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# Micro-Ferry Scheduling Problem with Charging and Embarking Times

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**Abstract:** This paper considers a variant of the travelling salesman problem where both energy consumption and variable travel speeds are taken into account. The problem describes the scheduling of pick-ups and deliveries of passengers with micro-ferries, where the energy consumption is dependent on the speed of the ferries. The schedule should be such that the ferries do not run out of energy during a trip, and time-window misfits, travel times, and waiting times for passengers are minimised. Scheduling of many transportation requests is made possible by including the charging of the ferries in the scheduling procedure, whereas the inclusion of embarking and disembarking times ensures that the passengers can board the ferry comfortably.

*Keywords:* Scheduling, Energy Consumption, Variable Speed, Travelling Salesman Problem

## 1. INTRODUCTION

In this paper the modelling of a scheduling problem for micro-ferries is discussed. The ferries are used to transport customers between several locations along the water, and the customers can provide a desired time-window for picking them up. The aim of the micro-ferry scheduling problem is to find a schedule that minimises the energy consumption, while assuring that the micro-ferries do not run out of energy. The challenge in the micro-ferry scheduling problem lies in the consideration of the energy consumption, and the possibility to vary the speed of the ferry for each transport. More common objectives as minimising travel times, waiting times, and time-window misfits are also included. It is a variant of the travelling salesman problem (TSP) (Bektaş, 2006; Laporte, 1992) with varying travel times, where the energy consumption is minimised and charging (refuelling) is taken into account.

Traditionally the TSP and its variants —like the vehicle routing problem (Kulkarni and Bhave, 1985; Toth and Vigo, 2001) and pick-up and delivery problem (Savelsbergh and Sol, 1995)— are concerned with minimising the travelled distance. These results do not take into account that vehicles can often move at different speeds, thereby affecting the travel times and possibly other characteristics that might be optimised. Recently the literature shows some work regarding routing and scheduling problems where the speed of vehicles and energy consumption becomes important. In (Bektaş and Laporte, 2011) the pollution routing problem has been proposed, which is a vehicle routing problem taking into account the pollution caused by the vehicles, depending on both the speed and load of a vehicle. The objective is to optimise the routing while con-

sidering the travel distance, greenhouse emissions, travel times, and costs. The speed of the vehicle is fixed for each road, and it is not explicitly used as an optimisation variable. Energy consumption is considered by Kara et al. (2007), who define the energy-minimising vehicle routing problem by considering the load of the vehicles, and arguing that minimising the energy consumption is similar to minimising the product of the load and the travelled distance. The speed of the vehicle is not taken into account in determining the energy consumption. In (Xiao et al., 2012) a vehicle routing problem is discussed where the fuel cost (= unit fuel cost x road-dependent fuel consumption rate x road length) instead of the road length is used as the constant cost term for travelling.

Instead of the (constant) distance that is used as a cost in the TSP, we consider a (variable) speed-dependent energy consumption as a cost. In (Burger et al., 2012) we first proposed the micro-ferry scheduling problem with soft time-windows. If feasible, the proposed method ensures that the transportation requests are spread over the micro-ferries such that they do not run out of energy while handling a request. In this paper the work is extended by including the possibility to recharge the micro-ferries in between the requests, and by taking into account the time needed for embarking and disembarking the micro-ferries. Including the charging ensures that there will never be too little energy for handling all requests, and therefore it allows the scheduling of more requests than the micro-ferries could handle based on the current energy levels.

This paper is organised as follows. Section 2 gives the formulation of the micro-ferry scheduling problem by introducing concepts and variables that will be used, based on two separate networks; one describes the physical network with stations as nodes and routes as arcs, the other describes a virtual network with transportations as nodes and relocations as arcs. Section 3 gives a summary of the work presented in (Burger et al., 2012); the extensions

introduced in this paper are given in Section 4. To solve the scheduling problem efficiently a linear programming approximation is used, as described in Section 5. The resulting mixed-integer linear program is tested in simulation, and the results are discussed in Section 6, followed by conclusions in Section 7.

## 2. PROBLEM FORMULATION

The problem of scheduling the micro-ferries can be solved by considering two distinct networks. With micro-ferries we mean small, autonomous water-taxis that can transport a small number of passengers from and to multiple locations along the water. They receive their orders of whom to pick up where and when and where to deliver the passengers from a central location, for which we propose a model in this paper. First we describe the physical network in which the micro-ferries travel to transport the customers. Next a virtual network is used to model the transportation requests (by nodes) and the order in which they are handled (by arcs).

### 2.1 Description of the physical network

We consider a network consisting of $L$ locations along the water where the $M$ micro-ferries can moor to pick-up and deliver customers. The number of transportation requests is denoted by $R$. The sets $\mathcal{L} = \{1, \ldots, L\}$, $\mathcal{M} = \{1, \ldots, M\}$, and $\mathcal{R} = \{1, \ldots, R\}$ are associated with the locations, micro-ferries, and requests respectively.

The operational speed of the micro-ferries is variable, and bounded by the interval $[\underline{u}, \overline{u}]$, with $0 < \underline{u} < \overline{u}$. The path lengths between locations $p, q \in \mathcal{L}$ is given by $\mathsf{l}_{pq} \geq 0$; we have $\mathsf{l}_{pq} = 0$ if and only if $p = q$.

Within this network the customers can make transportation requests to be brought from location $p_j \in \mathcal{L}$ to location $q_j \in \mathcal{L}$ within a desired time interval $[\mathsf{t}_{\mathsf{a},j}, \mathsf{t}_{\mathsf{b},j}]$ for the pick-up to take place, where $j \in \mathcal{R}$ denotes the request number.

The set $\mathcal{R}$ consists of two types of requests: *current requests* and *new requests*. The set $\mathcal{M}$ corresponds to the current requests that the $M$ micro-ferries are handling at the moment the scheduling problem is to be solved; if the micro-ferry is waiting at a location we model this as an 'empty request' with both the pick-up and delivery location equal to the current location of the micro-ferry. The set $\mathcal{N} = \{M + 1, \ldots, M + N\}$ denotes the new request that are not handled yet. The set $\mathcal{R}$ is defined as

$$\mathcal{R} := \mathcal{M} \cup \mathcal{N} = \{1, \ldots, M, M + 1, \ldots, R\}, \qquad (1)$$

with $R = M + N$ the total number of requests.

### 2.2 Description of the virtual network

The scheduling problem associated with the physical network described above consists of finding assignments of requests to micro-ferries such that

(1) each request is handled by one (and only one) ferry;
(2) the energy consumption of the ferries is minimised;
(3) it is guaranteed that ferries do not run out of energy while handling a request;
(4) the pick-ups for the requests should (preferably) be within the desired time-interval.

The problem can be represented by a graph $\mathcal{G} = (\mathcal{R}, \mathcal{A})$ where $\mathcal{R} = \{1, \ldots, R\}$ is a set of nodes associated with the requests, and $\mathcal{A} = \{(i, j) : i, j \in \mathcal{R}\}$ is a set of arcs connecting the nodes. There are two types of nodes; one associated with the $M$ current requests and one associated with the $N$ new requests.

*Node properties* Each node $j \in \mathcal{R}$ is associated with several variables, such as the index $k_j \in \mathcal{M}$ of the micro-ferry that will handle the request, the energy level $e_j \in \mathbb{R}_+$ of the micro-ferry after completion of request $j$, and the scheduled starting time $t_j \in \mathbb{R}$ (the time at which the customer is picked up). Furthermore, we associate a cost

$$c_{jj} := \mathsf{l}_{p_j q_j}, \qquad (2)$$

indicating the distance from the pick-up location $p_j$ to the delivery location $q_j$ of request $j$.

*Arc properties* Associated with each arc $a \in \mathcal{A}$ are binary variables $x_{ij} \in \{0, 1\}$ indicating whether ($x_{ij} = 1$) or not ($x_{ij} = 0$) request $j$ is handled directly after request $i$ (by the same micro-ferry), and constants $c_{ij} \in \mathbb{R}_+$ indicating the 'cost' to schedule request $j$ after request $i$. This cost equals the distance needed to travel from the delivery location $q_i$ of request $i$ towards the pick-up location $p_j$ of request $j$; when request $j$ directly succeeds request $i$, the micro-ferry has to travel without a passenger aboard over a distance

$$c_{ij} := \mathsf{l}_{q_i p_j}. \qquad (3)$$

If the locations $q_i$ and $p_j$ are the same, we have $c_{ij} = 0$. When $x_{jj} = 1$ for $j \in \mathcal{M}$ a micro-ferry is not assigned a new request after completing its current one. We say that the specific micro-ferry was assigned an *empty request*.

## 3. THE MICRO-FERRY SCHEDULING PROBLEM

Based on the network description given in Section 2, the mathematical model of the micro-ferry scheduling problem is developed. First a summary of the results described in (Burger et al., 2012) is given, split into the objective function and constraints of the problem. In Section 4 two extensions will be introduced, namely the modelling of charging of the micro-ferries, and the inclusion of embarking and disembarking times.

### 3.1 Objective function

Four distinct objectives are considered for the optimisation problem at hand, each of which can be modelled by a function that should be minimised. The optimisation problem becomes a trade-off between the *energy consumption*, the *empty-travel distance* of the micro-ferries, the *total travel time* of the customers, and the *time-window misfit* of the schedule. The relative importance of the four objectives can be influenced by using weighting variables $\alpha_{\mathsf{ec}}, \alpha_{\mathsf{et}}, \alpha_{\mathsf{tt}}, \alpha_{\mathsf{tw}} \geq 0$ in the objective function

$$J = \alpha_{\mathsf{ec}} J_{\mathsf{ec}} + \alpha_{\mathsf{et}} J_{\mathsf{et}} + \alpha_{\mathsf{tt}} J_{\mathsf{tt}} + \alpha_{\mathsf{tw}} J_{\mathsf{tw}}, \qquad (4)$$

where the details of the objective functions $J_{\mathsf{ec}}$, $J_{\mathsf{et}}$, $J_{\mathsf{tt}}$ and $J_{\mathsf{tw}}$ of the energy consumption, empty-travel distance, total travel time, and time-window misfit are given next.

*Energy consumption* The power of a micro-ferry can be modelled by a second-order polynomial in the vehicle speed $u_j$, written as (Burger et al. (2012))

$$P(u_j) := \mathsf{p}_2 u_j^2 + \mathsf{p}_1 u_j + \mathsf{p}_0, \qquad (5)$$

where $\mathsf{p}_0, \mathsf{p}_1, \mathsf{p}_2 \geq 0$ are constants for a specific micro-ferry model. The group of micro-ferries is assumed to be uniform; the extension to multiple types with different characteristics is considered for future work. The index $j$ denotes the request number the speed is associated with; the speed $u_j$ is an optimisation variable bounded by the interval $[\underline{\mathsf{u}}, \overline{\mathsf{u}}]$, and it is constant during request $j$. Therefore, we can calculate the energy consumption by multiplying the power (5) with the duration of the request. The time $T_{ij}(u_j)$ it takes to perform request $j$ when it succeeds request $i$ can be found by dividing the travelled distance $\mathsf{C}_{ij}$ by the travel speed $u_j$; the distance $\mathsf{C}_{ij}$ is the sum of the distance from the delivery location of request $i$ towards the pick-up location of request $j$ (the *relocation distance* $c_{ij}$ defined in (3)), and the distance from the pick-up location towards the delivery location of request $j$ (the *transportation distance* $c_{jj}$ defined in (2)). The energy consumption $\epsilon_{ij}$ is given by

$$\epsilon_{ij}(u_j) := P(u_j)T_{ij}(u_j) = (\mathsf{p}_2 u_j^2 + \mathsf{p}_1 u_j + \mathsf{p}_0)\frac{c_{ij} + c_{jj}}{u_j}$$

$$= (\mathsf{p}_2 u_j + \mathsf{p}_1 + \mathsf{p}_0\frac{1}{u_j})\mathsf{C}_{ij}. \qquad (6)$$

The energy consumption $\epsilon_{ij}$ in (6) represents the energy that *would be consumed* when request $i$ precedes request $j$. The binary optimisation variable $x_{ij}$ introduced in Section 2.2.2 indicates whether ($x_{ij} = 1$) or not ($x_{ij} = 0$) request $j$ succeeds request $i$. Therefore, the amount of energy that *will be consumed* during request $j$ can be calculated as

$$\epsilon_j(u_j) := (\mathsf{p}_2 u_j + \mathsf{p}_1 + \frac{\mathsf{p}_0}{u_j}) \sum_{i=1}^{R} \mathsf{C}_{ij} x_{ij}, \qquad (7)$$

resulting in the total amount of consumed energy given by

$$J_{\text{ec}} := \sum_{j=1}^{R} \epsilon_j = \sum_{j=1}^{R}\sum_{i=1}^{R}(\mathsf{p}_2 u_j + \mathsf{p}_1 + \frac{\mathsf{p}_0}{u_j})\mathsf{C}_{ij} x_{ij} \qquad (8)$$

$$= \sum_{j=1}^{R}\sum_{i=1}^{R} \epsilon_{ij}(u_j)x_{ij},$$

which is the objective function associated with the energy consumption. Note that the 'cost' terms $\epsilon_{ij}$ as defined in (6) are not constant, since they depend on the optimisation variable $u_j$.

*Empty-travel distance*    The distance a micro-ferry has to travel to relocate from the delivery location of one request towards the pick-up location of the next request results in undesired costs for the owner, since there are no passengers on board during these trips. Therefore, another objective is to minimise this empty travelling. Since the vehicles will consume energy while relocating, the objective function (8) already penalises empty travel, but one might want to put more emphasis on it.

The relocation distance between request $i$ and $j$ is given by the constant $c_{ij}$; the total empty-travel distance is found by summing up the travel costs [1]

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[1] This function includes the terms $c_{jj}x_{jj}$ with $c_{jj}$ the transportation distance of request $j$. Nonetheless, the terms $c_{jj}x_{jj}$ will equal zero for all $j \in \mathcal{R}$: when $j \in \mathcal{M}$ we can have $x_{jj} = 1$ (indicating an empty request) but then $c_{jj} = 0$ since the pick-up and delivery location

$$J_{\text{et}} := \sum_{j=1}^{R}\sum_{i=1}^{R} c_{ij} x_{ij}, \qquad (9)$$

which is the objective function penalising empty travel.

*Travel time*    The energy consumption given in (6) will be minimal when the function

$$\mathsf{p}_2 u_j + \mathsf{p}_1 + \mathsf{p}_0\frac{1}{u_j} \qquad (10)$$

is minimal; this happens when the speed $u_j$ satisfies [2]

$$u_j^* = \sqrt{\frac{\mathsf{p}_0}{\mathsf{p}_2}}, \qquad (11)$$

and hence minimising the objective function (8) will force the speeds towards this optimum. Since it might be considered a better service towards the customers to travel at higher speeds, thereby lowering the travel time, we can include a separate penalty on the travel times. The travel time of a customer for request $j$ is given by the transportation distance $c_{jj}$ divided by the speed $u_j$, hence

$$J_{\text{tt}} := \sum_{j=1}^{R} \frac{c_{jj}}{u_j} \qquad (12)$$

is the objective function associated with the travel time.

*Time-window misfit*    For each request $j$ we have a desired time-window $[\mathsf{t}_{\mathsf{a},j}, \mathsf{t}_{\mathsf{b},j}]$ wherein the pick-up of the customer should take place preferably. The misfit of the scheduled pick-up time $t_j$ for this request will be given by a slack time variable $s_j$; if $s_j = 0$ the pick-up is scheduled within the time-window, if $s_j > 0$ it represents the amount of time the pick-up is scheduled outside the time-window. To provide a good service to the customers the misfit should be minimised, which can be accomplished by using the cost function

$$J_{\text{tw}} := \sum_{j=1}^{R} s_j. \qquad (13)$$

### 3.2 Constraints

In order to obtain a meaningful solution to the micro-ferry scheduling problem, the optimisation variables should satisfy several constraints. We give a summary of the variables used in the optimisation problem; details can be found in (Burger et al., 2012).

*Scheduling variables*    The binary variables $x_{ij}$ reflect the order in which the requests are scheduled; when $x_{ij} = 1$ request $j$ is handled directly after request $i$ by the same micro-ferry. To ensure that all requests are handled by *one and only one* micro-ferry, the variables should satisfy the equality constraints (Bektaş, 2006; Laporte, 1992)

$$\sum_{i=1}^{R} x_{ij} = 1 \quad \forall j \in \mathcal{R}, \qquad (14a)$$

$$\sum_{j=1}^{R} x_{ij} = 1 \quad \forall i \in \mathcal{R}. \qquad (14b)$$

Constraints (14a) ensure that each request $j \in \mathcal{R}$ precedes exactly one other request; constraints (14b) ensure that

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are the same; when $j \in \mathcal{N}$ we will have $x_{jj} = 0$ due to (17) acting as sub-tour elimination constraints.

[2] That is, provided that $\underline{\mathsf{u}} \leq u_j^* \leq \overline{\mathsf{u}}$ and $\mathsf{p}_0 \neq 0$.

each request $i \in \mathcal{R}$ succeeds exactly one other request. Hence all requests are scheduled in between two requests. If there are less transportation requests than micro-ferries ($N < M$), some loops must exist represented by $x_{jj} = 1$.

*Pace variables*   Although for practical use it could be more convenient to work with the vehicle speed $u_j$, for the optimisation problem it is more convenient to work with the vehicle pace $w_j := u_j^{-1}$ (i.e., the reciprocal of speed (Daganzo (1997))) as an optimisation variable. Let $\underline{u}$ and $\overline{u}$ denote the minimum and maximum speed of the micro-ferries respectively, the pace variables should then satisfy

$$\underline{w} := \frac{1}{\overline{u}} \leq w_j \leq \overline{w} := \frac{1}{\underline{u}} \; \forall j \in \mathcal{R}. \tag{15}$$

To assign the speeds $u_{0,j}$ of the micro-ferries for the requests they are currently handling, we use the constraints

$$w_j = w_{0,j} := \frac{1}{u_{0,j}} \; \forall j \in \mathcal{M}. \tag{16}$$

*Pick-up time variables*   The variable $t_j$ denotes the pick-up time of request $j \in \mathcal{R}$, and it should be consistent with the schedule. To be more precise, if request $i$ precedes request $j$ —that is, if $x_{ij} = 1$— time $t_j$ should be at least larger than the pick-up time $t_i$ of request $i$, plus the time it takes to handle request $i$ and relocate the vehicle afterwards. Using a large constant $\mathsf{T}$ (based on the Big-M method (Taha, 1987)), we can enforce the pick-up times to be consistent using the inequality constraint

$$t_i - t_j + c_{ii}w_i + c_{ij}w_j + \mathsf{T}x_{ij} \leq \mathsf{T} \; \forall i \in \mathcal{R}, j \in \mathcal{N}. \tag{17}$$

If the micro-ferries are handling requests at the time the optimisation algorithm is started, the time variables associated with these micro-ferries should have the start time of the currently handled request. If a micro-ferry is not handling a request, the current time can be assigned. Assigning these times can be done using the constraints

$$t_j = t_{0,j} \; \forall j \in \mathcal{M}. \tag{18}$$

*Slack time variables*   For each request $j$ there is a desired time interval $[\mathsf{t}_{a,j}, \mathsf{t}_{b,j}]$ for picking up the customer. The scheduler should try to find a solution in which the pick-up time $t_j$ is within this interval, but it might not always be possible. Therefore, pick-up time $t_j$ may be scheduled outside the desired time interval by a value of $s_j \geq 0$, and use the inequality constraints

$$t_j + s_j \geq t_{a,j} \; \forall j \in \mathcal{R}, \tag{19a}$$
$$t_j - s_j \leq t_{b,j} \; \forall j \in \mathcal{R}. \tag{19b}$$

*Energy level variables*   The available energy of a micro-ferry after completion of request $j$ is denoted by $e_j$. If request $i$ is the preceding request, the energy level should be $e_j = e_i - \epsilon_j$, where for $x_{ij} = 1$ the amount of energy necessary to handle request $j$ becomes $\epsilon_j \triangleq \epsilon_{ij}$, with $\epsilon_{ij}$ defined in (6). Using a large constant $\mathsf{E}$ this can be accomplished by the inequality constraints

$$e_j - e_i + \epsilon_j + \mathsf{E}x_{ij} \leq \mathsf{E} \; \forall i \in \mathcal{R}, j \in \mathcal{N} \tag{20a}$$
$$e_i - e_j - \epsilon_j + \mathsf{E}x_{ij} \leq \mathsf{E} \; \forall i \in \mathcal{R}, j \in \mathcal{N}, \tag{20b}$$

whereas the initial energy levels can be assigned using

$$e_j = e_{0,j} \; \forall j \in \mathcal{M}. \tag{21}$$

*Assignment variables*   Using the constraints (14) we assure that each request is preceded and succeeded by

exactly one request, but on its own it is not enough to avoid schedules where one or more requests are not assigned to a micro-ferry, nor does it avoid that a request is scheduled to be handled by multiple micro-ferries. In terms of graph theory, the former means that there might exist sub-tours in the graph. To avoid this, we could use the sub-tour elimination constraints as developed by Miller et al. (1960) (and extended and improved by Desrochers and Laporte (1991)). The method is based on the idea of assigning potentials to each node in the network (as in an electric circuit), and increase the potentials along each arc. If there is a sub-tour, the potentials of the nodes will continue to increase, whereas when there are no sub-tours a maximum value can be assigned to the node potentials. In the micro-ferry scheduling problem the start times $t_j$ can be considered as the node potentials, and the time should increase along the route as enforced by (17).

In (Burger et al., 2012) a method is proposed that can be considered to be the dual of the sub-tour elimination constraints. To assure that no request is assigned to more than one micro-ferry, we enforce the existence of $M$ sub-tours, where $M$ denotes the number of micro-ferries in the network. By assigning a unique node current to the first $M$ nodes (those associated with the micro-ferries) using

$$k_j = j \; \forall j \in \mathcal{M}, \tag{22}$$

the existence of sub-tours is imposed by using

$$k_i - k_j + M(x_{ij} + x_{ji}) \leq M \; \forall (i,j) \in \mathcal{K}, \tag{23}$$

where the set $\mathcal{K}$ can be chosen as

$$\mathcal{K} = \{(i,j) : (i,j \in \mathcal{R}, i < j) \vee (i,j \in \mathcal{N}, i = j)\}$$

to allow loops on the nodes associated with the micro-ferries. This allows for the possibility that certain micro-ferries are not assigned any transportation requests (instead they are assigned an empty request).

Since inequality (23) ensures that the node currents cannot exceed the value $M$, and (22) assures that the minimum node current is 1, each sub-tour —representing the order in which a micro-ferry will handle the requests— will have a unique node current corresponding to the micro-ferry number the requests are assigned to.

## 4. EXTENSION OF THE MICRO-FERRY SCHEDULING PROBLEM

### 4.1 Charging of the micro-ferries

The first extension of the micro-ferry scheduling problem compared to the work in (Burger et al., 2012), is the modelling of the charging of the micro-ferries. The vehicles will work continuously, and after some time the batteries (or fuel) will run out, and charging (or refuelling) is necessary before they can continue handling requests.

First the micro-ferry will need to dock to a charging facility before the actual charging can begin. Likewise, the micro-ferry will need time after the charging before it will resume its service. We combine these two times in a constant value $\mathsf{t}_{ch}$ that will be necessary every time a micro-ferry will charge. This promotes to charge fewer times for longer durations, since the micro-ferry looses less time on docking when charging fewer times. Secondly, the duration of the actual charging $\tau_j$ will be variable, and it will be used as one of the optimisation variables.
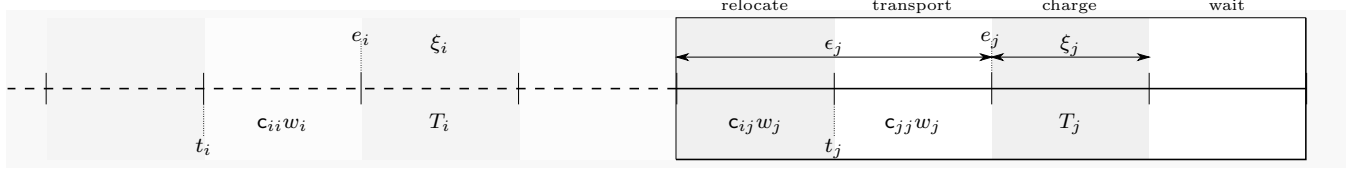
Fig. 1. *Overview of the four phases of a request, with the associated energy (top) and time (bottom) variables*

In the process of handling a request $j$, four phases are identified: *relocating, transporting, charging,* and *waiting.* Figure 1 gives a schematic view of these phases, along with the associated variables. In the relocation phase the micro-ferry is empty, and it travels from the delivery location of request $i$ to the pick-up location of request $j$. The passengers board at the beginning of the transportation phase, and leave the micro-ferry at the end of the phase when they have reached the delivery location. Next the micro-ferry can charge, possibly followed by a period of waiting before it starts handling the next request. Notice that the relocation phase, the charging phase, and the waiting phase may all have a zero duration, and thus no change in energy level.

The micro-ferries are given the opportunity to charge after each delivery of a request $j \in \mathcal{R}$, and represent the choice of whether or not to charge by a decision variable $y_j$. Using

$y_j = 0$: no charging after request $j$,
$y_j = 1$: charging after request $j$,

the charging time *after* request $j$ can be written as

$$T_j = (\mathsf{t}_{\mathrm{ch}} + \tau_j)y_j. \qquad (24)$$

Note that (24) is a non-linear equation since it contains a multiplication of the optimisation variables $\tau_j$ and $y_j$.

The charging time $\tau_j$ is an optimisation variable, and it should comply to certain constraints. First, the time should always have a positive value.[3] Secondly, the charging time should not exceed the time it takes to fully charge the micro-ferry.

Let $\mathsf{r}_{\mathrm{ch}}$ denote the constant charging rate for the vehicles. For micro-ferries powered by fossil fuels, this rate is proportional to the rate at which the micro-ferry can be filled with the fuel. When the micro-ferries are powered by batteries, the charging of the battery can be considered to be proportional to the charging current up to about 90% of the state of charge for both lithium-ion-based batteries (Jiang and Dougal, 2003) and lead-acid-based batteries (Cugnet and Liaw, 2011). Charging the batteries to a state of charge higher than 90% becomes inefficient, and we consider the micro-ferry fully charged ($e_j = \bar{\mathsf{e}}$) at this value of 90% of the state of charge. Then the added energy becomes proportional to the charging time, given as

$$\xi_j = \mathsf{r}_{\mathrm{ch}}\tau_j. \qquad (25)$$

The micro-ferries should not be charged to more than a maximum level $\bar{\mathsf{e}}$, resulting in the inequality constraints

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[3] One could enforce the charging time to be at least a certain duration to avoid short periods of charging, and hence strictly positive. We choose not to do so, since the fixed cost of the docking time will already help to avoid this. Furthermore, if a short charging time will be better for the overall performance of the system, it should be possible to have this opportunity.

$$e_j + \xi_j = e_j + \mathsf{r}_{\mathrm{ch}}\tau_j \le \bar{\mathsf{e}}, \quad \forall j \in \mathcal{R}. \qquad (26)$$

From (26) we can derive both the maximum and minimum charging times when $e_j = 0$ and $e_j = \bar{\mathsf{e}}$ respectively; we have

$$\underline{\mathsf{t}} := 0 \ \le \tau_j \le \ \bar{\mathsf{t}} := \frac{\bar{\mathsf{e}}}{\mathsf{r}_{\mathrm{ch}}}, \quad \forall j \in \mathcal{R}. \qquad (27)$$

When the micro-ferry is not scheduled to charge after request $j$ —that is, if $y_j = 0$— the charging time $\tau_j$ should be zero, to avoid that the added energy in (25) becomes non-zero. By adding the inequality constraints

$$0 \le \tau_j \le \bar{\mathsf{t}}y_j, \quad \forall j \in \mathcal{R}, \qquad (28)$$

it is assured that $y_j = 0 \Rightarrow \tau_j = 0$. Therefore, a linear equivalent to the non-linear equation (24) is given by

$$T_j = \mathsf{t}_{\mathrm{ch}}y_j + \tau_j. \qquad (29)$$

The charging time in (29) should be taken into account in the scheduling of the pick-ups. The pick-up time for request $j$ should satisfy (see Figure 1)

$$t_j \ge t_i + c_{ii}w_i + T_i + c_{ij}w_j \quad \text{if } x_{ij} = 1, \qquad (30)$$

which is accomplished by replacing (17) by

$$t_i - t_j + c_{ii}w_i + c_{ij}w_j + \tau_i + \mathsf{t}_{\mathrm{ch}}y_i + \mathsf{T}x_{ij} \le \mathsf{T} \ \forall i \in \mathcal{R}, j \in \mathcal{N}. \qquad (31)$$

The increase in energy levels after charging can be added by replacing (20) by (see Figure 1)

$$e_j - e_i - \mathsf{r}_{\mathrm{ch}}\tau_i + \epsilon_j + \mathsf{E}x_{ij} \le \mathsf{E} \ \forall i \in \mathcal{R}, j \in \mathcal{N}, \qquad (32\mathrm{a})$$
$$e_i - e_j + \mathsf{r}_{\mathrm{ch}}\tau_i - \epsilon_j + \mathsf{E}x_{ij} \le \mathsf{E} \ \forall i \in \mathcal{R}, j \in \mathcal{N}, \qquad (32\mathrm{b})$$

where $\mathsf{r}_{\mathrm{ch}}\tau_i = \xi_i$ is the energy increase (that can be zero) due to charging at the end of request $i$, defined in (25).

### 4.2 Inclusion of embarking and disembarking times

In the model proposed in (Burger et al., 2012) the time it takes to embark and disembark the micro-ferries has not been taken into account explicitly. Assume $\mathsf{t}_{\mathrm{p}}$ time units are allowed for customers to enter the micro-ferry at the pick-up location, and $\mathsf{t}_{\mathrm{d}}$ time units for customers to exit the micro-ferry at the delivery location. It takes $\mathsf{t}_{\mathrm{e}} = \mathsf{t}_{\mathrm{p}} + \mathsf{t}_{\mathrm{d}}$ time units at each request to embark and disembark the micro-ferries. This time should be taken into account in the scheduling of the pick-up times. This can be done by replacing (31) by

$$t_i - t_j + c_{ii}w_i + c_{ij}w_j + \tau_i + \mathsf{t}_{\mathrm{ch}}y_i + \mathsf{T}x_{ij} \le \mathsf{T} - \mathsf{t}_{\mathrm{e}} \ \forall i \in \mathcal{R}, j \in \mathcal{N} \qquad (33)$$

where the constant time $\mathsf{t}_{\mathrm{e}}$ for embarking and disembarking can be chosen by the network operator.

## 5. LINEAR PROGRAMMING APPROXIMATION

In the model described so far there are two difficulties for solving the problem exactly due to the objective function (8). First, both the speed $u_j$ and its reciprocal are used in the formulation. The objective function would become linear by replacing the latter with pace variable $w_j$,

but this requires a non-linear constraint $u_j w_j = 1$. Furthermore, the energy consumption term $\epsilon_j$ defined in (7) contains multiplications of variables, making the problem difficult to solve using standard optimisation programs. Therefore, we will use linear approximations to obtain a mixed-integer linear program (MILP).

### 5.1 Approximation of the speeds

The speed $u_j$ is the reciprocal of the pace $w_j$, or

$$u_j = \frac{1}{w_j}, \qquad (34)$$

which will be approximated by a piece-wise affine (PWA) function in the variable $w_j$ with $P$ sections. This gives

$$\hat{u}_j = \begin{cases} a_1 w_j + b_1, & \omega_0 \leq w_j \leq \omega_1 \\ \vdots & \vdots \\ a_P w_j + b_P, & \omega_{P-1} \leq w_j \leq \omega_P \end{cases} \qquad (35)$$

where $\omega_0 = \underline{w}$, $\omega_P = \overline{w}$ as defined in (15), and the scalars $\omega_p$ are constants for all $p \in \{1, \ldots, P-1\}$ and satisfy

$$\omega_p < \omega_{p+1} \quad \forall p \in \{1, \ldots, P-1\}. \qquad (36)$$

The constants $\omega_1, \ldots, \omega_{P-1}, a_1, \ldots, a_P$, and $b_1, \ldots, b_P$ can be found by minimising the error $u_j - \hat{u}_j$ in a least-squares sense, as discussed in (Burger et al., 2012).

Using methods as described in (Bemporad and Morari, 1999) we can transform (35) into a single function by introducing $R \cdot P$ binary variables $z_{jp}$ associated with the $R$ speeds $u_j$ and the $P$ constants $\omega_p$, representing

$$[z_{jp} = 1] \iff [w_j \leq \omega_p]. \qquad (37)$$

The relation (37) can be enforced by the constraints

$$w_j - \omega_p \leq \mathsf{W}(1 - z_{jp}) \quad \forall j \in \mathcal{R}, p \in \mathcal{P}, \qquad (38a)$$
$$\omega_p - w_j \leq \mathsf{W} z_{jp} \qquad \forall j \in \mathcal{R}, p \in \mathcal{P}, \qquad (38b)$$

with $\mathsf{W} := \overline{w} - \underline{w}$. Notice that (38a) ensures $z_{jp} = 0$ when $w_j > \omega_p$, and (38b) ensures $z_{jp} = 1$ when $w_j < \omega_p$. Furthermore we have $z_{jq} = 1$ for $q = p+1, \ldots, P$ if $z_{jp} = 1$, since $\omega_p < \omega_{p+1} < \cdots < \omega_P$ by (36). The PWA function (35) can then be written as

$$\hat{u}_j = (\mathsf{A}_1 z_{j1} + \mathsf{A}_2 z_{j2} + \cdots + \mathsf{A}_P z_{jP}) w_j \qquad (39)$$
$$\qquad + (\mathsf{B}_1 z_{j1} + \mathsf{B}_2 z_{j2} + \cdots + \mathsf{B}_P z_{jP}),$$
$$= (\mathsf{A}_1 w_j + \mathsf{B}_1) z_{j1} + \cdots + (\mathsf{A}_P w_j + \mathsf{B}_P) z_{jP},$$

where $\mathsf{A}_1, \ldots, \mathsf{A}_P$ and $\mathsf{B}_1, \ldots, \mathsf{B}_P$ are constants given as

$$\mathsf{A}_p = a_p - a_{p+1} \quad \forall p \in \{1, \ldots, P-1\}, \qquad (40a)$$
$$\mathsf{B}_p = b_p - b_{p+1} \quad \forall p \in \{1, \ldots, P-1\}, \qquad (40b)$$
$$\mathsf{A}_P = a_P, \qquad \mathsf{B}_P = b_P. \qquad (40c)$$

### 5.2 Linearised formulation of the energy consumption

Using the speed approximation $\hat{u}_j$ of (39), the energy consumption can be approximated by a linear function as well. We substitute (7) by

$$\hat{\epsilon}_j = \left( \mathsf{p}_2 \hat{u}_j + \mathsf{p}_1 + \mathsf{p}_0 \frac{1}{u_j} \right) \sum_{i=1}^{R} (c_{ij} + c_{jj}) x_{ij}, \qquad (41)$$

$$= \left( \mathsf{p}_2 \sum_{p=1}^{P} \{(\mathsf{A}_p w_j + \mathsf{B}_p) z_{jp}\} + \mathsf{p}_1 + \mathsf{p}_0 w_j \right) \sum_{i=1}^{R} \mathsf{C}_{ij} x_{ij}$$

with $\mathsf{C}_{ij} := c_{ij} + c_{jj}$ a constant representing the total distance travelled when request $i$ precedes request $j$. This

equation contains many multiplications of variables, making it a non-linear function, but they can be removed by introducing auxiliary variables $f_j$, $g_{jp}$ and $h_{jp}$.

Using (14a), $\mathsf{F} := \overline{\mathsf{f}} - \underline{\mathsf{f}}$, and the inequality constraints

$$f_j \leq \mathsf{C}_{ij} w_j + \mathsf{F}(1 - x_{ij}) \quad \forall i \in \mathcal{R}, \qquad (42a)$$
$$f_j \geq \mathsf{C}_{ij} w_j + \mathsf{F}(x_{ij} - 1) \quad \forall i \in \mathcal{R}, \qquad (42b)$$

where $\underline{\mathsf{f}}$ and $\overline{\mathsf{f}}$ are a lower bound and an upper bound on the product $\mathsf{C}_{ij} w_j$ respectively, we obtain

$$f_j \triangleq w_j \sum_{i=1}^{R} \mathsf{C}_{ij} x_{ij}. \qquad (43)$$

The inequality constraints

$$g_{jp} \leq \overline{\mathsf{g}} z_{jp}, \quad g_{jp} \leq \underline{\mathsf{g}}(z_{jp} - 1) + \sum_{i=1}^{R} \mathsf{C}_{ij} x_{ij}, \qquad (44a)$$

$$g_{jp} \geq \underline{\mathsf{g}} z_{jp}, \quad g_{jp} \geq \overline{\mathsf{g}}(z_{jp} - 1) + \sum_{i=1}^{R} \mathsf{C}_{ij} x_{ij}, \qquad (44b)$$

with $\underline{\mathsf{g}}$ and $\overline{\mathsf{g}}$ a lower bound and an upper bound on the constants $\mathsf{C}_{ij}$ respectively, ensure that we obtain

$$g_{jp} \triangleq z_{jp} \sum_{i=1}^{R} \mathsf{C}_{ij} x_{ij}. \qquad (45)$$

Finally, the inequality constraints

$$h_{jp} \leq \overline{\mathsf{f}} z_{jp}, \quad h_{jp} \leq f_j + \underline{\mathsf{f}}(z_{jp} - 1), \qquad (46a)$$
$$h_{jp} \geq \underline{\mathsf{f}} z_{jp}, \quad h_{jp} \geq f_j + \overline{\mathsf{f}}(z_{jp} - 1), \qquad (46b)$$

enforce the relationships

$$h_{jp} \triangleq z_{jp} f_j. \qquad (47)$$

Substitution of (43), (45), and (47) in (41) gives

$$\hat{\epsilon}_j = \mathsf{p}_0 f_j + \mathsf{p}_1 \sum_{i=1}^{R} \mathsf{C}_{ij} x_{ij} + \mathsf{p}_2 \sum_{p=1}^{P} (\mathsf{A}_p h_{jp} + \mathsf{B}_p g_{jp}), \qquad (48)$$

which is a linear function in the auxiliary variables $f_j$, $g_{jp}$ and $h_{jp}$, and hence all constraints are linear. The (non-linear) objective function (8) related to the energy consumption can be replaced by the linear approximation

$$\hat{J}_{\text{ec}} = \sum_{j=1}^{N} [\mathsf{p}_0 f_j + \mathsf{p}_1 \sum_{i=1}^{N} \mathsf{C}_{ij} x_{ij} + \mathsf{p}_2 \sum_{p=1}^{P} (\mathsf{A}_p h_{jp} + \mathsf{B}_p g_{jp})]. \qquad (49)$$

Both the objective functions (9) and (13) —related to empty travel and time-window misfit respectively— are linear functions. The objective function (12) is non-linear in its current form, but by substituting the speed variables $u_j$ by their associated pace variables $w_j$ we have

$$J_{\text{tt}} = \sum_{j=1}^{R} c_{jj} w_j, \qquad (50)$$

which is linear, such that the optimisation problem becomes a mixed-integer linear program (MILP).

## 6. SIMULATIONS

We will demonstrate the use of the proposed method for micro-ferry scheduling using a small case study. For the simulations we consider a network with four locations ($L = 4$; $\mathcal{L} = \{s_1, s_2, s_3, s_4\}$) and three micro-ferries ($M = 3$; $\mathcal{M} = \{m_1, m_2, m_3\}$). The MILP problem was implemented in Matlab using CPLEX as solver.
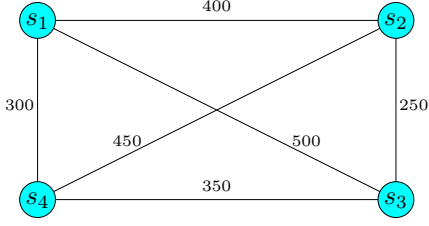
Fig. 2. *The physical network with four stations*

A schematic drawing of this network is shown in Figure 2, along with the path lengths $l_{ij}$ for $i, j \in \mathcal{L}$ (in $[m]$). The first micro-ferry is waiting at location $s_1$, while the other two are transporting customers. The initial values for the start times (in $[s]$), the speeds (in $[m/s]$) and the energy levels (in $[\%]$; $\underline{e} = 0$, $\overline{e} = 100$) are given by

$$t_0 = \begin{bmatrix} -120 \\ -60 \\ 0 \end{bmatrix}, \quad u_0 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \quad e_0 = \begin{bmatrix} 100 \\ 30 \\ 40 \end{bmatrix}. \quad (51)$$

The parameters associated with embarking and charging are chosen as

$$t_e = 60 \, [s], \quad t_{ch} = 30 \, [s], \quad r_{ch} = 1 \, [\%/s]. \quad (52)$$

There are fifteen new requests ($N = 15$) for which the pick-up locations, delivery locations, desired pick-up time windows, and transportation distances are shown in Table 1. Requests 1, 2 and 3 are associated with the micro-ferries.

Table 1. *Requests that need to be scheduled*

| request | pick-up | delivery | time window | | | transp. dist. |
|---|---|---|---|---|---|---|
| 1 | $s_1$ | $s_1$ | -120 | – | -60 | 0 |
| 2 | $s_4$ | $s_2$ | -60 | – | 0 | 450 |
| 3 | $s_4$ | $s_3$ | 0 | – | 60 | 350 |
| 4 | $s_1$ | $s_2$ | 0 | – | 60 | 400 |
| 5 | $s_4$ | $s_3$ | 180 | – | 240 | 350 |
| 6 | $s_2$ | $s_1$ | 240 | – | 300 | 400 |
| 7 | $s_2$ | $s_4$ | 360 | – | 420 | 450 |
| 8 | $s_4$ | $s_2$ | 720 | – | 780 | 450 |
| 9 | $s_2$ | $s_1$ | 720 | – | 780 | 400 |
| 10 | $s_1$ | $s_3$ | 660 | – | 720 | 500 |
| 11 | $s_3$ | $s_4$ | 780 | – | 840 | 350 |
| 12 | $s_1$ | $s_3$ | 960 | – | 1020 | 500 |
| 13 | $s_2$ | $s_3$ | 1020 | – | 1080 | 250 |
| 14 | $s_3$ | $s_4$ | 1080 | – | 1140 | 350 |
| 15 | $s_4$ | $s_2$ | 1080 | – | 1140 | 450 |
| 16 | $s_2$ | $s_1$ | 1440 | – | 1500 | 400 |
| 17 | $s_1$ | $s_4$ | 1440 | – | 1500 | 300 |
| 18 | $s_4$ | $s_2$ | 1500 | – | 1560 | 450 |

Using these initial conditions, we investigate the influence of changing the emphasis on minimising the time-window misfit towards minimising the energy consumption, both for fixed and variable speeds. In the cost function defined in (4) we choose $\alpha_{et} = 0$ and $\alpha_{tt} = 0$ (hence, no extra penalty is imposed on empty travel and long travel times), and $\alpha_{ec}$ and $\alpha_{tw}$ are varied. The power (5) is chosen to be

$$P(u) = p_2 u^2 + p_1 u + p_0 = 0.01u^2 + 0.01u + 0.09, \quad (53)$$

and by (11) the optimal speed is $u^* = 3 \, [m/s]$.

### 6.1 Fixed speeds

First the influence of changing the emphasis of either minimising the time-window misfit or the energy consumption while keeping the travel speeds fixed at $u_j = 3.0 \, [m/s]$ is considered for all $j \in \mathcal{R}$. For different values of $\alpha_{ec}$ and $\alpha_{tw}$ the values of the energy consumption (8), the time-window

misfit (13), the empty-travel distance (9), and the total travel time (12) are given in Table 2.

Table 2. *Varying objective with fixed speeds*

| $\alpha_{ec}$ | $\alpha_{tw}$ | $J_{ec}$ | $J_{tw}$ | $J_{et}$ | $J_{tt}$ | #rel |
|---|---|---|---|---|---|---|
| 0 | 100 | 588 | 162 | 1600 | 2267 | 5 |
| 1 | 100 | 553 | 162 | 1100 | 2267 | 3 |
| 10 | 100 | 553 | 162 | 1100 | 2267 | 3 |
| 100 | 100 | 553 | 162 | 1100 | 2267 | 3 |
| 100 | 10 | 518 | 296 | 600 | 2267 | 2 |
| 100 | 1 | 476 | 2177 | 0 | 2267 | 0 |
| 100 | 0 | 476 | 9520 | 0 | 2267 | 0 |

As can be expected the energy consumption reduces as the value of $\alpha_{ec}$ becomes larger relative to $\alpha_{tw}$, and the time-window misfit (i.e., the total amount of seconds the pick-up times are scheduled outside the desired time-windows) increases. The empty-travel distance also decreases when energy consumption is penalised more, since less empty travel means less energy consumption. The total travel time (i.e., the time the passengers spent on the micro-ferries) remains the same, since both the transportation distance and the travel speed are constant.

An example of a schedule is given in Figure 3 for $\alpha_{ec} = 100$ and $\alpha_{tw} = 100$. The time in seconds passes along the horizontal axis, and the energy level percentage increases along the vertical axis. The figure shows the scheduled pick-up times $t_i$ and the desired time windows at the bottom; the green or red colour indicates whether or not the time window is met. Here the desired time windows have not been met three times, and three relocations are necessary. Note that the slope of the energy level reduction during relocation and transportation is always the same, indicating a constant power, as expected for fixed speeds.

### 6.2 Variable speeds

In the following we repeat the simulations done before, but this time the travel speed may vary. The minimum and maximum speed are given by $\underline{u} = 2$ and $\overline{u} = 5$. The results are summarised in Table 3.

Table 3. *Varying objective with variable speeds*

| $\alpha_{ec}$ | $\alpha_{tw}$ | $J_{ec}$ | $J_{tw}$ | $J_{et}$ | $J_{tt}$ | #rel |
|---|---|---|---|---|---|---|
| 0 | 100 | 648 | 0 | 2100 | 1913 | 5 |
| 1 | 100 | 577 | 0 | 1250 | 1913 | 4 |
| 10 | 100 | 577 | 0 | 1250 | 2183 | 4 |
| 100 | 100 | 577 | 0 | 1250 | 2183 | 4 |
| 100 | 10 | 522 | 156 | 600 | 2253 | 2 |
| 100 | 1 | 476 | 2021 | 0 | 2284 | 0 |
| 100 | 0 | 475 | 7557 | 0 | 2436 | 0 |

Due to the variable speed, also the total travel time $J_{tt}$ changes. All desired time-windows are met for the upper four cases ($J_{tw} = 0$), but at the cost of more relocations and empty travel distance $J_{et}$.

Figure 4 shows the schedule for $\alpha_{ec} = 100$ and $\alpha_{tw} = 100$ when the travel speeds can vary. The slopes of the energy level reduction varies, consistent with the varying speeds. Due to the increased flexibility, it is possible to schedule all pick-ups within their desired time-windows. Both the number of relocations (4 instead of 3) and energy consumption (577 instead of 553) have increased, but the time-window misfit is lower (0 instead of 162).
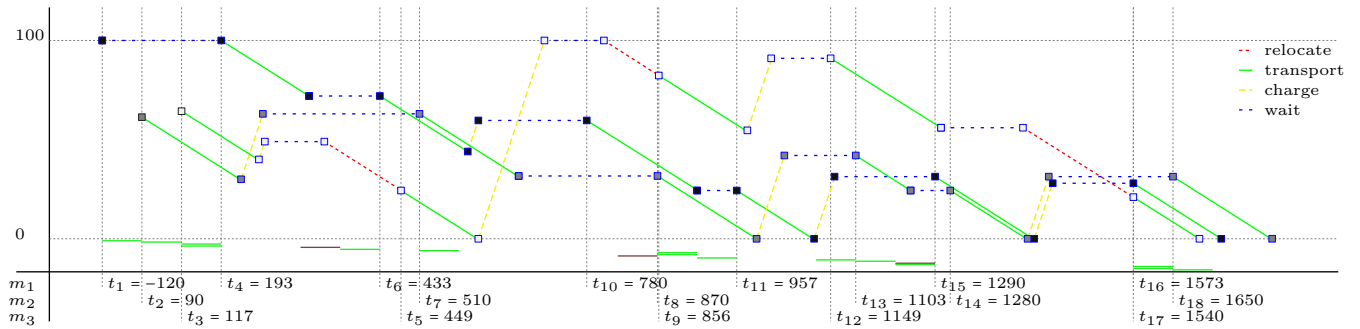
Fig. 3. *Schedule of the three micro-ferries $m_1, m_2, m_3$ when using fixed speeds, with pick-up times in seconds*
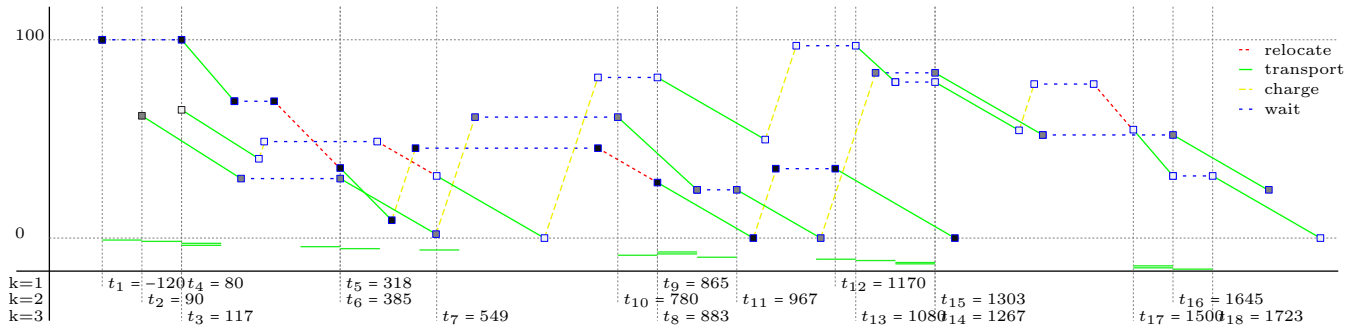


Fig. 4. *Schedule of the three micro-ferries $m_1, m_2, m_3$ when using variable speeds, with pick-up times in seconds*

## 7. CONCLUSION

In this paper we have proposed a modelling framework for solving micro-ferry scheduling problems, where energy consumption is taken into account. By using the travel speed as one of the optimisation variables, a trade-off can be made between minimising the total amount of consumed energy and the misfit between the scheduled and desired pick-up times. A mixed-integer linear programming approximation of the non-linear problem is developed, and numerical simulations are provided to give an example of the schedules one obtains using the proposed method.

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