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Optimal Trajectory Planning for Trains under Operational Constraints Using Mixed Integer Linear Programming ⋆

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Abstract: The optimal trajectory planning problem for trains under operational constraints is considered, which is essential for the success of the real-time operation and the rescheduling process for railway networks. The operational constraints caused by the timetable, real-time operation, or rescheduling often include target points and target window constraints. The approach proposed in this paper can take such constraints into account. In addition, the varying maximum traction force is approximated using a piecewise affine function and included in the trajectory planning problem. The optimal control problem is recast as a mixed integer linear programming problem, which can be solved efficiently by existing solvers. A case study is used to demonstrate the performance of the proposed approach.

Keywords: train operation, trajectory planning, MILP, operational constraints

1. INTRODUCTION

Transport plays a key role in the economy, the environment, and the society of regions and countries. Public transportation, especially the rail traffic, becomes more and more important for the sustainability of transportation systems (Peng, 2008). To strengthen the position of rail traffic, railway systems must become more efficient, more competitive, and satisfy expectations of customers, e.g. train services should be safe, fast, punctual, and comfortable (Lüthi, 2009).

Timetabling and traffic management are two basic elements of public transport operations, and this also includes railway systems (Hansen and Pachl, 2008). It is known that timetabling and traffic management are inherently linked, but the function of the timetable is often only loosely connected to its execution in practice (Hansen and Pachl, 2008). Therefore, real-time rescheduling is proposed to improve punctuality, to increase network capacity, and to reduce energy consumption by identifying and resolving conflicts arising during actual operations. Rescheduling may require railway systems with advanced GSM-R or European Train Control Systems equipment, so the decisions taken in railway control centers can be transmitted to Driver Machine Interfaces (DMI) or Automatic Train Operation (ATO) systems immediately (Caimi et al., 2009). Hence, train drivers or ATO systems can operate the train precisely according to the dynamic plan given by the rescheduling process.

An essential factor for the success of real-time rescheduling is the calculation of optimal reference trajectories for trains, because the reference trajectory has an impact on driving behavior and achievable accuracy, especially after rescheduling. There are two possible strategies to define a reference trajectory (Lüthi, 2009). One possibility is to calculate the trajectories precisely in the railway control center as part of the rescheduling loop and then transmit them to DMI or ATO systems. The other possibility is that the rescheduling system only specifies operational constraints (e.g. reference passing times at particular points) and the calculation of the optimal reference trajectory is performed on the DMI or ATO system. The transmitted data volume and the architecture of DMI or ATO system depend on the strategies described above. For more information of these two strategies, see Lüthi (2009). The focus of this paper is on the calculation of optimal reference trajectories under operational constraints. The approach proposed in this paper is suitable for both strategies for defining reference trajectories.

Many researchers explored the optimal control problem for trains, since it has significant effects for energy saving, punctuality, and riding comfort. They applied various methods, which can be grouped into the following two main categories: analytical solutions and numerical optimization. For analytical solutions, a comprehensive analysis of the optimal trajectory planning problem is given in Howlett (2000); Khmelnitsky (2000); Liu and Golovicher (2003). The maximum principle can be applied to obtain optimal operation regimes (i.e. maximum traction, cruising, coasting, and maximum braking) and their sequence. It is difficult to obtain the analytical solution

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if more realistic conditions are considered, which introduce more complex nonlinear terms into the model equations and the constraints (Ko et al., 2004). Because of the comparable high computing power available nowadays, numerical optimization approaches are applied more and more to the train optimal control problem. A number of advanced techniques such as fuzzy and genetic algorithms have been proposed to calculate the optimal reference trajectory of train operation, see e.g. Chang and Xu (2000), Chang and Sim (1997), and Han et al. (1999). But in these approaches, the optimal solution is not always guaranteed. Therefore, in Wang et al. (2011) a mixed integer linear programming (MILP) approach has been proposed to solve the optimal trajectory problem. The resulting MILP problem can be solved efficiently using existing commercial and free solvers that guarantee finding the global optimum.

The varying line resistance, variable speed restrictions, and constant maximum traction force are taken into account in the optimal trajectory planning problem in Wang et al. (2011). However, there exist other constraints that can result from the timetable or from real-time rescheduling process (Lüthi, 2009; Albrecht et al., 2010). For example, in order not to get hindered by a preceding train, a train may be required not to pass a location earlier than a certain time. On the other hand, in order not to hinder a following train, a train may have to be at a certain location not later than a scheduled time. In addition, the maximum traction force is a nonlinear function of the train's speed due to the characteristics of the power equipment and the maximum adhesion between wheel rim and rail (Hansen and Pachl, 2008). In this paper, we will extend our MILP approach of Wang et al. (2011) to solve the optimal trajectory planning problem under operational constraints and a varying maximum traction force.

The remainder of this paper is structured as follows. In Section 2, the optimal control problem of train operations and the MILP approach proposed in Wang et al. (2011) are summarized. Section 3 introduces several extensions to the proposed MILP approach, i.e. more accurate line resistance, new piecewise affine approximations in the time differential equation, and varying maximum traction force. Section 4 presents the operational constraints caused by the timetable, real-time operation, or the rescheduling and shows how to include these operational constraints to the MILP problem. Section 5 illustrates the calculation of the optimal reference trajectory under operational constraints and a varying maximum traction force by the proposed MILP approach with a case study. We conclude with a short discussion of some topics for future work in Section 6.

2. PROBLEM DEFINITION AND THE MILP APPROACH

In this section, the formulation of the optimal control problem and the MILP approach we proposed in Wang et al. (2011) are summarized.

2.1 Optimal control problem

The mass-point model of train is widely used in the literature on optimal control of trains (Franke et al., 2003). The continuous-time model of train operation is described as (Howlett, 2000; Liu and Golovicher, 2003):

$$m\rho \frac{\mathrm{d}v}{\mathrm{d}t} = u(t) - R_\mathrm{b}(v) - R_\mathrm{l}(s,v), \quad (1)$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = v, \quad (2)$$

where m is the mass of the train, ρ is a factor to consider the rotating mass (Hansen and Pachl, 2008), v is the velocity of the train, s is the position of the train, u is the control variable, i.e. the traction or braking force, which is bounded by the maximum traction force u_max and the maximum braking force u_min: $u_\mathrm{min} \leq u \leq u_\mathrm{max}$, $R_\mathrm{b}(v)$ is the basic resistance including roll resistance and air resistance, and $R_\mathrm{l}(s,v)$ is the line resistance caused by track grade, curves, and tunnels. In practice, according to the Strahl formula (Rochard and Schmid, 2000) the basic resistance $R_\mathrm{b}(v)$ can be described as

$$R_\mathrm{b}(v) = m(a_1 + a_2 v^2),$$

where the coefficients a_1 and a_2 depend on the train characteristics and the wind speed, which can be calculated from the data known about the train. The line resistance $R_\mathrm{l}(s,v)$ caused by track slope, curves, and tunnels can be described by Mao (2008)

$$R_\mathrm{l}(s,v) = mg\sin\alpha(s) + f_\mathrm{c}(r(s)) + f_\mathrm{t}(l_\mathrm{t}(s),v), \quad (3)$$

where g is the gravitational acceleration, $\alpha(s)$, $r(s)$, and $l_\mathrm{t}(s)$ are the slope, the radius of the curve, and the length of the tunnel along the track, respectively. For more information about these terms, see Mao (2008). It is worth to note that in Wang et al. (2011) the maximum traction force u_max is considered as constant. However, in reality it is a nonlinear function of the train's speed and in Section 3 below nonlinear function will be approximated by piecewise affine (PWA) function.

Franke et al. (2003) choose kinetic energy per mass unit $\tilde{E} = 0.5v^2$ and time t as states, and the position s as the independent variable. The reference trajectory planning problem for trains is formulated as (Wang et al., 2011):

$$J = \int_{s_\mathrm{start}}^{s_\mathrm{end}} \left(u(s) + \lambda \cdot \left| \frac{\mathrm{d}u(s)}{\mathrm{d}s} \right| \right) \mathrm{d}s \quad (4)$$

subject to the model (1) and (2), the constraints

$$u_\mathrm{min} \leq u(s) \leq u_\mathrm{max}, \quad (5)$$

$$0 < \tilde{E}(s) \leq \tilde{E}_\mathrm{max}(s), \quad (6)$$

and the boundary conditions,

$$\tilde{E}(s_\mathrm{start}) = \tilde{E}_\mathrm{start}, \;\; \tilde{E}(s_\mathrm{end}) = \tilde{E}_\mathrm{end}, \quad (7)$$

$$t(s_\mathrm{start}) = 0, \;\; t(s_\mathrm{end}) = T, \quad (8)$$

where the objective function J is a weighted sum of the energy consumption and the riding comfort of the train operation; $\tilde{E}_\mathrm{max}(s)$ is equal to $0.5V^2_\mathrm{max}(s)$ and $V_\mathrm{max}(s)$ is the maximum allowable velocity, which depends on the train characteristics and line conditions, and as such it is usually a piecewise constant function of the coordinate s (Khmelnitsky, 2000; Liu and Golovicher, 2003); s_start, $\tilde{E}(s_\mathrm{start})$, and $t(s_\mathrm{start})$ are the position, the kinetic energy per mass, and the time at the beginning of the route; s_end, $\tilde{E}(s_\mathrm{end})$, and t_end are the position, the kinetic energy per mass, and the time at the end of the route; the scheduled running time T is given by the timetable or the rescheduling process. It is assumed that the unit kinetic energy $\tilde{E}(s) > 0$, which means the train's speed is always strictly larger than zero, i.e. the train travels nonstop (Khmelnitsky, 2000).

The restrictions of maximum speed, maximum traction or braking force, scheduled running times, etc. are considered in Wang et al. (2011). However, there exist operational constraints, which result from the timetable, real-time operation restrictions, or the real-time rescheduling process (Lüthi, 2009; Albrecht et al., 2010). In this paper, these operational constraints are included in the MILP approach in Section 4.

2.2 The MILP approach

A discrete-space model is obtained in Wang et al. (2011) by splitting the position horizon $[s_{\text{start}}, s_{\text{end}}]$ into N intervals. It is assumed that the track and train parameters as well as traction or breaking force can be considered as constant in each interval $[s_k, s_{k+1}]$ with length $\Delta s_k = s_{k+1} - s_k$, for $k = 1, 2, \ldots, N$.

According to the transformation properties in Bemporad and Morari (1999) and by introducing a vector of logical variables $\delta(k)$ and a real-valued vector of auxiliary variables $z(k)$, the dynamics of the train operation can be transformed into a so-called mixed logical dynamic model of the following form (see Wang et al. (2011)):

$$\begin{aligned} x(k+1) = A_k x(k) + B_k u(k) + C_{1,k}\delta(k) + C_{2,k}\delta(k+1) \\ + D_{1,k}z(k) + D_{2,k}z(k+1) + e_k, \end{aligned} \tag{9}$$

$$\begin{aligned} R_{1,k}\delta(k) + R_{2,k}\delta(k+1) + R_{3,k}z(k) + R_{4,k}z(k+1) \\ \leq R_{5,k}u(k) + R_{6,k}x(k) + R_{7,k}, \end{aligned} \tag{10}$$

where $x(k) = [E(k)\ t(k)]^T$ and (10) also includes the upper bound and lower bound constraints for $E(k)$, $t(k)$, and $u(k)$. For the sake of simplicity, we use $E(k)$ as a short-hand notation for $\tilde{E}(s_k)$.

By introducing a new variable $\omega(k)$ to deal with the absolute value $|\Delta u(k)|$ in the objective function, the optimal control problem can be recast as a the following mixed integer linear programming (MILP) problem (Wang et al., 2011)

$$\min_{\tilde{V}} \quad C_J^T \tilde{V}, \tag{11}$$

subject to

$$F_1 \tilde{V} \leq F_2 x(1) + f_3 \tag{12}$$

$$F_4 \tilde{V} = F_5 x(1) + f_6 \tag{13}$$

where $C_J = [\,\Delta s_1 \ \cdots \ \Delta s_N \ 0 \ \cdots \ 0 \ 1 \ \cdots \ 1\,]^T$, $\tilde{V} = [\,\tilde{u}^T \ \tilde{\delta}^T \ \tilde{z}^T \ \tilde{\omega}^T\,]^T$,

$$\tilde{u} = \begin{bmatrix} u(1) \\ u(2) \\ \vdots \\ u(N) \end{bmatrix}, \tilde{\delta} = \begin{bmatrix} \delta(1) \\ \delta(2) \\ \vdots \\ \delta(N+1) \end{bmatrix}, \tilde{\omega} = \begin{bmatrix} \omega(1) \\ \omega(2) \\ \vdots \\ \omega(N-1) \end{bmatrix},$$

and $\tilde{z}$ is defined in a similar way as $\tilde{\delta}$. The MILP problem (11)-(13) can be solved by several existing commercial and free solvers, such as CPLEX, Xpress-MP, GLPK (see e.g. Linderoth and Ralphs (2004); Atamturk and Savelsbergh (2005)).

3. EXTENSIONS

In this section, we will introduce several extensions to our previous work in Wang et al. (2011). The line resistance is considered more accurately. New piecewise affine (PWA) approximations with two subfunctions are introduced to approximate the nonlinear function $f(E(k))$ in the time equation. In addition, a varying maximum traction force is included in the MILP approach.

3.1 More accurate line resistance

In Wang et al. (2011) the line resistance is considered as a piecewise constant function. However, when running in tunnels, the train experiences a higher air resistance, which is a function of v^2. In order to consider the line resistance more accurately, in this paper we rewrite $R_l(s,v)$ as

$$R_l(s,v) = \tilde{R}_l(s) + a_t(s)v^2, \tag{14}$$

where $\tilde{R}_l(s)$ includes the terms that do not depend on the train's speed. By defining the discretization of the interval $[s_{\text{start}}, s_{\text{end}}]$ properly, we can assume without loss of generality that $\tilde{R}_l(s)$ and $a_t(s)$ are of the following form:

$$\begin{aligned} \tilde{R}_l(s) &= \tilde{R}_{l,k} \quad \text{for} \quad s \in [s_k, s_{k+1}], \\ a_t(s) &= a_{t,k} \quad \text{for} \quad s \in [s_k, s_{k+1}]. \end{aligned}$$

for $k = 1, 2, \cdots, N$. The quadratic term v^2 is linear in the kinetic energy E and it can be easily included in the mixed logical dynamic model of train operation.

3.2 New PWA approximations in the time differential equation

The difference equation of time obtained using a trapezoidal integration rule in Wang et al. (2011) is

$$t(k+1) = t(k) + \frac{1}{2}\left(\frac{1}{\sqrt{2E(k)}} + \frac{1}{\sqrt{2E(k+1)}}\right)\Delta s_k \tag{15}$$

with $t(1) = 0$. The nonlinear function $f(E(k)) = \frac{1}{2\sqrt{2E(k)}}$ is approximated by PWA function with 3 subfunctions in Wang et al. (2011). In order to reduce the calculation time of the MILP approach, we now consider approximations with 2 affine subfunctions (cf. Figure 1). It is known that the approximation error can be reduced by taking more regions. Therefore, the approximation error of PWA function with three subfunctions is better than that with two subfunctions. However, this will be offset by the huge difference in CPU time to solve the resulting MILP problem (see also Section 5).

The coefficients of the PWA function do not depend on the space interval index k in Wang et al. (2011). In this paper, we adapt these coefficients of PWA approximations with 2 affine subfunctions depending on the space interval index k to reduce the approximation error, i.e. we can have different PWA subfunctions for different space intervals. According to the piecewise constant function of speed limits, different approximations of $f(E(k))$ are obtained by using proper weighting functions. In addition, we introduce two additional PWA approximations of $f(E(k))$ for the first segment and the last segment. Since the lowest speed in these two segments is a small positive number near zero, the highest weight should be given to the low speed interval. Here, even though we approximate $f(E(k))$ by 2 affine subfunctions, the approximation error is still small and better than the approximation in Wang et al. (2011). The PWA approximation of the nonlinear function $f(E(k)) = \frac{1}{2\sqrt{2E(k)}}$ depending on the space interval index of can be written as

$$f_{\text{PWA}}(E(k)) = \begin{cases} \alpha_{1,k}E(k) + \beta_{1,k} & \text{for } E_{\min,k} \leq E(k) \leq E_{1,k}, \\ \alpha_{2,k}E(k) + \beta_{2,k} & \text{for } E_{1,k} \leq E(k) \leq E_{\max,k}, \end{cases}$$

with $E_{\min,k}$ and $E_{\max,k}$ for the interval $[s_k, s_{k+1}]$. Furthermore, the values of the coefficients and $E_{1,k}$ are determined by the least-squares optimization (Azuma et al., 2010). A logical variable $\delta_1(k)$ and a new auxiliary variable $z_1(k)$ are introduced to formulate the PWA constraints into linear inequality constraints, see Wang et al. (2011).

3.3 Varying maximum traction force

The maximum traction force $u_{\max}$ is often considered as constant in the literature (Howlett, 2000). However, in reality it is

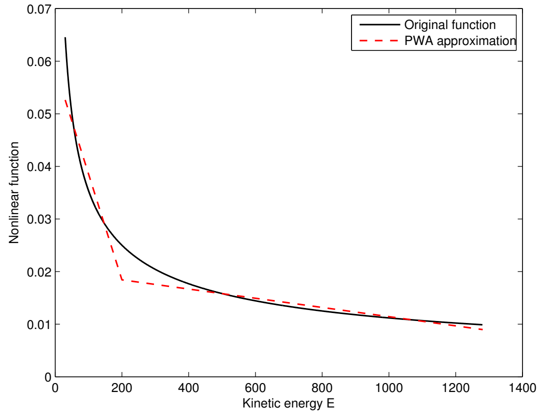

Fig. 1. The PWA approximation of the nonlinear function $f(E(k))$

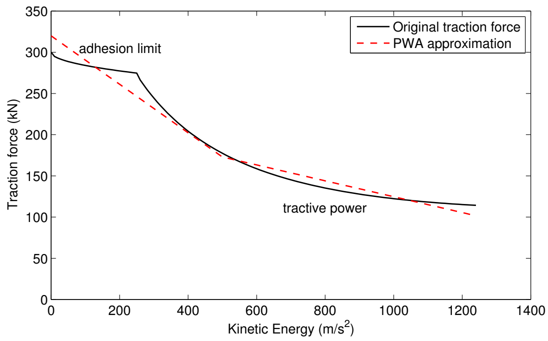

Fig. 2. Maximum traction force in dependency of kinetic energy and its PWA approximation

a function of the velocity v. Due to the maximum adhesion and the characteristics of the power equipment (Hansen and Pachl, 2008; Schank, 2011), the function of the maximum traction force $u_{\max}(v)$ is described as a group of hyperbolic or parabolic formulas in Hansen and Pachl (2008), where each formula approximates the actual traction force for a certain speed interval. For example, if the train speed v belongs to interval $[v_j, v_{j+1}]$, then the maximum traction force can be written as

$$u_{\max}(v) = c_{1,j} + c_{2,j}v + c_{3,j}v^2, \quad v \in [v_j, v_{j+1}], \tag{16}$$

or

$$u_{\max}(v) = c_{\mathrm{h},j}/v, \ v \in [v_j, v_{j+1}], \tag{17}$$

where v_j, v_{j+1}, $c_{1,j}$, $c_{2,j}$, $c_{3,j}$, and $c_{\mathrm{h},j}$ are determined by the characteristics of the train for $j = 1, 2, \cdots, M-1$.

The varying maximum traction force can be reformulated as a nonlinear function of the kinetic energy as shown in Figure 2 (Schank, 2011). In a similar way as the approximation of the nonlinear function $f(E(k))$, we can obtain PWA approximations of the maximum traction force depending on the space interval index k. If we consider an approximation using 2 affine subfunctions (cf. Figure 2), then the approximation can be written as

$$u_{\max,\mathrm{PWA}}(E(k)) = \begin{cases} \lambda_{1,k}E(k) + \mu_{1,k} & \text{for } E_{\min,k} \le E(k) \le E_{2,k}, \\ \lambda_{2,k}E(k) + \mu_{2,k} & \text{for } E_{2,k} \le E(k) \le E_{\max,k}, \end{cases} \tag{18}$$

where values of $E_{2,k}$ and coefficients of affine functions are decided by weighted least-squares optimization (Azuma et al., 2010).

In order to deal with the PWA constraints of the maximum traction force, a logical variable $\delta_2(k)$ is introduced, which is defined by

$$[E(k) \le E_{2,k}] \Leftrightarrow [\delta_2(k) = 1].$$

Since $E_{\min,k}$ and $E_{\max,k}$ are the minimum and maximum values of $E(k)$ for $k = 1, 2, \cdots, N$, by applying the transformation properties of Bemporad and Morari (1999), this logical condition can be rewritten as

$$\begin{aligned} (E_{\max,k} - E_{2,k})\delta_2(k) &\le E_{\max,k} - E(k), \\ (E_{\min,k} - E_{2,k} - \varepsilon)\delta_2(k) &\le E(k) - E_{2,k} - \varepsilon, \end{aligned} \tag{19}$$

where ε is a small positive number, typically the machine precision (Bemporad and Morari, 1999). By defining a new auxiliary variable $z_2(k) = \delta_2(k)E(k)$, which according to Bemporad and Morari (1999) can be expressed as

$$\begin{aligned} z_2(k) &\le E_{\max,k}\delta_2(k), \\ z_2(k) &\ge E_{\min,k}\delta_2(k), \\ z_2(k) &\le E(k) - E_{\min,k}(1-\delta_2(k)), \\ z_2(k) &\ge E(k) - E_{\max,k}(1-\delta_2(k)), \end{aligned} \tag{20}$$

the PWA constraints $u(k) \le u_{\max,\mathrm{PWA}}(E(k))$ can be written as

$$u(k) \le \delta_2(k)[\lambda_{1,k}E(k) + \mu_{1,k}] + (1-\delta_2(k))[\lambda_{2,k}E(k) + \mu_{2,k}],$$

which can be reformulated as

$$\begin{aligned} -(\lambda_{1,k} - \lambda_{2,k})z_2(k) - (\mu_{1,k} - \mu_{2,k})\delta_2(k) + \\ u(k) - \lambda_{2,k}E(k) - \mu_{2,k} \le 0. \end{aligned} \tag{21}$$

By redefining the coefficient matrices and extending the variables $\delta(k)$ and $z(k)$ of the MLD model (9) and (10) properly, the constraints caused by the varying maximum traction force can be included into (9) - (10).

It is important to note that all the extensions presented in this section still lead to an MILP problem.

4. OPERATIONAL CONSTRAINTS

Train operations are restricted by the maximum speed limits, the characteristics of trains (e.g. the maximum traction force, the mass of train, and the maximum speed of the train), the properties of lines (e.g. the grade profile, tunnels, and curves), and so on (Liu and Golovicher, 2003).

There also exist some other constraints that result from the timetable, real-time operation restrictions, or the real-time rescheduling process. Albrecht et al. (2010, 2011) classified these operational constraints into two groups: target points and target windows. Target points correspond to fixed passing times, which could be arrival and departure times at stations. In dense networks, target points could also be passing times at certain places where overtaking and crossing of trains is planned. If the passing time is not that strict but is characterized by an earliest arrival time and a tolerated delay, then it forms a target window constraint. The scheduled arrival times at minor stations without connections with other trains can be regarded as target windows. If the train reaches a certain place exactly on time according to the defined target point or in the target window, then conflicts can be avoided.

It is assumed that the positions corresponding to target points or target window constraints are $s_{\mathrm{t},j}$, for $j \in \{1, 2, \ldots, N_{\mathrm{t}}\}$ with $s_{\mathrm{t},j} = s_{k_j}$ for some k_j in $\{1, 2, \ldots, N\}$. The operational constraints can be included in the optimal control problem as follows:

- for target points:

$$t(s_{\mathrm{t},j}) = T_{\mathrm{target},j}, \tag{22}$$

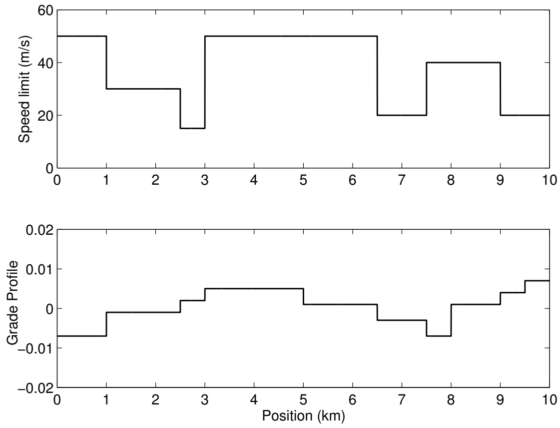

Fig. 3. The speed limits and the grade profile of the line

- for target windows:

$$T_{\text{target}_{\min},j} \leq t(s_{\text{t},j}) \leq T_{\text{target}_{\max},j}, \tag{23}$$

where $T_{\text{target},j}$ is the fixed passing time for train to pass position $s_{\text{t},j}$, and $T_{\text{target}_{\min},j}$ and $T_{\text{target}_{\max},j}$ are the minimum and maximum passing time at position $s_{\text{t},j}$ respect to the target window constraints. In addition, $t(s_{\text{t},j})$ is corresponding to $t(k_j)$, which is one of the state variables of the model (9) - (10).

Note that (22) and (23) are linear constraints. Hence, we still have an MILP problem.

5. CASE STUDY

As a benchmark, we use an extended case study of Vašak et al. (2009) and Schank (2011). Because the parameters for the train in Schank (2011) are derived from existing rolling stock data, in this paper we will use those parameters as shown in Table 1. The speed limits and grade profile are shown in Figure 3. We assume the track sections between 2500 m to 3000 m are the most critical bottleneck on this line, where the intercity train overtakes the regional train. As stated in Albrecht et al. (2010), there exists an exact time for the intercity train to pass this bottleneck, so that the intercity train and the regional train will not hinder each other. It is assumed that the scheduled running time for the intercity train to run the whole journey is 500 s and the exact passing time for a train to pass the entrance of the bottleneck, i.e. 2500 m, could be 130 s, 140 s, or 150 s.

The length Δs_k for interval $[s_k, s_{k+1}]$ depends on the speed limits, gradient profile, tunnels, operational constraints, and so on. In addition, if the number of space intervals N is large, then the computation time of the MILP approach will be long, but the accuracy will be better. According to the speed limits, grade profile, and operational constraints, the length of each interval is chosen to equal 500 m, i.e. $\Delta s_k = 500$ m for $k = 1, 2, \ldots, 20$, which provides a good balance between the computation time and the accuracy. The nonlinear function $f(E(k)) = \frac{1}{2\sqrt{2E(k)}}$ is approximated by using different PWA approximations for different space intervals. The maximum traction force is a nonlinear function of the train's speed as stated in Section 3, which is approximated by PWA functions as shown in (18). The coefficients of (18) depend on the space interval index k. Here, for simplicity, we just use one PWA approximation with two affine subfunctions for all k as shown in Figure 2. The parameters of the PWA function are listed in Table 2. The

Table 1. Parameters of train and line path (Schank, 2011)

Property	Symbol	Value
Train mass [kg]	m	$5.07 \cdot 10^5$
Basic resistance [N/kg]	R_b	$0.014 + 2.564 \cdot 10^{-5} v^2$
Mass factor	ρ	1.06
Line length [m]	s_T	10^4
Minimum kinetic energy [J]	$E_{\min}$	0.1
Maximum braking force (regular)[N]	$u_{\min}$	$-4.475 \cdot 10^5$

Table 2. The coefficients of the PWA approximation of maximum traction force

Segment m	λ_m [kg/m]	μ_m [kg· m/s^2]	$E_m - E_{m+1}$ [(m/s)^2]
1	$-2.9396 \cdot 10^2$	$4.1992 \cdot 10^5$	$0.1 - 500$
2	$-0.9637 \cdot 10^2$	$3.2112 \cdot 10^5$	$500 - 1250$

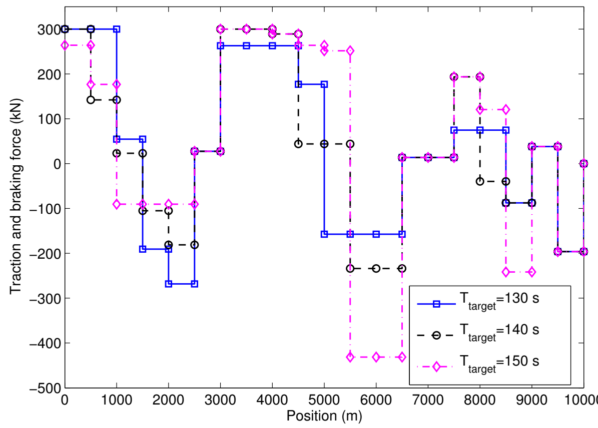

Fig. 4. The optimal control inputs corresponding to different operational constraints

Table 3. The actual energy consumption corresponding to different operational constraints

Target point constraints (Passing time) [s]	130	140	150
Energy consumption [MJ]	324.17	325.99	340.59

objective function in this paper is a trade-off between energy consumption and riding comfort. The value of λ in (4) could be chosen properly according to the requirements, and is taken equal to 25 in this case study.

The optimal control inputs obtained by the MILP approach corresponding to target points 130 s, 140 s, and 150 s are shown in Figure 4. The computing time for each scenario is about 0.55 seconds on a 1.8 GHz Intel Core2 Duo CPU running a 64-bit Linux operating system. When applying these inputs to the nonlinear model of the nonlinear model of the train (1) and (2), we can obtain the optimal trajectories shown in Figure 5. The actual energy consumptions corresponding to these three scenarios are shown in Table 3. The case study shows that the proposed MILP approach can solve the optimal trajectory planning problem under operational constraints, which has little affect to the calculation time.

6. CONCLUSIONS AND FUTURE WORK

We have considered the optimal trajectory planning problem for trains under operational constraints. The nonlinear train operation model is formulated as a mixed logical dynamical model

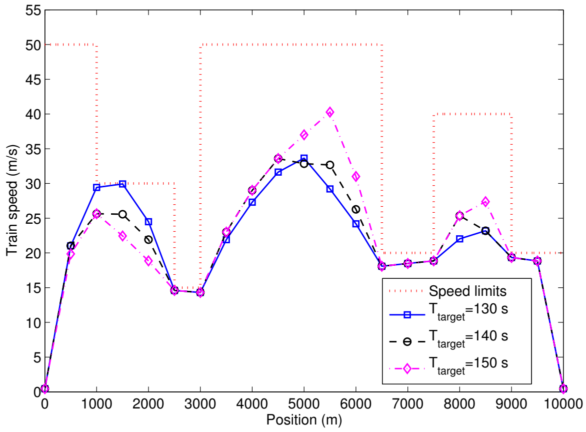

Fig. 5. The optimal trajectories based on nonlinear train model for control signals computed using the MILP approach

by using piecewise affine approximation. The variable line resistance, speed limits, varying maximum traction force, and operational constraints (including target window constraints and target point constraints) are considered. The optimal control problem then is recast as a mixed integer linear programming (MILP) problem, which can be solved efficiently by existing solvers.

From the mixed logical dynamical model transformation and the formulation of the MILP problem, it is worth to note that the operational constraints can be easily included in the problem definition. Furthermore, the operational constraints have little effect on the calculation time. The computation cost of the MILP problem with the operational constraints is less than 1 second in the case study. An extensive comparison and assessment of the MILP approach and other approaches from the literature for various case studies and a wide range of scenarios considering different operational constraints will be a topic for future work.

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