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A Full Characterization of the Set of Optimal Affine Solutions to the Reverse Stackelberg Game

Noortje Groot, Bart De Schutter, and Hans Hellendoorn

***Abstract*— The class of reverse Stackelberg games can be used as a structure for hierarchical decision making and can be adopted in multi-level optimization approaches for large-scale control problems like road tolling. In this game, a leader player acts first by presenting a leader function that maps the follower decision space into the leader decision space. Subsequently, the follower acts by presenting his optimal decision variables. In order to solve the – in general complex – reverse Stackelberg game, a specific structure of the leader function is considered in this paper, given a desired equilibrium that the leader strives to achieve. We present conditions for the existence of such an optimal affine leader function in the static reverse Stackelberg game and delineate the set of all possible solutions of the affine leader function structure. The parametrized characterization of such a set facilitates further optimization, e.g., when considering the sensitivity to deviations from the optimal follower response, as is illustrated by a simple example. Moreover, it can be used to verify the existence of an optimal affine leader function in a constrained decision space.**

I. Introduction

In the control of large-scale systems, a centralized approach is often computationally intractable. Moreover, it may be infeasible for the centralized controller to have access to all relevant information on the system. As an alternative to the more efficient decentralized and distributed optimization approaches, hierarchical methods can be adopted in control settings where a natural hierarchy applies, or as a way to structure the optimization problem [1], [2]. In this paper, we focus on a particular type of hierarchical game as a means to structure multi-level optimization problems. In the reverse Stackelberg game [3], also known as incentives [4], a leader player acts first by presenting a leader function that maps the follower decision space into the leader decision space. Subsequently, the follower acts by presenting his optimal decision variables. Applications of this hierarchical game can be found in e.g., road tolling [5] and network [6] and electricity pricing [7].

Unfortunately, already the deterministic, static single-leader single-follower reverse Stackelberg problem we study here is in general analytically and computationally complex, which is due to the composed functions apparent in the optimization problem [8]. Moreover, the game shares its structure with the strongly NP-hard bilevel programming problem [9]. Hence, to ease the solvability of the reverse Stackelberg problem, a particular leader optimum is generally assumed to be the desired reverse Stackelberg equilibrium [3], [4], [10]. Several leader function structures can then be investigated that lead to this desired equilibrium, i.e., the leader function is solved for in a parametrized approach.

However, the current literature mostly resides in specific games that have a strictly convex, differentiable follower objective function; under these conditions an affine leader function is automatically optimal under the absence of additional constraints [11], [12]. In order to generalize these strong assumptions, in earlier work [13], we have developed conditions for the existence of an optimal affine leader function for the unconstrained case, considering also nonsmooth and nonconvex sublevel sets. For the constrained case, only necessary conditions were developed.

In the current paper a complete parametrized characterization is made of the set of affine leader functions that can solve the reverse Stackelberg game to optimality. In addition, several examples show the importance of the characterization of the full set of solutions. E.g., nonuniqueness of the optimal leader function gives rise to the possibility of a secondary selection criterion, as is also suggested in [14], [15]: by minimizing a sensitivity function, a robust leader function could be selected. Moreover, with this complete set one can easily verify whether or not an optimal affine solution also exists in the constrained case. These results therefore add to the basis of a structured approach for solving more general subclasses of the complex reverse Stackelberg game.

The remainder of this paper is structured as follows. After the definition of the reverse Stackelberg game, some preliminary notation and assumptions are stated in Section II. In Section III the main existence results for an optimal affine leader function are presented for the unconstrained case with a nonconvex and nonsmooth sublevel set. Then, in Section IV we derive the full set of affine leader functions that solve the reverse Stackelberg game to optimality in the unconstrained case. These results are analyzed under the presence of constraints in Section V and an example is provided to show the relevance of the full characterization. A secondary objective is considered as a further motivation of the characterization in Section VI. The paper is concluded in Section VII.

II. Preliminaries

A. The Reverse Stackelberg Game

A common approach to the reverse Stackelberg problem is for the leader player to select a particular desired optimum $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$ that she seeks to achieve [4], [10], with the leader and follower decision variables denoted by $u_{\mathrm{L}} \in \Omega_{\mathrm{L}} \subseteq \mathbb{R}^{n_{\mathrm{L}}}$, $u_{\mathrm{F}} \in \Omega_{\mathrm{F}} \subseteq \mathbb{R}^{n_{\mathrm{F}}}$. A natural choice would be the leader's global optimum $\arg\min_{u_{\mathrm{L}} \in \Omega_{\mathrm{L}}, u_{\mathrm{F}} \in \Omega_{\mathrm{F}}} \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}, u_{\mathrm{F}})$,

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where $\mathcal{J}_p : \Omega_\mathrm{L} \times \Omega_\mathrm{F} \to \mathbb{R}, p \in \{\mathrm{L}, \mathrm{F}\}$ denotes respectively the leader's and follower's objective function.

The single-leader single-follower, static, deterministic reverse Stackelberg game can now be defined as follows. Let Γ_L denote the class of admissible leader functions in a particular game context. The problem then becomes for the leader to determine an optimal leader function $\gamma_\mathrm{L} : \Omega_\mathrm{F} \to \Omega_\mathrm{L}$ that leads to the desired equilibrium $(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$:

$$\text{To find:} \quad \gamma_\mathrm{L} \in \Gamma_\mathrm{L}, \tag{1}$$

$$\text{s.t.} \quad \arg \min_{u_\mathrm{F} \in \Omega_\mathrm{F}} \mathcal{J}_\mathrm{F}(\gamma_\mathrm{L}(u_\mathrm{F}), u_\mathrm{F}) = u_\mathrm{F}^\mathrm{d}, \tag{2}$$

$$\gamma_\mathrm{L}(u_\mathrm{F}^\mathrm{d}) = u_\mathrm{L}^\mathrm{d}. \tag{3}$$

The leader should thus construct her function γ_L such that it passes through the desired optimum but such that it does not contain any other point in the sublevel set

$$\Lambda_\mathrm{d} := \{(u_\mathrm{L}, u_\mathrm{F}) \in \Omega_\mathrm{L} \times \Omega_\mathrm{F} | \mathcal{J}_\mathrm{F}(u_\mathrm{L}, u_\mathrm{F}) \le \mathcal{J}_\mathrm{F}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})\}.$$

For such γ_L the follower will select u_F^d under the minimization of his objective function.

The problem (1)–(3) is often solved by determining the optimal parameters of a fixed structure of the leader function γ_L [4], often considering the affine function structure [11], [12]. In the remainder of this paper, the full set of such affine leader functions will be derived under which the leader is able to induce the follower to choose the input u_F^d and thus reach the desired solution.

B. Notation

In addition to some commonly adopted concepts like strictly convex sets and supporting hyperplanes, we will use the following definitions. See also [16], [17], [18].

A set X is an *affine subspace* [17] if $y, z \in X \iff \alpha y + (1-\alpha) z \in X \forall \alpha \in \mathbb{R}$.

A *vertex point* [18] or exposed point of a convex set intersects with a strictly supporting hyperplane. Similarly, a point $\tilde{x}$ of a nonconvex set is a vertex point if there exists a neighborhood of $\tilde{x}$, $\mathcal{N}(\tilde{x})$, such that $\tilde{x}$ intersects with a strictly supporting hyperplane to $\mathcal{N}(\tilde{x})$.

The *generalized gradient* [19] $\partial f(x)$ of a locally Lipschitz continuous function $f : \mathbb{R}^n \to \mathbb{R}$ at x is defined as follows: $\partial f(x) := \mathrm{conv}(\{\lim_{m \to \infty} \nabla f(x_m) | x_m \to x, x_m \in \mathrm{dom}(f) \setminus \Omega_f\})$, with Ω_f being the set of points where f is nondifferentiable and where no limit $\lim_{m \to \infty} \nabla f(x_m)$ exists [19]. By $\mathcal{V}(X(x))$ we denote the *generalized normal* to the set X at the point $x \in \bar{X}$, defined as the set of normal vectors to the possible tangent hyperspaces to X at x.

Finally, $\Pi_X(x)$ denotes a *supporting hyperplane* to the set X at the point $x \in X$. The *projection* of a vector v on the vector x is denoted by $\mathrm{proj}_x(v)$ Similarly, the projection of the set $P \subseteq \mathbb{R}^n$ onto the space $X = \mathbb{R}^m, m \le n$ is denoted by $\mathrm{proj}_X(P)$. The *null space* of X is denoted by $\mathrm{null}(X)$. We denote by $\mathcal{B}_X$ the matrix, the columns of which form a basis of X. Let $n_{\mathcal{B}_X}$ be the number of columns of $\mathcal{B}_X$, then $\dim(\mathcal{B}_X) = n_{\mathcal{B}_X}$.

Remark 1: Note that in this paper the leader function γ_L is defined as a mapping $\Omega_\mathrm{F} \to \Omega_\mathrm{L}$ that can also be represented by the set of points $\{(u_\mathrm{L}, u_\mathrm{F}) | u_\mathrm{F} \in \Omega_\mathrm{F}, u_\mathrm{L} = \gamma_\mathrm{L}(u_\mathrm{F})\}$ [17]. Hence, we adopt both the mapping and the set representation of γ_L depending on the context.

C. Assumptions

Throughout the paper the following assumptions are made:

[A.1] Let $\Omega_\mathrm{L}, \Omega_\mathrm{F}$ be convex sets.
[A.2] Let Λ_d be a connected set.
[A.3] Let $n_\mathrm{L}, n_\mathrm{F}$ be finite.
[A.4] Let $\Lambda_\mathrm{d} \neq \{(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})\}$.

Assumption [A.1] is taken from the literature [11], [20] and ensures convexity of $\mathcal{J}_\mathrm{F}$ and Λ_d. The strong assumption of taking $\mathcal{J}_\mathrm{F}$ and therefore also Λ_d to be strictly convex [11] is replaced by the relaxed condition [A.2]. In case $\mathcal{J}_\mathrm{F}$ is a convex or quasiconvex function, [A.2] is automatically satisfied. Further, assumption [A.3] is needed in order to be able to adopt the concept of a supporting hyperplane. This assumption is also accepted in many control applications [21]. Finally, [A.4] excludes the special case in which $(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$ is automatically optimal for the follower, in which case the game has a trivial solution.

III. Necessary and Sufficient Conditions

In this section, conditions for the existence of an optimal affine leader function in the unconstrained case are stated, as derived in [13]. These existence conditions form the basis for a characterization of the set of optimal solutions. Here, the most general case of Λ_d nonconvex and nonsmooth is considered, where two cases are distinguished. Propositions 2 and 3 consider respectively the case in which the desired leader equilibrium $(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$ is a vertex point of $\mathrm{conv}(\Lambda_\mathrm{d})$, or is in the interior of $\mathrm{conv}(\Lambda_\mathrm{d})$ for $n_\mathrm{L} \ge 1$ and $n_\mathrm{L} > 1$. Note that for $n_\mathrm{L} = 1$ and $(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$ not a vertex point of $\mathrm{conv}(\Lambda_\mathrm{d})$, no optimal affine leader function exists.

Proposition 2 ([13]): Let $(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$ be a vertex point of $\mathrm{conv}(\Lambda_\mathrm{d})$ and assume that $n_L \ge 1$. Additionally, allow Λ_d to be nonsmooth at $(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$ and assume that $\Omega_\mathrm{L} = \mathbb{R}^{n_\mathrm{L}}, \Omega_\mathrm{F} = \mathbb{R}^{n_\mathrm{F}}$. Then the desired equilibrium $(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$ can be reached under an affine $\gamma_\mathrm{L} : \Omega_\mathrm{F} \to \Omega_\mathrm{L}$ if and only if $\mathrm{proj}_{\Omega_\mathrm{L}}(\mathcal{V}(\mathrm{conv}(\Lambda_\mathrm{d}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})))) \neq \{0\}$.

Proposition 3 ([13]): Let $n_L > 1$ and assume that $(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d}) \in \mathrm{int}(\mathrm{conv}(\Lambda_\mathrm{d}))$. Allow Λ_d to be nonsmooth at $(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$ and assume that $\Omega_\mathrm{L} = \mathbb{R}^{n_\mathrm{L}}, \Omega_\mathrm{F} = \mathbb{R}^{n_\mathrm{F}}$. Then the desired equilibrium $(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$ can be reached under an affine $\gamma_\mathrm{L} : \Omega_\mathrm{F} \to \Omega_\mathrm{L}$ if and only if there exists an n_F-dimensional tangent, affine subspace $\Pi_\mathrm{d}^\mathrm{t}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$ to Λ_d at $(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$ such that $\Pi_\mathrm{d}^\mathrm{t}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d}) \cap \Lambda_\mathrm{d} = \{(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})\}$ and such that $\mathrm{proj}_{\Omega_\mathrm{L}}(\mathcal{V}(\Lambda_\mathrm{d}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d}))) \neq \{0\}$.

It is important to note that an optimal affine γ_L should satisfy (1) and (2). Constraint (3) is easily satisfied, i.e., for any affine function passing through the desired equilibrium. The main elements of the necessary and sufficient conditions w.r.t. (1) and (2) are therefore:

- $\mathrm{dom}(\gamma_\mathrm{L}) = \Omega_\mathrm{F}$, i.e., it is not allowed for the function γ_L represented by a set of points to be perpendicular to the follower decision space. This orthogonality requirement is necessary and sufficient for the coverage of Ω_F by an affine function if $\Omega_\mathrm{L} = \mathbb{R}^{n_\mathrm{L}}, \Omega_\mathrm{F} = \mathbb{R}^{n_\mathrm{F}}$. Hence, $\mathrm{proj}_{\Omega_\mathrm{L}}(\mathcal{V}(\Lambda_\mathrm{d}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d}))) \neq \{0\}$ or $\mathrm{proj}_{\Omega_\mathrm{L}}(\mathcal{V}(\mathrm{conv}(\Lambda_\mathrm{d}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})))) \neq \{0\}$ is used in the above propositions.
- u_F^d should be the optimal follower response to γ_L, i.e., the set γ_L should not intersect with the sublevel set Λ_d in any other point than $(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$. Hence, the concepts of a vertex point and a tangent hyperplane are adopted in the above propositions.

IV. Characterization of Optimal Affine γ_L

As for the derivation of an affine leader function mapping $\Omega_\mathrm{F} \to \Omega_\mathrm{L}$, the results of Section IV-A were first derived in [11] to which the reader is referred for details. As this result only captures the case of $\mathcal{J}_\mathrm{F}$ differentiable and moreover as only one particular solution is specified, in Section IV-B this characterization is extended to cover the nondifferentiable case and moreover, to capture the full set of optimal leader functions of the affine structure.

A. Under Differentiability Assumptions

The derivation of the affine leader function

$$u_\mathrm{L} = \gamma_\mathrm{L}(u_\mathrm{F}) = u_\mathrm{L}^\mathrm{d} + B(u_\mathrm{F}^\mathrm{d} - u_\mathrm{F}), \tag{4}$$

reduces to the computation of an $n_\mathrm{L} \times n_\mathrm{F}$ matrix B. In order to make sure that B exists as defined next, it is assumed in [11] that $\Omega_\mathrm{L}, \Omega_\mathrm{F}$ are Hilbert spaces and that $\mathcal{J}_\mathrm{F}$ is Fréchet differentiable on $\Omega_\mathrm{L} \times \Omega_\mathrm{F}$. Additionally, $\mathcal{J}_\mathrm{F}$ is assumed to be strictly convex. It is known that for $\Omega_\mathrm{L}, \Omega_\mathrm{F}$ Banach spaces there exists a continuous linear operator B such that $Bu_\mathrm{F} = u_\mathrm{L}, u_\mathrm{F} \neq 0$ [11]. Then, for $n_\mathrm{L}, n_\mathrm{F}$ finite, B should satisfy

$$[\nabla_{u_\mathrm{L}} \mathcal{J}_\mathrm{F}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})]^\mathrm{T} B = [\nabla_{u_\mathrm{F}} \mathcal{J}_\mathrm{F}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})]^\mathrm{T}, \tag{5}$$

which can be verified by taking the inner product of the expression $u_\mathrm{L} = 0$ conform (4) and $[\nabla_{u_\mathrm{L}} \mathcal{J}_\mathrm{F}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})]^\mathrm{T}$, which is required to be nonzero as follows from the conditions for the existence of an optimal affine γ_L. This product

$$\begin{aligned}
0 &= [\nabla_{u_\mathrm{L}} \mathcal{J}_\mathrm{F}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})]^\mathrm{T} [(u_\mathrm{L}^\mathrm{d} - u_\mathrm{L}) + B(u_\mathrm{F}^\mathrm{d} - u_\mathrm{F})] \\
&= [\nabla_{u_\mathrm{L}} \mathcal{J}_\mathrm{F}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})]^\mathrm{T} (u_\mathrm{L}^\mathrm{d} - u_\mathrm{L}) \\
&\qquad + [\nabla_{u_\mathrm{L}} \mathcal{J}_\mathrm{F}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})]^\mathrm{T} B (u_\mathrm{F}^\mathrm{d} - u_\mathrm{F}) \\
&= [\nabla_{u_\mathrm{L}} \mathcal{J}_\mathrm{F}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})]^\mathrm{T} (u_\mathrm{L}^\mathrm{d} - u_\mathrm{L}) \\
&\qquad + [\nabla_{u_\mathrm{F}} \mathcal{J}_\mathrm{F}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})]^\mathrm{T} (u_\mathrm{F}^\mathrm{d} - u_\mathrm{F})
\end{aligned} \tag{6}$$

corresponds exactly to the expression of a tangent hyperplane $\Pi^\mathrm{t}_{\Lambda_\mathrm{d}}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$ to Λ_d at $(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$, from which it is concluded that if (5) holds, the affine function γ_L indeed lies on the hyperplane $\Pi^\mathrm{t}_{\Lambda_\mathrm{d}}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$.

Under the condition $\nabla_{u_\mathrm{L}} \mathcal{J}_\mathrm{F}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d}) \neq 0$ the following expression is mentioned in [11]:

$$B = \nabla_{u_\mathrm{L}} \mathcal{J}_\mathrm{F}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d}) \nabla^\mathrm{T}_{u_\mathrm{F}} \mathcal{J}_\mathrm{F}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d}) / ||\nabla^\mathrm{T}_{u_\mathrm{L}} \mathcal{J}_\mathrm{F}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})||^2. \tag{7}$$

Note that this is only one of many possible expressions for $n_\mathrm{L} > 1$. Moreover, in some constrained cases this expression does not yield an optimal leader function, while an alternative, optimal affine solution does exist as will be illustrated by an example in Section V. A generalized characterization of the optimal affine leader function is therefore developed in Section IV-B below.

B. The General Case

A more general approach that does not require differentiability of $\mathcal{J}_\mathrm{F}$ is by characterizing γ_L as a linear combination of matrices $R^\mathrm{T} = \begin{bmatrix} R_\mathrm{L}^\mathrm{T} & R_\mathrm{F}^\mathrm{T} \end{bmatrix}$, $R \in \mathbb{R}^{(n_\mathrm{L}+n_\mathrm{F}) \times n_\mathrm{F}}, R_\mathrm{L} \in \mathbb{R}^{n_\mathrm{L} \times n_\mathrm{F}}, R_\mathrm{F} \in \mathbb{R}^{n_\mathrm{F} \times n_\mathrm{F}}$, i.e.,

$$\gamma_\mathrm{L} : \begin{bmatrix} u_\mathrm{L} \\ u_\mathrm{F} \end{bmatrix} = \begin{bmatrix} u_\mathrm{L}^\mathrm{d} \\ u_\mathrm{F}^\mathrm{d} \end{bmatrix} + \begin{bmatrix} R_\mathrm{L} \\ R_\mathrm{F} \end{bmatrix} \cdot s, \tag{8}$$

where $s \in \mathbb{R}^{n_\mathrm{F}}$ represents the free parameters of the affine function. Now, for R_F invertible – which automatically follows from the necessary conditions as will be proven in Lemma 4 – it follows that:

$$\begin{aligned}
&u_\mathrm{F} = u_\mathrm{F}^\mathrm{d} + R_\mathrm{F} \cdot s \qquad \Rightarrow s = R_\mathrm{F}^{-1}(u_\mathrm{F} - u_\mathrm{F}^\mathrm{d}), \\
&u_\mathrm{L} = u_\mathrm{L}^\mathrm{d} + \underbrace{R_\mathrm{L} R_\mathrm{F}^{-1}}_{B}(u_\mathrm{F} - u_\mathrm{F}^\mathrm{d}),
\end{aligned} \tag{9}$$

i.e., one arrives at the explicit form of leader function (4). The problem left in order to arrive at a full characterization of an optimal affine γ_L is to determine the set of possible basis vectors.

Lemma 4: In order for a leader function γ_L characterized by (8) to be optimal, for $R^\mathrm{T} = \begin{bmatrix} R_\mathrm{L}^\mathrm{T} & R_\mathrm{F}^\mathrm{T} \end{bmatrix}$ the following should hold:

1) $\exists \nu \in \mathcal{V}(X) : \nu^\mathrm{T} R = 0^\mathrm{T}$, with $\mathcal{V}(X)$ the generalized normal to X at $(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$, $X = \mathrm{conv}(\Lambda_\mathrm{d}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d}))$ (Prop. 2) or $X = \Lambda_\mathrm{d}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$ (Prop. 3).
2) The columns of R_F should be a basis for Ω_F, i.e., R_F should be of full rank n_F and thus invertible.

Proof:

1) By definition of a tangent hyperplane $\Pi_\mathrm{d}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$ to X at $(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$, $\Pi_\mathrm{d}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d}) \perp \nu$ for some $\nu \in \mathcal{V}(X)$. Since we require γ_L characterized by (8) to lie on $\Pi_\mathrm{d}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$, it follows that it is needed that also $R \perp \nu$, i.e., $\nu^\mathrm{T} R = 0^\mathrm{T}$.
   Note that in case $\mathcal{J}_\mathrm{F}$ is differentiable, i.e., $\nu^\mathrm{T} = \begin{bmatrix} \nu_\mathrm{L}^\mathrm{T} & \nu_\mathrm{F}^\mathrm{T} \end{bmatrix}$ with $\nu_\mathrm{L} = \nabla_{u_\mathrm{L}} \mathcal{J}_\mathrm{F}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$ and $\nu_\mathrm{F} = \nabla_{u_\mathrm{F}} \mathcal{J}_\mathrm{F}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})$, this condition is equivalent to (6).
2) the n_F columns of R_F are independent basis vectors spanning Ω_F. Thus, R_F is of full rank hence invertible.

■

In fact, we can select w.l.o.g. $R_\mathrm{F} := I_{n_\mathrm{F}} = \begin{bmatrix} e_1 & \ldots & e_{n_\mathrm{F}} \end{bmatrix}$ as shown in Lemma 5.

Lemma 5: If there exists an optimal affine γ_L characterized by (8), one can select w.l.o.g. $R_\mathrm{F} = I_{n_\mathrm{F}}$.

Proof: Consider

$$\mathcal{S} := \left\{ \gamma_\mathrm{L} \;\middle|\; \begin{bmatrix} u_\mathrm{L} \\ u_\mathrm{F} \end{bmatrix} = \begin{bmatrix} u_\mathrm{L}^\mathrm{d} \\ u_\mathrm{F}^\mathrm{d} \end{bmatrix} + \begin{bmatrix} R_\mathrm{L} \\ I_{n_\mathrm{F}} \end{bmatrix} \cdot s, \; s \in \mathbb{R}^{n_\mathrm{F}}, \text{with} \right.$$

$R_{\mathrm{L}}, R_{\mathrm{F}} = I_{n_{\mathrm{F}}}$ satisfying conditions 1) and 2) of Lemma 4$\}$,

$$\tilde{\mathcal{S}} := \left\{ \gamma_{\mathrm{L}} \;\middle|\; \begin{bmatrix} u_{\mathrm{L}} \\ u_{\mathrm{F}} \end{bmatrix} = \begin{bmatrix} u_{\mathrm{L}}^{\mathrm{d}} \\ u_{\mathrm{F}}^{\mathrm{d}} \end{bmatrix} + \begin{bmatrix} \tilde{R}_{\mathrm{L}} \\ \tilde{R}_{\mathrm{F}} \end{bmatrix} \cdot \tilde{s},\ \tilde{s} \in \mathbb{R}^{n_{\mathrm{F}}}, \text{with } \tilde{R}_{\mathrm{L}}, \tilde{R}_{\mathrm{F}} \text{ satisfying conditions 1) and 2) of Lemma 4 for } \tilde{R} := R \right\}.$$

To prove that $\mathcal{S} \equiv \tilde{\mathcal{S}}$ we will show that for each possible 3-tuple $(s, I_{n_{\mathrm{F}}}, R_{\mathrm{L}})$ according to (8) with $\nu^{\mathrm{T}} \begin{bmatrix} R_{\mathrm{L}}^{\mathrm{T}} & I_{n_{\mathrm{F}}} \end{bmatrix}^{\mathrm{T}} = 0^{\mathrm{T}}$ that yields some $u_{\mathrm{L}}, u_{\mathrm{F}}$, one can find an equivalent tuple $(\tilde{s}, \tilde{R}_{\mathrm{F}}, \tilde{R}_{\mathrm{L}})$, $\tilde{s} \in \mathbb{R}^{n_{\mathrm{F}}}$ for which additionally it holds that $\nu^{\mathrm{T}} \begin{bmatrix} \tilde{R}_{\mathrm{L}}^{\mathrm{T}} & \tilde{R}_{\mathrm{F}} \end{bmatrix}^{\mathrm{T}} = 0^{\mathrm{T}}$, yielding the same $u_{\mathrm{L}}, u_{\mathrm{F}}$.

It can be easily seen that the expression $u_{\mathrm{F}} = u_{\mathrm{F}}^{\mathrm{d}} + I_{n_{\mathrm{F}}} \cdot s$ is equivalent to $u_{\mathrm{F}} = u_{\mathrm{F}}^{\mathrm{d}} + \tilde{R}_{\mathrm{F}} \cdot \tilde{s}$ with $s = \tilde{R}_F \cdot \tilde{s}$: as shown in Lemma 4 it follows from the existence of an optimal affine γ_{L} that R_{F} is invertible. Then, for a given s there exists a unique $\tilde{s}$ and vice versa. From $B = \tilde{R}_{\mathrm{L}} \tilde{R}_{\mathrm{F}}^{-1}$ according to (9) and from the substitution to $B = R_{\mathrm{L}} I_{n_{\mathrm{F}}}$, for equivalence it should hold that $R_{\mathrm{L}} = \tilde{R}_{\mathrm{L}} \tilde{R}_{\mathrm{F}}^{-1}$. Finally, we have that

$$\begin{aligned} \nu_{\mathrm{L}}^{\mathrm{T}} R_{\mathrm{L}} + \nu_{\mathrm{F}}^{\mathrm{T}} = 0 &\Rightarrow \nu_{\mathrm{L}}^{\mathrm{T}} \tilde{R}_{\mathrm{L}} \tilde{R}_{\mathrm{F}}^{-1} + \nu_{\mathrm{F}}^{\mathrm{T}} = 0 \\ &\overset{\cdot \tilde{R}_{\mathrm{F}}}{\Rightarrow} \nu_{\mathrm{L}}^{\mathrm{T}} \tilde{R}_{\mathrm{L}} + \nu_{\mathrm{F}}^{\mathrm{T}} \tilde{R}_{\mathrm{F}} = 0, \\ \nu_{\mathrm{L}}^{\mathrm{T}} \tilde{R}_{\mathrm{L}} + \nu_{\mathrm{F}}^{\mathrm{T}} \tilde{R}_{\mathrm{F}} = 0 &\Rightarrow \nu_{\mathrm{L}}^{\mathrm{T}} R_{\mathrm{L}} \tilde{R}_{\mathrm{F}} + \nu_{\mathrm{F}}^{\mathrm{T}} \tilde{R}_{\mathrm{F}} = 0 \\ &\overset{\cdot \tilde{R}_{\mathrm{F}}^{-1}}{\Rightarrow} \nu_{\mathrm{L}}^{\mathrm{T}} R_{\mathrm{L}} + \nu_{\mathrm{F}}^{\mathrm{T}} = 0, \end{aligned}$$

hence, $\mathcal{S} = \tilde{\mathcal{S}}$. ■

Now, given $R_{\mathrm{F}} := I_{n_{\mathrm{F}}}$ we still need to identify the set of R_{L} that satisfy $\nu^{\mathrm{T}} R = 0^{\mathrm{T}}$ for some normal vector ν, which reduces to $\begin{bmatrix} \nu_{\mathrm{L}}^{\mathrm{T}} & \nu_{\mathrm{F}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} R_{\mathrm{L}} \\ I_{n_{\mathrm{F}}} \end{bmatrix} = 0^{\mathrm{T}} \Rightarrow \nu_{\mathrm{L}}^{\mathrm{T}} R_{\mathrm{L}} = -\nu_{\mathrm{F}}^{\mathrm{T}}$. Due to the necessary condition $\mathrm{proj}_{\Omega_{\mathrm{L}}}(\mathcal{V}(\mathrm{conv}(\Lambda_{\mathrm{d}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})))) \neq \{0\}$ (Prop. 2) or $\mathrm{proj}_{\Omega_{\mathrm{L}}}(\mathcal{V}(\Lambda_{\mathrm{d}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}))) \neq \{0\}$ (Prop. 3), $\nu_{\mathrm{L}}^{\mathrm{T}}$ must contain at least a nonzero entry. Hence, the expressions $\nu_{\mathrm{L}}^{\mathrm{T}} R_{\mathrm{L},j} = -\nu_{\mathrm{F},j}, j = 1, ..., n_{\mathrm{F}}$ can indeed be solved for. Proposition 7 below provides a parametrized characterization of this problem that will be needed for further optimization.

From the previous derivations, the following theorem automatically follows.

Theorem 6: Let $\Omega_{\mathrm{L}} = \mathbb{R}^{n_{\mathrm{L}}}, \Omega_{\mathrm{F}} = \mathbb{R}^{n_{\mathrm{F}}}$. Assume that the conditions in Proposition 2 or Proposition 3 are satisfied and that therefore an optimal affine leader function of the form (4) exists. Then, the set $\Gamma_{\mathrm{L}}^{*} := \{\gamma_{\mathrm{L}} : \Omega_{\mathrm{F}} \to \Omega_{\mathrm{L}} | \gamma_{\mathrm{L}} \text{ satisfies } (1) - (3)\}$ of such optimal affine solutions according to (4) is characterized by $B := R_{\mathrm{L}} I_{n_{\mathrm{F}}}$, with $\nu_{\mathrm{L}}^{\mathrm{T}} R_{\mathrm{L}} = -\nu_{\mathrm{F}}^{\mathrm{T}}$, for some $\nu \in \mathcal{V}(\mathrm{conv}(\Lambda_{\mathrm{d}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})))$ (Prop. 2) or $\nu \in \mathcal{V}(\Lambda_{\mathrm{d}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}))$ (Prop. 3), with $\mathcal{V}(X(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}))$ the generalized normal to X at $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$.

For the sake of conciseness, in the remainder of this section we will assume $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$ to be a vertex point of $\mathrm{conv}(\Lambda_{\mathrm{d}})$. As a result, we consider the case in which Proposition 2 is satisfied rather than Proposition 3, i.e., we consider the generalized normal $\mathcal{V}(\mathrm{conv}(\Lambda_{\mathrm{d}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})))$. For

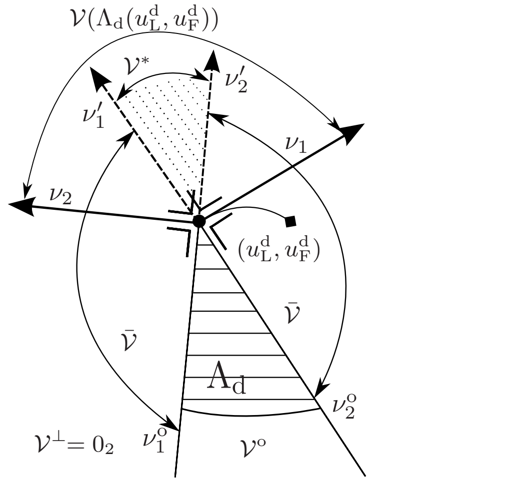

Fig. 1: The normal cone $\mathcal{V}(\Lambda_{\mathrm{d}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}))$ and the associated cone $\bar{\mathcal{V}}(\Lambda_{\mathrm{d}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})) = \left(\mathbb{R}^{n_{\mathrm{L}}+n_{\mathrm{F}}} \setminus \left(\mathcal{V}^{0} \cup \mathcal{V}^{*}\right)\right) \cup \mathcal{V}^{\perp}$.

the case in which Proposition 3 holds, the generalized normal should be substituted by $\mathcal{V}(\Lambda_{\mathrm{d}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}))$ in the following.

In order to be able to optimize over the set of possible leader functions and to select a function that is optimal with respect to some criteria, Proposition 7 now provides a parametrized characterization of the optimal affine leader function, the proof of which can be found in [22].

Proposition 7: Let $\Gamma_{\mathrm{L}}^{*} := \{\gamma_{\mathrm{L}} : \Omega_{\mathrm{F}} \to \Omega_{\mathrm{L}} | \gamma_{\mathrm{L}} \text{ satisfies } (2) - (3), (8)\}$. For Λ_{d} nonsmooth at $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$,

$$\left.\begin{aligned} R \in \mathcal{R} &:= \left\{ \begin{bmatrix} R_1 & \ldots & R_{n_{\mathrm{F}}} \end{bmatrix} \middle| R_j \in \mathcal{R}_j, j = 1, ..., n_{\mathrm{F}} \right\}, \\ \mathcal{R}_j &:= \left\{ W \cdot p_j^{+} | (10) \right\} \cup \left\{ (-W) \cdot p_j^{-} | (10) \right\}, \\ p_j^{k} &:= \left(\sum_{i=1}^{N_{k,j}^{\mathrm{f}}} \alpha_{i,k,j} \beta_{i,k,j}^{\mathrm{f}} + \sum_{i=1}^{N_{k,j}^{\mathrm{e}}} \mu_{i,k,j} \beta_{i,k,j}^{\mathrm{e}} \right) \\ & \sum_{i} \alpha_{i,k,j} = 1, \alpha_{i,k,j} \in \mathbb{R}_{+}, \mu_{i,k,j} \in \mathbb{R}_{+}, k \in \{+, -\} \end{aligned}\right\} \quad (10)$$

for $j = 1, ..., n_{\mathrm{F}}$, with $W = \begin{bmatrix} w_1 & \ldots & w_m \end{bmatrix}$, $w_i \in \mathbb{R}^{n_{\mathrm{L}}+n_{\mathrm{F}}}$ one of the $m \in \mathbb{N}$ generators such that $\bar{\mathcal{V}}(\mathrm{conv}(\Lambda_{\mathrm{d}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}))) := \{\sum_{i=1}^{m} \beta_i w_i | \beta_i \in \mathbb{R}_{+}\} \cup \{\sum_{i=1}^{m} \beta_i (-w_i) | \beta_i \in \mathbb{R}_{+}\}$, which is assumed to be finitely generated.

Further, $\beta_{i,k,j}^{\mathrm{f}}$ is one of the $N_{k,j}^{\mathrm{f}}$ finite vertices and $\beta_{i,k,j}^{\mathrm{e}}$ is one of the $N_{k,j}^{\mathrm{e}}$ extreme rays of the polyhedron $\mathcal{P}_j^{+} = \{\beta | PW\beta = e_j, \beta \in \mathbb{R}_{+}^{m}\}$ or $\mathcal{P}_j^{-} = \{\beta | P(-W)\beta = e_j, \beta \in \mathbb{R}_{+}^{m}\}$, $k \in \{+, -\}$.

For $\mathcal{J}_{\mathrm{F}}$ differentiable at $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$, R_{L} belongs to the affine space of the form

$$\mathcal{R}_{\mathrm{L}} := \left\{ R_{\mathrm{L}} | R_{\mathrm{L}} = R_{\mathrm{L}}^{0} + \mathcal{B}_N \cdot T, T \in \mathbb{R}^{\dim(\mathcal{B}_N) \times n_{\mathrm{F}}} \right\},$$

with R_{L}^{0} a particular solution of $\nabla_{u_{\mathrm{L}}}^{\mathrm{T}} \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}) R_{\mathrm{L}} = \nabla_{u_{\mathrm{F}}} \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$ and with $N := \mathrm{null}\left(\nabla_{u_{\mathrm{L}}}^{\mathrm{T}} \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})\right)$.

Remark 8: So far a static, single-stage reverse Stackelberg game has been considered. Whereas this basic case serves for developing the conditions summarized in Section III and the characterization of the present section, real-life control settings will often have a dynamic, multi-stage nature [23].

In the dynamic game, a state variable $x \in \mathcal{X} \subseteq \mathbb{R}^{n_x}$ is considered with an associated update equation, i.e., in discrete-time: $x(k+1) = A_{\mathrm{L}}(k)u_{\mathrm{L}}(k) + A_{\mathrm{F}}(k)u_{\mathrm{F}}(k) + B(k)x(k)$ with $k \in \mathcal{K} = \{1, 2, ..., K\}, K \in \mathbb{N}$. In addition, this state variable is integrated in the objective functions $\mathcal{J}_p : \Omega_{\mathrm{L}} \times \Omega_{\mathrm{F}} \times \mathcal{X} \to \mathbb{R}, p \in \{\mathrm{L}, \mathrm{F}\}$.

As it is also done in e.g., [11], the current static results can be simply applied to the dynamic case with open-loop information. In other words, at the start of the game the open-loop values $(u_{\mathrm{L}}^{\mathrm{d}}(k), u_{\mathrm{F}}^{\mathrm{d}}(k))$ are computed, for which the mappings $\gamma_{\mathrm{L}}(u_{\mathrm{F}}(k), k)$ can be computed as done in the static case, for each $k \in \mathcal{K}$.

V. Presence of Constraints

Note that for the constrained reverse Stackelberg game with $u_{\mathrm{L}} \in \Omega_{\mathrm{L}} \subset \mathbb{R}^{n_{\mathrm{L}}}, u_{\mathrm{F}} \in \Omega_{\mathrm{F}} \subset \mathbb{R}^{n_{\mathrm{F}}}$ only necessary but no sufficient conditions have been developed for the existence of an optimal affine γ_{L} in [13]. However, with the current set of feasible solutions that is essentially developed for the unconstrained decision space[1], constraints can simply be incorporated to verify whether the elements of Γ_{L}^* are still optimal under the constrained condition. Thus, while the derivation of the initial set Γ_{L}^* for the unconstrained case is a local process that uses the locally defined hyperplane $\Pi_{\mathrm{conv}(\Lambda_{\mathrm{d}})}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$, in order to verify whether the global conditions (a) $\mathrm{dom}(\gamma_{\mathrm{L}}) = \Omega_{\mathrm{F}}$ and (b) $\gamma_{\mathrm{L}}(\Omega_{\mathrm{F}}) \subseteq \Omega_{\mathrm{L}}$ are satisfied, Γ_{L}^* may be reduced to exclude infeasible elements.

To summarize the solution approach: first, an initial set Γ_{L}^* of optimal affine functions that are locally feasible is derived from the necessary and sufficient conditions of the existence of an optimal affine leader function for unconstrained decision spaces. Then, we deal with additional constraints in the game by restricting this set to include only those elements that satisfy $\gamma_{\mathrm{L}}(\Omega_{\mathrm{F}}) \subseteq \Omega_{\mathrm{L}}$.

Expression (7) Subject to Constraints

We now provide a situation in which the specific expression of B proposed in [11] for $\mathcal{J}_{\mathrm{F}}$ differentiable at $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$ does not yield a feasible leader function in the constrained case, but in which an optimal leader function does exist.

Let

$$\mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}, u_{\mathrm{F}}) = (u_{\mathrm{F}} - 6)^2 + (u_{\mathrm{L},1} - 1)^2 + (u_{\mathrm{L},2} - 5)^2$$

and let $(u_{\mathrm{L},1}^{\mathrm{d}}, u_{\mathrm{L},2}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}) = (4, 0.5, 6)$. Then, $\nabla_{u_{\mathrm{F}}} \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}) = 2u_F - 12$, $\nabla_{u_{\mathrm{L}}} \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}) = \begin{bmatrix} 2u_{L,1} - 10 \\ 2u_{L,2} - 2 \end{bmatrix}$, leading to

$$B := \frac{\nabla_{u_{\mathrm{L}}} \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}}) \nabla_{u_{\mathrm{F}}}^{\mathrm{T}} \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})}{||\nabla_{u_{\mathrm{L}}}^{\mathrm{T}} \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})||^2} =$$

$$\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot (-4) \right) / \left(\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 4/5 \\ -8/5 \end{bmatrix}.$$

[1] Naturally, constraints applicable to a game should immediately be incorporated in the computation of $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$ and of the set Λ_{d} and not only when deriving B. However, $\Pi_{\mathrm{conv}(\Lambda_{\mathrm{d}})}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$ is computed based on $\Omega_{\mathrm{L}} = \mathbb{R}^{n_{\mathrm{L}}}, \Omega_{\mathrm{F}} = \mathbb{R}^{n_{\mathrm{F}}}$; it thus still has to be verified whether a solution γ_{L} exists in the bounded decision space such that $\mathrm{dom}(\gamma_{\mathrm{L}}) = \Omega_{\mathrm{F}}$ and $\gamma_{\mathrm{L}}(\Omega_{\mathrm{F}}) \subseteq \Omega_{\mathrm{L}}$.

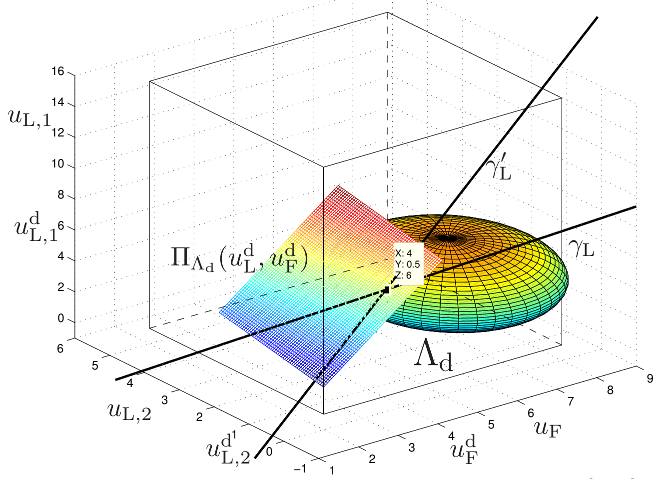

Fig. 2: Situation with a supporting hyperplane $\Pi_{\Lambda_{\mathrm{d}}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$ that is unique due to the differentiability of $\mathcal{J}_{\mathrm{F}}$ at $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$. The bounds of the decision space are indicated by a box.

As can be seen in Fig. 2 this mapping γ_{L} does not return values for all $u_{\mathrm{F}} \in \Omega_{\mathrm{F}}$ within the bounding box imposed by the constraints $u_{\mathrm{L},1} \in [0, 16], u_{\mathrm{L},2} \in [0, 5], u_{\mathrm{F}} \in [2, 8]$. Thus, B as defined by (7) and does not belong to the characterization of an optimal γ_{L}.

However, there do exist optimal affine mappings $\Omega_{\mathrm{F}} \to \Omega_{\mathrm{L}}$ through $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$ that lie on $\Pi_{\Lambda_{\mathrm{d}}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$. A suitable leader function that also lies on the tangent hyperplane defined by the relation

$$-u_{\mathrm{L},1} + 1/2 \cdot u_{\mathrm{L},2} + 2 \cdot u_{\mathrm{F}} - 9/4 = 0$$

would be:

$$u_{\mathrm{L}} = \gamma_{\mathrm{L}}'(u_{\mathrm{F}}) = \begin{bmatrix} 6 \\ 1/2 \end{bmatrix} + \begin{bmatrix} (-9/4 - 6)/4 \\ -1/8 \end{bmatrix} (4 - u_{\mathrm{F}}).$$

VI. Secondary Objective

Another motivation for the complete set Γ_{L}^* of optimal leader functions is the consideration of a secondary objective within the reverse Stackelberg game. E.g., if one would consider suboptimal solutions around the desired leader equilibrium $(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$ due to a suboptimal response of the follower to γ_{L}^*, one may look at the sensitivity of this follower response as a criterion for adopting a certain leader function. Such a secondary objective is relevant when one expects small deviations from the optimal decisions, or noise around the desired equilibrium. In [14], [15] such a secondary objective is considered for the case the leader is uncertain regarding the parameter values in $\mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}, u_{\mathrm{F}})$.

A graphical example is given in Fig. 3, where the optimal leader function γ_{L} is subject to sensitivity to deviations of the follower from $\arg\min_{u_{\mathrm{F}} \in \Omega_{\mathrm{F}}} \mathcal{J}_{\mathrm{F}}(\gamma_{\mathrm{L}}(u_{\mathrm{F}}), u_{\mathrm{F}})$. The solutions on γ_{L} for $u_{\mathrm{F}} > u_{\mathrm{F}}^d, u_{\mathrm{L}} > u_{\mathrm{L}}^d$ are close to $\mathrm{bd}(\Lambda_{\mathrm{d}})$ and therefore they return almost the same value $\mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}^{\mathrm{d}}, u_{\mathrm{F}}^{\mathrm{d}})$ to the follower. Hence, significantly sensitive solutions may be removed from Γ_{L}^*, where sensitivity of a solution γ_{L} can

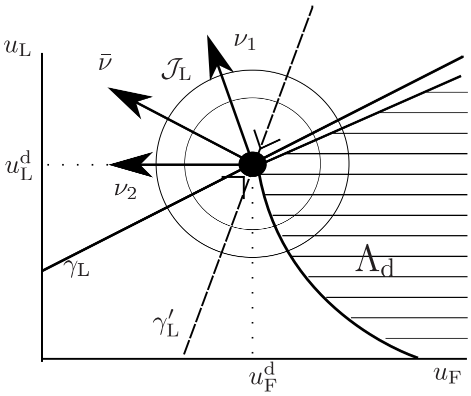

Fig. 3: Situation in which the optimal leader function γ_L is sensitive to deviations of the follower from $\arg\min_{u_\mathrm{F}\in\Omega_\mathrm{F}} \mathcal{J}_\mathrm{F}(\gamma_\mathrm{L}(u_\mathrm{F}), u_\mathrm{F})$.

be defined through its vicinity to $\mathrm{bd}(\mathrm{conv}(\Lambda_\mathrm{d}))$, i.e., by the angle between γ_L and the mean normal vector

$$\bar{\nu} := \sum_{i=1}^{n} \frac{1}{n}\nu_i \text{ or } \bar{\nu} := \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\nu_i$$

for the case with $n \in \mathbb{N}$ or with infinitely many generators of $\mathcal{V}(\mathrm{conv}(\Lambda_\mathrm{d}(u_\mathrm{L}^\mathrm{d}, u_\mathrm{F}^\mathrm{d})))$, respectively. The alternative solution $\gamma'_\mathrm{L} \in \Gamma^*_\mathrm{L} : \gamma'_\mathrm{L} \perp \tilde{\nu}$ is the least sensitive to such deviations from $\arg\min_{u_\mathrm{F}\in\Omega_\mathrm{F}} \mathcal{J}_\mathrm{F}(\gamma_\mathrm{L}(u_\mathrm{F}), u_\mathrm{F})$ in Fig. 3.

VII. Conclusions and Further Research

A parametrized characterization of the set of possible affine leader functions is provided that solve the single-leader single-follower reverse Stackelberg game to optimality. This complete set of optimal affine solutions of the reverse Stackelberg game can be derived for cases with a nondifferentiable follower objective function and a nonconvex sublevel set, also under the addition of constraints. While the current literature mostly resides with special cases in which the follower objective function is strictly convex and differentiable, we have thus relaxed these assumptions. Moreover, by specifying this full set, secondary optimization objectives can be considered, e.g., selecting the solution that is least sensitive to deviations from the optimum. If the set of feasible affine leader functions is however empty, alternative leader functions need to be found. Further steps are therefore to consider more diverse nonlinear, e.g., polynomial structures. In this way, we build towards a systematic approach for solving more general classes of the reverse Stackelberg game, aiming for application in control problems like road tolling.

Acknowledgments

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References

[1] J. B. Cruz, Jr., "Leader-follower strategies for multilevel systems," *IEEE Transactions on Automatic Control*, vol. 23, no. 2, pp. 244–255, 1978.

[2] R. Scattolini, "Architectures for distributed and hierarchical model predictive control: A review," *Journal of Process Control*, vol. 19, no. 5, pp. 723–731, 2009.

[3] Y.-C. Ho, P. B. Luh, and R. Muralidharan, "Information structure, Stackelberg games, and incentive controllability," *IEEE Transactions on Automatic Control*, vol. 26, no. 2, pp. 454–460, 1981.

[4] Y.-C. Ho, P. B. Luh, and G. J. Olsder, "A control-theoretic view on incentives," *Automatica*, vol. 18, no. 2, pp. 167–179, 1982.

[5] K. Staňková, G. J. Olsder, and M. C. J. Bliemer, "Comparison of different toll policies in the dynamic second-best optimal toll design problem: Case study on a three-link network," *European Journal of Transport and Infrastructure Research*, vol. 9, no. 4, pp. 331–346, 2009.

[6] H. Shen and T. Başar, "Incentive-based pricing for network games with complete and incomplete information," in *Advances in Dynamic Game Theory*, ser. Annals of the International Society of Dynamic Games, S. Jørgensen, M. Quincampoix, and T. Vincent, Eds. New York, NY, USA: Birkhäuser Boston, 2007, vol. 9, no. V, pp. 431–458.

[7] P. B. Luh, Y.-C. Ho, and R. Muralidharan, "Load adaptive pricing: An emerging tool for electric utilities," *IEEE Transactions on Automatic Control*, vol. 27, no. 2, pp. 320–329, 1982.

[8] G. J. Olsder, "Phenomena in inverse Stackelberg games, part 1: Static problems," *Journal of Optimization Theory and Applications*, vol. 143, no. 3, pp. 589–600, 2009.

[9] A. Chinchuluun, P. M. Pardalos, and H.-X. Huang, *Multilevel (Hierarchical) Optimization: Complexity Issues, Optimality Conditions, Algorithms*, ser. Advances in Mechanics and Mathematics. New York, NY, USA: Springer, 2009, vol. 17, ch. 6, pp. 197–221.

[10] T. Başar and H. Selbuz, "Closed-loop Stackelberg strategies with applications in the optimal control of multilevel systems," *IEEE Transactions on Automatic Control*, vol. 24, no. 2, pp. 166–179, 1979.

[11] Y.-P. Zheng and T. Başar, "Existence and derivations of optimal affine incentive schemes for Stackelberg games with partial information: A geometric approach," *International Journal of Control*, vol. 35, no. 6, pp. 997–1011, 1982.

[12] H. Ehtamo and R. P. Hämäläinen, "Incentive strategies and equilibria for dynamic games with delayed information," *Journal of Optimization Theory and Applications*, vol. 63, no. 3, pp. 355–369, 1989.

[13] N. Groot, B. De Schutter, and H. Hellendoorn, "Existence conditions for an optimal affine leader function in the reverse Stackelberg game," in *Proceedings of the 15th IFAC Workshop on Control Applications of Optimization*, Rimini, Italy, Sep. 2012.

[14] D. H. Cansever and T. Başar, "A minimum sensitivity approach to incentive design problems," *Large Scale Systems*, vol. 5, pp. 233–244, 1983.

[15] ——, "Optimum/near optimum incentive policies for stochastic decision problems involving parametric uncertainty," *Automatica*, vol. 21, no. 5, pp. 575–584, 1985.

[16] R. T. Rockafellar, *Convex Analysis*. Princeton, NJ, USA: Princeton University Press, 1970.

[17] A. Auslender and M. Teboulle, *Asymptotic Cones and Functions in Optimization and Variational Inequalities*, ser. Springer monographs in mathematics. New York, NY, USA: Springer, 2003.

[18] J. Dattorro, *Convex Optimization & Euclidean Distance Geometry*. Palo Alto, CA, USA: Meboo Publishing USA, 2005.

[19] F. H. Clarke, *Optimization and Nonsmooth Analysis*. New York, NY, USA: Wiley-Interscience, 1983.

[20] T. Başar and G. J. Olsder, *Dynamic Noncooperative Game Theory*, 2nd ed., ser. Classics in Applied Mathematics. Philadelphia, PA, USA: SIAM, 1999.

[21] K. J. Åström and B. Wittenmark, *Computer-Controlled Systems – Theory and Applications*, 3rd ed. Upper Saddle River, NJ, USA: Prentice Hall, 1997.

[22] N. Groot, B. De Schutter, and H. Hellendoorn, "Proof of characterization of optimal affine leader functions in the reverse Stackelberg game," Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands, Tech. Rep. 12-024, Mar. 2012.

[23] G. J. Olsder, "Phenomena in inverse Stackelberg games, part 2: Dynamic problems," *Journal of Optimization Theory and Applications*, vol. 143, no. 3, pp. 601–618, 2009.