumvent by considering the reverse Stackelberg game.

As for the formal complexity of the reverse Stackelberg game, it should be noted that the Stackelberg game can be written into a bilevel programming problem, in which the follower's lower-level optimization problem is considered as a constraint to the higher level optimization problem [26], [27]. Different from the perspective of the Stackelberg game, bilevel programming is focused rather on the computation of a Stackelberg solution, where the sequential nature of the game is translated into constraints. Similarly, cases of incomplete information should be translated into a formal multilevel mathematical program: in the original bilevel program, perfect information is assumed [28]. While the resemblance with the original Stackelberg game is often mentioned in the literature on multilevel programming, a link with the reverse game does not appear. Nonetheless, the reverse game is subject to the same hierarchical structure, where in addition the relation between u_{L} and u_{F} is captured by γ_{L}. In other words, the Stackelberg game is a special case of the reverse game: for a relation $\gamma_{\mathrm{L}} : \Omega_{\mathrm{F}} \to u_{\mathrm{L}}^{\mathrm{d}}$, the reverse game reduces to a Stackelberg game. Hence, results on multilevel programming [29], [27] could prove useful for the analysis of the reverse Stackelberg game.

In general, linear bilevel hence multilevel programming is proven NP-hard [30] and later strongly NP-hard [31]. A more elaborate complexity analysis of multilevel programming can be found in [32]. In other words, it is generally assumed that no polynomial-time algorithm exists that can solve the general problem to optimality. Hence, the problem instance should be sufficiently small in order to find a global optimum, or one should adopt heuristic methods. In Section III-B below an overview of possible algorithms for multilevel programming problems is presented.

A. Analytic Solution Approach

In order to do be able to solve the reverse Stackelberg game, a common, indirect approach is the following [4], [6], [7]. Given a desired solution that the leader seeks to achieve, i.e., a globally optimal solution in case of minimization:

$$(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}) \in \arg \min_{u_{\mathrm{L}} \in \Omega_{\mathrm{L}}, u_{\mathrm{F}} \in \Omega_{\mathrm{F}}} \mathcal{J}_{\mathrm{L}}(u_{\mathrm{L}}, u_{\mathrm{F}}),$$

the reverse Stackelberg problem reduces to finding a function $\gamma_{\mathrm{L}} : \Omega_{\mathrm{F}} \to \Omega_{\mathrm{L}}$ such that the follower's unique response coincides with the desired decision variable. Thus, (1) $u_{\mathrm{L}}^{\mathrm{d}} = \gamma_{\mathrm{L}}(u_{\mathrm{F}}^{\mathrm{d}})$ and (2) $u_{\mathrm{F}}^{\mathrm{d}} = \arg\min_{u_{\mathrm{F}} \in \Omega_{\mathrm{F}}} \mathcal{J}_{\mathrm{F}}(\gamma_{\mathrm{L}}(u_{\mathrm{F}}), u_{\mathrm{F}})$. This optimum is often referred to as 'team optimum' according the theory of teams [33] where it refers to the best the leader can obtain if the other players support her. The term team optimum is therefore slightly misleading as a substitute for the leader's global optimum. Further, it should be noted that it may be difficult to compute such a globally optimal equilibrium point in the case of incomplete information on e.g., the follower's decision space.

If the leader is able to induce the follower to arrive at the desired solution $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$ (by an affine function γ_{L}), the problem is called (linearly) incentive-controllable [7]. This term stems from the theory of incentives, where it is used to indicate whether the follower can be induced to reveal his true information in case the leader is unable to observe his actions. Similarly, 'incentive compatibility' indicates whether a game or strategy is strategy-proof, i.e., whether the players are induced to act truthfully in spite of asymmetric information [18]. It should thus be noted that the term is used differently in the context of reverse Stackelberg games in which the leader may have full information concerning the follower.

B. Numerical Solution methods

While research on the reverse Stackelberg game from a game-theoretical or even control-theoretical perspective has focused rather on obtaining analytical solutions of the leader function, for available numerical solution methods, inspiration can be gained from multilevel programming. Available solution methods for such problems can be categorized as extreme point algorithms, branch-and-bound algorithms, complementary pivot algorithms, descent methods, and penalty function methods [26]. More references to algorithms for multilevel programming can be found in [34].

Alternatively, genetic programming can be adopted, as described for the case of incomplete information in Section II-D of the companion paper. An overview with references on genetic algorithms for multilevel programs can be found in [34]. Here, a genetic algorithm was developed for general multilevel Stackelberg games in which players on a single level play a noncooperative simultaneous (Nash) game, without assumptions regarding linearity, convexity, continuity and differentiability. Here, however, both the follower's response is assumed to be a singleton and the leader is assumed to be indifferent in case her response is nonunique; the relaxation thereof leads exactly to the need for a reverse Stackelberg formulation as explained in Section I. Although the method is able to find a global optimum for the general game, the approach is computationally still rather inefficient.

A more recent development is in using multi-parametric programming methods for multilevel optimization [35], in which each subproblem is stated as a multi-parametric programming problem with parameters linked to other subproblems. The complexity of the overall problem is thus broken down to the computation of the reaction set of each subproblem. Under study of the linear quadratic case, this approach results in a single-level convex optimization problem; efficient methods for general nonlinear and nonconvex multilevel problems are however far from widespread.

IV. Application Areas

Hierarchical control can be roughly divided in cases in which a natural division in multiple levels exists and those where a hierarchical structure is adopted as an alternative to strictly distributed optimization in order to facilitate the problem solving [5]. The latter approach arises in large-scale control problems where information is not automatically available to all controllers and difficult or costly to communicate. Hence, the overall problem is decomposed where the higher-level controller acts as a coordinator.

Looking into more specific applications of the reverse Stackelberg game; next to contracting and pricing problems as studied in the area of incentives and management science (Section II-B.2, [36]) the following application areas have been considered:

- Network pricing. In [37] a reverse Stackelberg game is used to model a situation in which an internet service provider is considered as a leader that sets a price for the bandwidth used by followers, where the price is dependent on the actual bandwidth used. Both complete and incomplete information on the type of users as denoted by a constant parameter $w \in \mathbb{R}$ is studied, both for which $\epsilon-$optimal, nonlinear strategies are obtained, with ϵ an arbitrarily small constant. Such strategy will lead to an equilibrium that is within an ϵ deviation from the leader's optimal solution. The ϵ-optimal solutions are obtained by making a small deviation in the leader function from the desired but unattainable optimum, i.e., by substituting $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$ by $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}-\epsilon)$, which becomes the new equilibrium point.
  For the rather specific instance of the problem with $u_{\mathrm{L}}, u_{\mathrm{F}}$ scalar and $\mathcal{J}_{\mathrm{F}} = w\log(1+u_{\mathrm{F}}) - \frac{1}{1-u_{\mathrm{F}}} - u_{\mathrm{L}}$, it is shown in [37] that such approximate solutions can always be found, e.g., for a function $\gamma_{\mathrm{L}}(u_{\mathrm{F}}) = a_1 u_{\mathrm{F}} + a_2 u_{\mathrm{F}}^2$, a_1, a_2 constants. It should be noted that the proposed $\epsilon-$optimal solution is therefore not feasible in general, i.e., for vectors $u_{\mathrm{L}}, u_{\mathrm{F}}$.
- Road tolling. In [17] a dynamic toll design problem is considered on a three-link highway network where one of the links is not subject to a toll. In order to minimize total travel time and thus reduce congestion, a toll proportional to the traffic flow is proposed. It is shown that this reverse Stackelberg game formulation yields a better performance for the road authority compared to a constant or time-varying toll. However, the three links have the same origin and destination, where the route choice of the homogeneous traffic considered is solely based on the tolls. Hence, the road authority does not face any uncertainties in the studied setting.
- Electricity pricing. In [38] reverse Stackelberg games are used to model the pricing of electricity in a stochastic peak load problem where the leader proposes an electricity price p consisting of a fixed charge c and a unit price c^{var} that is based on the varying demand d that represents the follower's decision, i.e., $p := u_{\mathrm{L}} = \gamma_{\mathrm{L}}(u_{\mathrm{F}}) = c^{\mathrm{var}} u_{\mathrm{F}} + c$. As opposed to the previous application, now the game is played for several cycles n, each consisting of an m-stage game. The values of c_{nm}^{var} and c_{nm} are computed as functions of the values of d, p, and the randomly distributed state ξ of previous stages in the current cycle, or in the previous cycle.
  When the load adaptive pricing strategy is adopted within a dynamic infinite-horizon setting, stability of the system to small disturbances in u_{L} and u_{F} is proven as well as the convergence of the coefficients that determine p and c within γ_{L}.

It should be noted that a large number of publications is available in the areas enumerated above that consider a standard hierarchical game, i.e., that adopt the original Stackelberg structure. An overview of applications of Stackelberg (differential) games in supply chain management and marketing settings can be found in [39]. Also the areas of road pricing [40] and electricity markets [41] are considered.

Overall, while the use of Stackelberg games in large-scale control systems is proposed [3], [4], not many results actually consider this; applications are rather adopted as a means to illustrate the Stackelberg concept. Nonetheless, as indicated in [5] hierarchical methods can indeed pose a viable approach to structure also large-scale control problems. We therefore see potential in applying the reverse Stackelberg game also in those areas to which the original Stackelberg game is applied, while real-life application remains a challenge for both the Stackelberg and reverse Stackelberg framework.

V. Conclusion

An overview has been presented to clarify the concept of the reverse Stackelberg game within several research areas, as well as to emphasize its potential for application in the field of control, while taking into account the complexity of the game. An overview of main results in the literature concerning reverse Stackelberg games as well as an analysis of open issues for further research is provided in the companion paper 'Reverse Stackelberg Games, Part II: Results and Open Issues'.

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