Delft University of Technology

Delft Center for Systems and Control

Technical report 14-002

Model predictive control of manufacturing systems with max-plus algebra*

T.J.J. van den Boom and B. De Schutter

If you want to cite this report, please use the following reference instead:
T.J.J. van den Boom and B. De Schutter, "Model predictive control of manufacturing systems with max-plus algebra," Chapter 12 in *Formal Methods in Manufacturing* (J. Campos, C. Seatzu, and X. Xie, eds.), *Industrial Information Technology*, CRC Press, ISBN 978-1466561557, pp. 343–378, Feb. 2014. doi:10.1201/9781315216140-12

Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
phone: +31-15-278.24.73 (secretary)
URL: https://www.dcsc.tudelft.nl

Model predictive control of manufacturing systems with max-plus algebra

Ton J.J. van den Boom* and Bart De Schutter*

February 15, 2025

Abstract

Manufacturing systems can often be modeled as max-plus-linear (MPL) systems. MPL systems are discrete-event systems with synchronization but no choice and they are linear in the so-called max-plus algebra, which has addition maximization as its basic operations.

In this chapter we present an in-depth account of the model predictive control (MPC) framework for MPL systems. MPC is an on-line model based controller design method that is very popular in the process industry and that can also be extended to MPL systems. A key advantage of MPC is that it can accommodate constraints on the inputs and outputs of the controlled system. In MPC the optimal control signal is obtained by an optimization over all possible future control sequences. In general, the resulting MPL-MPC optimization problem is nonlinear and nonconvex. However, we show that if the control objective is piecewise affine, the constraints are linear, and if the control objective and the constraints depend monotonically on the outputs of the system, which is a frequently occurring situation for manufacturing systems, the MPL-MPC optimization can be recast into a linear programming problem, which can be solved very efficiently.

Subsequently we focus on implementation and timing aspects, closed-loop behavior, and tuning rules for MPL-MPC. We derive sufficient conditions for stability and formulate a closed-loop expression for the unconstrained MPL-MPC controller. In the case of perturbed operation due to modeling errors and/or noise we need a robust MPL-MPC controller. We show that under quite general conditions the resulting optimization problems can be solved very efficiently. For the bounded error case we also derive an MPL-MPC controller by optimizing over feedback policies, rather than open-loop input sequences. In general, this results in increased feasibility and a better performance. Finally we discuss robust MPC for MPL systems with stochastic uncertainty.

Keywords: Discrete-event systems; Predictive control; Model-based control; Generalized predictive control; Max-plus-linear systems; Max-plus algebra.

*Delft University of Technology, Delft Center for Systems and Control, Mekelweg 2, 2628 CD Delft, The Netherlands, e-mail: a.j.j.vandenboom@tudelft.nl, b.deschutter@tudelft.nl

Contents

1 Introduction

Discrete-event systems are event-driven dynamical systems (i.e., their dynamics are due to asynchronous occurrences of discrete events and so the state transitions are initiated by events, rather than a clock), that often arise in the context of manufacturing systems, telecommunication networks, railway networks, parallel computing, etc. Discrete-event systems with synchronization but no choice can be described by models that are "linear" in the max-plus algebra, and they are called max-plus-linear (MPL) systems. In the last decades there has been an increasing amount of research on MPL systems. Most literature on this class of systems addresses the performance analysis of MPL systems [2, 13, 14, 22, 28]. Design of optimal controllers for MPL systems using residuation techniques are given in [16, 37, 41]. Another approach for control of MPL systems is the Model Predictive Control (MPC) approach.

In MPC [35, 40] dynamical models are used to predict the system dynamics. The MPC problem is usually formulated as solving on-line a finite horizon open-loop optimal control problem subject to system dynamics and constraints involving states and controls. Many successful applications of MPC have been reported for conventional time-driven systems, and it is now one of the most applied advanced control technique in the process industry. MPC has also been extended to MPL discrete event systems [19, 25, 26, 44, 45, 47, 46, 55, 64].

This chapter considers the problem of designing an MPC controller for the class of MPL discrete event systems, and gives and extensive overview of the available results. We consider the case where the input, output and state sequence must satisfy a given set of linear inequality constraints. In Section 1.2 and 1.3 we start with some background in MPC for time-driven systems and give some basic results in max-plus algebra and max-plus-linear systems. In Section 2 we introduce the basics of MPC for MPL systems. The MPC optimization problem is introduced and appropriate choices for the cost criterion and the constraints will be discussed. We introduce the prediction and control horizon and present the problem in a standard form. The resulting optimization problem can be solved using various algorithms. In Section 3 the performance of the MPC controller is analyzed. For the unconstrained case we give an analytic expression for the controller and provide sufficient conditions for stability. Section 4 treats robust MPC in the case of perturbed operation due to modeling errors and/or noise. We consider both the bounded perturbation case and the stochastic perturbation case and we show that under quite general conditions the resulting optimization problems can be solved very efficiently. Section 5 provides a final discussion on the status and perspectives of MPC for MPL systems.

1.1 Model predictive control for time-driven systems

In this section we give a short introduction to MPC for time-driven discrete-time systems. More extensive information on MPC can be found in [11, 35].

Consider a plant with n_u inputs and n_y outputs that can be modeled by a state space description (in conventional linear algebra) of the form

$$\mathbf{x}(k) = \mathbf{A}\,\mathbf{x}(k-1) + \mathbf{B}\,\mathbf{u}(k) \tag{1}$$

$$\mathbf{y}(k) = \mathbf{C}\,\mathbf{x}(k)\ . \tag{2}$$

The vector $\mathbf{x} \in \mathbb{R}^n$ represents the state, $\mathbf{u} \in \mathbb{R}^{n_u}$ the input, $\mathbf{y} \in \mathbb{R}^{n_y}$ the output, and $k \in \mathbb{Z}$ is the discrete time counter. In order to distinguish systems that can be described by a model of the form (1) – (2) from the max-plus-linear systems that will be considered later on, a system that can be modeled by (1) – (2) will be called a *plus-times-linear* system.

In MPC a performance index or cost criterion J is formulated that reflects the reference tracking error (J_{out}) and the control effort (J_{in}):

$$\begin{aligned} J(k) &= J_{\text{out}}(k) + \beta J_{\text{in}}(k) \\ &= \sum_{j=0}^{N_{\text{p}}-1} \sum_{i=1}^{n_y} \Big(\hat{\mathbf{y}}_i(k+j|k) - \mathbf{r}_i(k+j) \Big)^2 + \beta \sum_{j=0}^{N_{\text{p}}-1} \sum_{i=1}^{n_u} \Big(\mathbf{u}_i(k+j) \Big)^2 \end{aligned} \tag{3}$$

where $\hat{\mathbf{y}}(k+j|k)$ is the estimate of the output at time step $k+j$ based on the information available at time step k, $\mathbf{r} \in \mathbb{R}^{n_y}$ is a reference signal, β is a non-negative scalar, and N_{p} is the prediction horizon.

In MPC the input is often taken to be constant from a certain point on: $\mathbf{u}(k+j) = \mathbf{u}(k+N_{\text{c}}-1)$ for $j = N_{\text{c}}, \ldots, N_{\text{p}}-1$ where N_{c} is the control horizon. The use of a control horizon leads to a reduction of the number of optimization variables. This results in a decrease of the computational burden, a smoother controller signal (because of the emphasis on the average behavior rather than on aggressive noise reduction), and a stabilizing effect (since the output signal is forced to its steady-state value).

MPC uses a receding horizon principle. At time step k the future control sequence $\mathbf{u}(k)$, ..., $\mathbf{u}(k+N_{\text{c}}-1)$ is determined such that the cost criterion is minimized subject to the constraints. At time step k the first element of the optimal sequence ($\mathbf{u}(k)$) is applied to the process. At the next time step the horizon is shifted, the model is updated with new information of the measurements, and a new optimization at time step $k+1$ is performed.

By successive substitution of (1) in (2), estimates of the future values of the output can be computed [11]. In matrix notation we obtain:

$$\tilde{\mathbf{y}}(k) = \tilde{\mathbf{C}}\, \mathbf{x}(k-1) + \tilde{\mathbf{D}}\, \tilde{\mathbf{u}}(k)$$

with

$$\tilde{\mathbf{y}}(k) = \begin{bmatrix} \hat{\mathbf{y}}(k|k) \\ \hat{\mathbf{y}}(k+1|k) \\ \vdots \\ \hat{\mathbf{y}}(k+N_{\text{p}}-1|k) \end{bmatrix}, \; \tilde{\mathbf{r}}(k) = \begin{bmatrix} \mathbf{r}(k) \\ \mathbf{r}(k+1) \\ \vdots \\ \mathbf{r}(k+N_{\text{p}}-1) \end{bmatrix}, \; \tilde{\mathbf{u}}(k) = \begin{bmatrix} \mathbf{u}(k) \\ \mathbf{u}(k+1) \\ \vdots \\ \mathbf{u}(k+N_{\text{p}}-1) \end{bmatrix}, \tag{4}$$

$$\tilde{\mathbf{C}}(k) = \begin{bmatrix} \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{N_{\text{p}}} \end{bmatrix}, \quad \tilde{\mathbf{D}} = \begin{bmatrix} \mathbf{CB} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{CAB} & \mathbf{CB} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{CA}^{N_{\text{p}}-1} & \mathbf{CA}^{N_{\text{p}}-2}\mathbf{B} & \ldots & \mathbf{CB} \end{bmatrix}. \tag{5}$$

The MPC problem at time step k for plus-times-linear systems is defined as follows:

> Find the input sequence $\mathbf{u}(k), \ldots, \mathbf{u}(k+N_{\text{c}}-1)$ that minimizes the performance index $J(k)$ subject to the linear constraint
>
> $$\mathbf{E}(k)\tilde{\mathbf{u}}(k) + \mathbf{F}(k)\tilde{\mathbf{y}}(k) + \mathbf{G}(k)\tilde{\mathbf{r}}(k) \leqslant \mathbf{h}(k) \tag{6}$$
>
> with $\mathbf{E}(k) \in \mathbb{R}^{l \times n_u N_{\text{p}}}$, $\mathbf{F}(k) \in \mathbb{R}^{l \times n_y N_{\text{p}}}$, $\mathbf{G}(k) \in \mathbb{R}^{l \times n_y N_{\text{p}}}$, $\mathbf{h}(k) \in \mathbb{R}^{l}$ for some integer l, subject to the control horizon constraint
>
> $$\mathbf{u}(k+j) = \mathbf{u}(k+N_{\text{c}}-1) \qquad \text{for } j = N_{\text{c}}, N_{\text{c}}+1, \ldots, N_{\text{p}}-1 \tag{7}$$

Note that minimizing $J(k)$ subject to (6) and (7), boils down to a convex quadratic programming problem, which can be solved very efficiently.

The parameters N_p, N_c, and β are the three basic MPC tuning parameters: The prediction horizon N_p is related to the length of the step response of the process, and the time step set $\{1, 2, \ldots, N_\mathrm{p}\}$ should contain the crucial dynamics of the process. The control horizon $N_\mathrm{c} \leqslant N_\mathrm{p}$ is usually taken equal to the system order. The parameter $\beta \geqslant 0$ makes a trade-off between the tracking error and the control effort, and is usually chosen as small as possible (while still getting a stabilizing controller).

1.2 Max-plus algebra and max-plus-linear systems

Max-plus algebra

The basic operations of the max-plus algebra (see also Chapter 2) are maximization and addition [2, 17, 28], which will be represented by $\oplus$ and $\otimes$ respectively:

$$x \oplus y = \max(x, y) \quad \text{and} \quad x \otimes y = x + y$$

for $x, y \in \mathbb{R}_\varepsilon \stackrel{\text{def}}{=} \mathbb{R} \cup \{-\infty\}$. Define $\varepsilon = -\infty$. The structure $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ is called the max-plus algebra [2]. The operations $\oplus$ and $\otimes$ are called the max-plus-algebraic addition and max-plus-algebraic multiplication respectively since many properties and concepts from linear algebra can be translated to the max-plus algebra by replacing $+$ by $\oplus$ and $\times$ by $\otimes$.

The matrix $\boldsymbol{\mathcal{E}}_{m\times n}$ is the $m \times n$ max-plus-algebraic zero matrix: $(\boldsymbol{\mathcal{E}}_{m\times n})_{ij} = \varepsilon$ for all i, j; and $\mathbf{E}_n$ is the $n \times n$ max-plus-algebraic identity matrix: $(\mathbf{E}_n)_{ii} = 0$ for all i and $(\mathbf{E}_n)_{ij} = \varepsilon$ for all i, j with $i \neq j$. If $\mathbf{A}, \mathbf{B} \in \mathbb{R}_\varepsilon^{m\times n}$, $\mathbf{C} \in \mathbb{R}_\varepsilon^{n\times p}$ then

$$\begin{aligned}
(\mathbf{A} \oplus \mathbf{B})_{ij} &= \mathbf{A}_{ij} \oplus \mathbf{B}_{ij} = \max(\mathbf{A}_{ij}, \mathbf{B}_{ij}) \\
(\mathbf{A} \otimes \mathbf{C})_{ij} &= \bigoplus_{k=1}^{n} \mathbf{A}_{ik} \otimes \mathbf{C}_{kj} = \max_k (\mathbf{A}_{ik} + \mathbf{C}_{kj})
\end{aligned}$$

for all i, j. Note the analogy with the conventional definitions of matrix sum and product.

A max-plus diagonal matrix $\mathbf{S} = \mathrm{diag}_\oplus(s_1, \ldots, s_n)$ has elements $\mathbf{S}_{ij} = \varepsilon$ for $i \neq j$ and diagonal elements $\mathbf{S}_{ii} = s_i$ for $i = 1, \ldots, n$. $\mathbf{E} = \mathrm{diag}_\oplus(0, \ldots, 0)$ is the max-plus identity matrix. The matrix $\boldsymbol{\mathcal{E}}_{m\times n}$ is the $m \times n$ max-plus-algebraic zero matrix: $(\boldsymbol{\mathcal{E}}_{m\times n})_{ij} = \varepsilon$ for all i, j. The max-plus-algebraic matrix power of $\mathbf{A} \in \mathbb{R}_\varepsilon^{n\times n}$ is defined as follows: $\mathbf{A}^{\otimes^0} = \mathbf{E}$ and $\mathbf{A}^{\otimes^k} = \mathbf{A} \otimes \mathbf{A}^{\otimes^{k-1}}$ for $k = 1, 2, \ldots$ If for a max-plus diagonal matrix $\mathbf{S} = \mathrm{diag}_\oplus(s_1, \ldots, s_n)$ all s_i are finite, the inverse of $\mathbf{S}$ is equal to $\mathbf{S}^{\otimes^{-1}} = \mathrm{diag}_\oplus(-s_1, \ldots, -s_n)$. Then it holds that $\mathbf{S} \otimes \mathbf{S}^{\otimes^{-1}} = \mathbf{S}^{\otimes^{-1}} \otimes \mathbf{S} = \mathbf{E}$. For any matrix $\mathbf{A} \in \mathbb{R}_\varepsilon^{n\times n}$ we can define

$$\mathbf{A}^* = \mathbf{E} \oplus \mathbf{A} \oplus \mathbf{A}^{\otimes 2} \oplus \mathbf{A}^{\otimes 3} \oplus \ldots$$

Finally we introduce the max-plus-algebraic eigenvalue of a matrix $\mathbf{A} \in \mathbb{R}_\varepsilon^{n\times n}$. The scalar $\lambda \in \mathbb{R}_\varepsilon$ is a max-plus-algebraic eigenvalue if there exists a $\mathbf{v} \in \mathbb{R}_\varepsilon^{n}$ with at least one finite entry such that $\mathbf{A} \otimes \mathbf{v} = \lambda \otimes \mathbf{v}$ [2]. The vector $\mathbf{v}$ is called a max-plus-algebraic eigenvector.

Max-plus-linear systems

Discrete-event systems with only synchronization and no concurrency can be modeled by a max-plus-algebraic model of the following form [2, 17, 28]:

$$\mathbf{x}(k) = \mathbf{A} \otimes \mathbf{x}(k-1) \oplus \mathbf{B} \otimes \mathbf{u}(k) \tag{8}$$
$$\mathbf{y}(k) = \mathbf{C} \otimes \mathbf{x}(k) \tag{9}$$

with $\mathbf{A} \in \mathbb{R}_\varepsilon^{n\times n}$, $\mathbf{B} \in \mathbb{R}_\varepsilon^{n\times n_u}$, and $\mathbf{C} \in \mathbb{R}_\varepsilon^{n_y\times n}$ where n_u is the number of inputs, n_y the number of outputs, and n is the system order. Note the analogy of the description (8) – (9) with the state space model (1) – (2) for plus-times-linear systems. An important difference with the description (1) – (2) is that now the components of the input, the output, and the state are event times, and that the counter k in (8) – (9) is an event counter (and event occurrence instants are in general not equidistant), whereas in (1) – (2) k increases each clock cycle. For a manufacturing system, $u(k)$ would typically represent the time instants at which raw material is fed to the system for the kth time, $x(k)$ the time instants at which the machines start processing the kth batch of intermediate products, and $y(k)$ the time instants at which the kth batch of finished products leaves the system. The matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ consist of the event durations, such as production times or transportation times. A discrete-event system that can be modeled by (8) – (9) will be called a max-plus-linear time-invariant discrete-event system or *max-plus-linear* (MPL) system for short.

The MPL system (8)-(9) is called structurally controllable if all states are connected to some input [2]. It can be checked that the system is structurally controllable iff the matrix

$$\mathbf{\Gamma}_n = \left[\begin{array}{cccc} \mathbf{B} & \mathbf{A} \otimes \mathbf{B} & \dots & \mathbf{A}^{\otimes^{n-1}} \otimes \mathbf{B} \end{array} \right]$$

is row-finite (i.e., if it has at least one finite element in each row) [24].

The MPL system (8)-(9) is called structurally observable if all states are connected to some output [2]. It can be checked that the system is structurally observable iff the matrix

$$\boldsymbol{\mathcal{O}}_n = \left[\begin{array}{cccc} \mathbf{C}^T & \mathbf{A}^T \otimes \mathbf{C}^T & \dots & \mathbf{A}^{T^{\otimes^{n-1}}} \otimes \mathbf{C}^T \end{array} \right]$$

is row-finite (i.e., if it has at least one finite element in each row) [24].

Remark 1 For plus-times-linear systems the influence of noise is usually modeled by adding an extra noise term to the state and/or output equation. For MPL models the entries of the system matrices correspond to production times or transportation times. So instead of modeling noise (i.e., variation in the processing times) by adding an extra max-plus-algebraic term in (8) or (9), noise should rather be modeled as an additive term to these system matrices. We will discuss perturbed MPL models in Section 4.
Note that in some papers disturbances are seen as exogenous and uncontrollable inputs acting on the system [15, 32]. For more information we refer to these two papers. ◇

Max-plus-scaling functions and max-plus-non-negative-scaling functions

Let $\mathcal{S}_{\mathrm{mps}}$ be the set of max-plus-scaling functions, i.e., functions f of the form $f(\mathbf{z}) = \max_{i=1,\dots,m}(\mu_i + \nu_{i,1}\mathbf{z}_1 + \ldots + \nu_{i,n}\mathbf{z}_n)$, with variable $\mathbf{z} \in \mathbb{R}_\varepsilon^n$ and constants $\nu_{i,j} \in \mathbb{R}$ and $\mu_i \in \mathbb{R}$. If we want to stress that f is a function of $\mathbf{z}$ we will denote this by $f \in \mathcal{S}_{\mathrm{mps}}(\mathbf{z})$.

Let $\mathcal{S}_{\mathrm{mpns}}$ denote the set of *max-plus-non-negative-scaling* functions, i.e., max-plus-scaling functions $f(\mathbf{z}) = \max_{i=1,\dots,m}(\mu_i + \nu_{i,1}\mathbf{z}_1 + \ldots + \nu_{i,n}\mathbf{z}_n)$, with $\nu_{i,j} \geqslant 0$ for all $j = 1,\dots,n$.

Proposition 2 *[54]*
The set $\mathcal{S}_{\mathrm{mpns}}$ is closed under the operations max*,* $+$*, and scalar multiplication by a non-negative scalar.* ◇

Proposition 3 *[54]*
If $f \in \mathcal{S}_{\mathrm{mpns}}$ then f is a nondecreasing function of its arguments. ◇

2 Model predictive control for max-plus-linear systems

2.1 Evolution of the system

We assume that $\mathbf{x}(k)$, the state at event step k, can be measured or estimated using previous measurements. We can then use (8) – (9) to estimate the evolution of the output of the system for the input sequence $\mathbf{u}(k), \ldots, \mathbf{u}(k + N_\mathrm{p} - 1)$:

$$\mathbf{y}(k+j|k) = \mathbf{C} \otimes \mathbf{A}^{\otimes^{j+1}} \otimes \mathbf{x}(k-1) \oplus \bigoplus_{i=0}^{j} \mathbf{C} \otimes \mathbf{A}^{\otimes^{j-i}} \otimes \mathbf{B} \otimes \mathbf{u}(k+i) \ ,$$

or, in matrix notation, $\tilde{\mathbf{y}}(k) = \tilde{\mathbf{C}} \otimes \mathbf{x}(k-1) \oplus \tilde{\mathbf{D}} \otimes \tilde{\mathbf{u}}(k)$ with

$$\tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{C} \otimes \mathbf{A} \\ \mathbf{C} \otimes \mathbf{A}^{\otimes^2} \\ \vdots \\ \mathbf{C} \otimes \mathbf{A}^{\otimes^{N_\mathrm{p}}} \end{bmatrix} ,$$

$$\tilde{\mathbf{D}} = \begin{bmatrix} \mathbf{C} \otimes \mathbf{B} & \boldsymbol{\varepsilon} & \ldots & \boldsymbol{\varepsilon} \\ \mathbf{C} \otimes \mathbf{A} \otimes \mathbf{B} & \mathbf{C} \otimes \mathbf{B} & \ldots & \boldsymbol{\varepsilon} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C} \otimes \mathbf{A}^{\otimes^{N_\mathrm{p}-1}} \otimes \mathbf{B} & \mathbf{C} \otimes \mathbf{A}^{\otimes^{N_\mathrm{p}-2}} \otimes \mathbf{B} & \ldots & \mathbf{C} \otimes \mathbf{B} \end{bmatrix} ,$$

where $\tilde{\mathbf{y}}(k)$ and $\tilde{\mathbf{u}}(k)$ are defined by (4).

2.2 Cost criterion

Also in MPC for MPL systems a performance index or cost criterion J is formulated that reflects the reference tracking error (J_out) and the control effort (J_in):

$$J(k) = J_\mathrm{out}(k) + \beta J_\mathrm{in}(k) \ .$$

A straightforward translation of the cost criterion used in MPC for plus-times-linear systems is not very useful in practice. We therefore discuss more appropriate choices for the output cost criterion J_out and the input cost criterion J_in.

Tracking error or output cost criterion J_out

If the reference signal $\mathbf{r}(k)$ with the due dates for the finished products is known and if we have to pay a penalty for every delay, a well-suited cost criterion is the tardiness:

$$J_{\mathrm{out},1}(k) = \sum_{j=0}^{N_\mathrm{p}-1} \sum_{i=1}^{n_y} \max(\mathbf{y}_i(k+j|k) - \mathbf{r}_i(k+j), 0) \ . \quad (10)$$

If we have perishable goods, then we could want to minimize the differences between the reference signal and the actual output time instants. This leads to

$$J_{\mathrm{out},2}(k) = \sum_{j=0}^{N_\mathrm{p}-1} \sum_{i=1}^{n_y} |\mathbf{y}_i(k+j|k) - \mathbf{r}_i(k+j)| \ . \quad (11)$$

For a smooth production in manufacturing systems it may be important that the difference between two consecutive deliveries $\Delta\mathbf{y}_i(k) = \mathbf{y}_i(k) - \mathbf{y}_i(k-1)$ shows little variation. This means we want to balance the output rates, and we can consider

$$J_{\text{out},3}(k) = \sum_{j=0}^{N_\text{p}-1} \sum_{i=1}^{n_y} |\Delta^2 \mathbf{y}_i(k+j|k)| \tag{12}$$

where $\Delta^2 s(k) = \Delta s(k) - \Delta s(k-1) = s(k) - 2s(k-1) + s(k-2)$ for a signal $s(\cdot)$.

Input cost criterion J_{in}

A straightforward translation of the plus-times-linear input cost criterion $\tilde{\mathbf{u}}^T(k)\tilde{\mathbf{u}}(k)$ would lead to a minimization of the input time instants. Since this could result in input buffer overflows, a better objective is to *maximize* the input time instants. For a manufacturing system, this would correspond to a scheme in which raw material is fed to the system as late as possible. As a consequence, the internal buffer levels are kept as low as possible. This also leads to a notion of stability if we let instability for the manufacturing system correspond to internal buffer overflows. So for MPL systems an appropriate cost criterion is $J_{\text{in},0}(k) = -\tilde{\mathbf{u}}^T(k)\tilde{\mathbf{u}}(k)$. Note that this is exactly the opposite of the input effort cost criterion for plus-times-linear systems. Another objective function that leads to a maximization of the input time instants is

$$J_{\text{in},1}(k) = -\sum_{j=0}^{N_\text{p}-1} \sum_{i=1}^{n_u} \mathbf{u}_i(k+j) \ . \tag{13}$$

Similar to the balancing of the output rates (12) it may be important to minimize the variation of the feeding rates of the raw material into the manufacturing system. This means we want to balance the input rates, and we can consider

$$J_{\text{in},2}(k) = \sum_{j=0}^{N_\text{p}-1} \sum_{i=1}^{n_u} |\Delta^2 \mathbf{u}_i(k+j)| \ . \tag{14}$$

We could replace the summations in (10) – (14) by max-plus-algebraic summations (i.e., maximizations), or consider weighted mixtures of several cost criteria.

2.3 Constraints

Just as in MPC for plus-times-linear systems we can consider the linear constraint

$$\mathbf{E}(k)\tilde{\mathbf{u}}(k) + \mathbf{F}(k)\tilde{\mathbf{y}}(k) + \mathbf{G}(k)\tilde{\mathbf{r}}(k) \leqslant \mathbf{h}(k) \ . \tag{15}$$

Furthermore, it is easy to verify that typical constraints for discrete-event systems are minimum or maximum separation between input and output events:

$$\mathbf{a}_1(k+j) \leqslant \Delta\mathbf{u}(k+j) \leqslant \mathbf{b}_1(k+j) \qquad \text{for } j = 0, \ldots, N_\text{c} - 1 \tag{16}$$

$$\mathbf{a}_2(k+j) \leqslant \Delta\mathbf{y}(k+j|k) \leqslant \mathbf{b}_2(k+j) \qquad \text{for } j = 0, \ldots, N_\text{p} - 1 \, , \tag{17}$$

or maximum due dates for the output events:

$$\mathbf{y}(k+j|k) \leqslant \mathbf{r}(k+j) \qquad \text{for } j = 0, \ldots, N_\text{p} - 1 \, , \tag{18}$$

can also be recast as a linear constraint of the form (15).

Since for MPL systems the input and output sequences correspond to occurrence times of consecutive events, they should be nondecreasing. Therefore, we should always add the condition $\Delta\mathbf{u}(k+j) \geqslant 0$ for $j = 0, \ldots, N_\text{c} - 1$ to guarantee that the input sequences are nondecreasing.

2.4 The evolution of the input beyond the control horizon

A straightforward translation of the conventional control horizon constraint would imply that the input should stay constant from event step $k+N_\mathrm{c}$ on, which is not very useful for MPL systems since there the input sequences should normally be increasing. Therefore, we change this condition as follows: the feeding rate should stay constant beyond event step $k+N_\mathrm{c}$, i.e.,

$$\Delta\mathbf{u}(k+j) = \Delta\mathbf{u}(k+N_\mathrm{c}-1) \qquad \text{for } j = N_\mathrm{c}, \ldots, N_\mathrm{p}-1\,, \tag{19}$$

or $\Delta^2\mathbf{u}(k+j) = 0$ for $j = N_\mathrm{c}, \ldots, N_\mathrm{p}-1$. This condition introduces regularity in the input sequence and it prevents the buffer overflow problems that could arise when all resources are fed to the system at the same time instant as would be implied by the conventional control horizon constraint (7).

2.5 The standard MPC problem for MPL systems

If we combine the material of previous subsections, we finally obtain the following problem:

$$\min_{\tilde{\mathbf{u}}(k),\tilde{\mathbf{y}}(k)} J(k) = \min_{\tilde{\mathbf{u}}(k),\tilde{\mathbf{y}}(k)} J_{\mathrm{out},p_1}(k) + \beta J_{\mathrm{in},p_2}(k) \tag{20}$$

for some J_{out,p_1}, J_{in,p_2} subject to

$$\tilde{\mathbf{y}}(k) = \tilde{\mathbf{C}} \otimes \mathbf{x}(k-1) \oplus \tilde{\mathbf{D}} \otimes \tilde{\mathbf{u}}(k) \tag{21}$$

$$\mathbf{E}(k)\tilde{\mathbf{u}}(k) + \mathbf{F}(k)\tilde{\mathbf{y}}(k) \leqslant \mathbf{h}(k) - \mathbf{G}(k)\tilde{\mathbf{r}}(k) \tag{22}$$

$$\Delta\mathbf{u}(k+j) \geqslant 0 \qquad \text{for } j = 0, \ldots, N_\mathrm{c}-1 \tag{23}$$

$$\Delta^2\mathbf{u}(k+j) = 0 \qquad \text{for } j = N_\mathrm{c}, \ldots, N_\mathrm{p}-1 \tag{24}$$

This problem will be called the MPL-MPC problem for event step k. MPL-MPC also uses a receding horizon principle.

Other design control design methods for MPL systems are discussed in [2, 8, 42]. However, the majority of the methods do not allow the inclusion of general linear constraints of the form (22) or even simple constraints of the form (16) or (17). Some recent papers on controller design for max-plus systems take constraints on the state space into account [1, 29, 36].

2.6 Algorithms to solve the MPL-MPC problem

Nonlinear optimization

In general the problem (20) – (24) is a nonlinear nonconvex optimization problem: although the constraints (22) – (24) are convex in $\tilde{\mathbf{u}}(k)$ and $\tilde{\mathbf{y}}(k)$, the constraint (21) is in general not convex. So we could use standard multi-start nonlinear nonconvex local optimization methods [9] to compute the optimal control policy.

The feasibility of the MPC-MPL problem can be verified by solving the system of (in)equalities (21) – (24)[1]. If the problem is found to be infeasible we can use the same techniques as in conventional MPC and use constraint relaxation [11]. Additional information on these topics are given in Section 3.

[1] In general this is a nonlinear system of equations but if the constraints depend monotonically on the output, the feasibility problem can be recast as a linear programming problem (cf. Theorem 4).

The ELCP approach

Now we discuss an alternative approach which is based on the Extended Linear Complementarity Problem (ELCP) [18]. Consider the ith row of (21) and define $\mathcal{J}_i^C = \{ j \mid \tilde{\mathbf{C}}_{ij} \neq \varepsilon\}$ and $\mathcal{J}_i^D = \{ j \mid \tilde{\mathbf{D}}_{ij} \neq \varepsilon\}$. We have $\tilde{\mathbf{y}}_i(k) = \max\Big(\max_{j\in\mathcal{J}_i^C} (\tilde{\mathbf{C}}_{ij} + \mathbf{x}_j(k-1)), \max_{j\in\mathcal{J}_i^D} (\tilde{\mathbf{D}}_{ij} + \tilde{\mathbf{u}}_j(k))\Big)$ or equivalently

$$\begin{aligned}
\tilde{\mathbf{y}}_i(k) &\geqslant \tilde{\mathbf{C}}_{ij} + \mathbf{x}_j(k-1) \qquad \text{for } j \in \mathcal{J}_i^C \\
\tilde{\mathbf{y}}_i(k) &\geqslant \tilde{\mathbf{D}}_{ij} + \tilde{\mathbf{u}}_j(k) \qquad \text{for } j \in \mathcal{J}_i^D
\end{aligned}$$

with the extra condition that at least one inequality should hold with equality (i.e., at least one residue should be equal to 0):

$$\prod_{j\in\mathcal{J}_i^C} (\tilde{\mathbf{y}}_i(k) - \tilde{\mathbf{C}}_{ij} - \mathbf{x}_j(k-1)) \cdot \prod_{j\in\mathcal{J}_i^D} (\tilde{\mathbf{y}}_i(k) - \tilde{\mathbf{D}}_{ij} - \tilde{\mathbf{u}}_j(k)) = 0 \ . \tag{25}$$

Hence, (21) can be rewritten as a system of equations of the form

$$\mathbf{A}_\mathrm{e}\tilde{\mathbf{y}}(k) + \mathbf{B}_\mathrm{e}\tilde{\mathbf{u}}(k) + \mathbf{c}_\mathrm{e}(k) \geqslant 0 \tag{26}$$

$$\prod_{j\in\phi_i} (\mathbf{A}_\mathrm{e}\tilde{\mathbf{y}}(k) + \mathbf{B}_\mathrm{e}\tilde{\mathbf{u}}(k) + \mathbf{c}_\mathrm{e}(k))_j = 0 \qquad \text{for } i = 1, \ldots, n_y N_\mathrm{p} \tag{27}$$

for appropriately defined matrices and vectors $\mathbf{A}_\mathrm{e}, \mathbf{B}_\mathrm{e}, \mathbf{c}_\mathrm{e}$, and index sets ϕ_i. We can rewrite the linear constraints (22) – (24) as

$$\mathbf{E}(k)\tilde{\mathbf{u}}(k) + \mathbf{F}(k)\tilde{\mathbf{y}}(k) + \mathbf{G}(k)\tilde{\mathbf{r}}(k) - \mathbf{h}(k) \geqslant 0 \tag{28}$$

$$\mathbf{L}(k)\tilde{\mathbf{u}}(k) + \ell(k) = 0 \ . \tag{29}$$

So the feasible set of the MPC problem (i.e., the set of feasible system trajectories) coincides with the set of solutions of the system (26) – (29), which is a special case of an Extended Linear Complementarity Problem (ELCP) [18]. In [18] we have developed an algorithm to compute a compact parametric description of the solution set of an ELCP. In order to determine the optimal MPC policy we can use nonlinear optimization algorithms to determine for which values of the parameters the objective function J over the solution set of the ELCP (26) – (29) reaches its global minimum. The algorithm of [18] to compute the solution set of a general ELCP requires exponential execution times, which implies that the ELCP approach is not feasible if N_c is large.

Monotonically nondecreasing objective functions

Now consider the *relaxed* MPC problem which is also defined by (20) – (24) but with the $=$-sign in (21) replaced by a $\geqslant$-sign. Note that whereas in the original problem $\tilde{\mathbf{u}}(k)$ is the only independent variable since $\tilde{\mathbf{y}}(k)$ can be eliminated using (21), the relaxed problem has both $\tilde{\mathbf{u}}(k)$ and $\tilde{\mathbf{y}}(k)$ as independent variables. It is easy to verify that the set of feasible solutions of the relaxed problem coincides with the set of solutions of the system of linear inequalities (26), (28), (29). So the feasible set of the relaxed MPC problem is convex.

A function : $\mathbf{y} \to \mathbf{f}(\mathbf{y})$ is a monotonically nondecreasing function if $\bar{\mathbf{y}} \leqslant \check{\mathbf{y}}$ implies that $\mathbf{f}(\bar{\mathbf{y}}) \leqslant \mathbf{f}(\check{\mathbf{y}})$. Now we show that if the objective function J and the linear constraints are monotonically nondecreasing as a function of $\tilde{\mathbf{y}}$ (this is the case for $J = J_{\mathrm{out},1}$, $J_{\mathrm{in},1}$, or $J_{\mathrm{in},2}$, and e.g., $\mathbf{F}_{ij} \geqslant 0$ for all i, j), then the optimal solution of the relaxed problem can be transformed into an optimal solution of the original MPC problem. If in addition the objective function is convex (e.g., $J = J_{\mathrm{out},1}$ or $J_{\mathrm{in},1}$), we finally get a convex optimization problem.

Theorem 4 *[19]*
Let the objective function J and mapping $\tilde{\mathbf{y}} \to \mathbf{F}(k)\tilde{\mathbf{y}}$ be monotonically nondecreasing functions of $\tilde{\mathbf{y}}$. Let $(\tilde{\mathbf{u}}^*, \tilde{\mathbf{y}}^*)$ be an optimal solution of the relaxed MPC problem. If we define $\tilde{\mathbf{y}}^\sharp = \tilde{\mathbf{C}} \otimes \mathbf{x}(k-1) \oplus \tilde{\mathbf{D}} \otimes \tilde{\mathbf{u}}^* \oplus$ then $(\tilde{\mathbf{u}}^*, \tilde{\mathbf{y}}^\sharp)$ is an optimal solution of the original MPC problem. $\diamond$

Proposition 5 *[19, 64]*
By introducing some additional dummy variables, the MPC-problem with linear output cost-function $J(k) = J_{\mathrm{out},1}(k)$ can be reduced to a linear programming problem, which can be solved very efficiently. $\diamond$

3 Implementation aspects and performance analysis

In this chapter we explore the performance of the MPL system in closed with an MPC controller. More specifically, we focus on implementation, stability, feasibility, timing aspects, and derive some tuning rules. Consider the standard MPC problem (20)-(24) for MPL systems (8)-(9) with

$$J_{\mathrm{out}}(k) = \sum_{j=0}^{N_\mathrm{p}-1} \sum_{i=1}^{n_y} \max(\mathbf{y}_i(k+j|k) - \mathbf{r}_i(k+j), 0) = \sum_{i=1}^{n_y N_\mathrm{p}} \max(\tilde{\mathbf{y}}_i(k) - \tilde{\mathbf{r}}_i(k), 0) \quad (30)$$

$$J_{\mathrm{in}}(k) = \sum_{j=0}^{N_\mathrm{p}-1} \sum_{i=1}^{n_u} (\mathbf{r}_i(k+j) - \mathbf{u}_i(k+j)) = \sum_{i=1}^{n_u N_\mathrm{p}} (\tilde{\mathbf{r}}_i(k) - \tilde{\mathbf{u}}_i(k)). \quad (31)$$

For a manufacturing system, cost functions (30)-(31) correspond to a scheme in which raw material is fed to the system as late as possible (Just-in-time control). Note that this implies that the internal buffer levels are kept as low as possible.

3.1 Unconstrained MPC

In this section we will take a closer look at the closed-loop behavior of MPC of an unconstrained MPL system. We consider the unconstrained MPL-MPC problem of minimizing (20), where J is given by (30)-(31), subject to (21) and (23).

Closed-loop expression

In conventional MPC theory, in the absence of inequality constraints, the closed loop consisting of the (conventional) linear time-invariant (LTI) process with the MPC controller, is again an LTI system (in the conventional algebra). Unfortunately, there is no analogous property for MPL systems. However, an analytic closed-loop expression can be formulated for the *unconstrained* MPC for MPL systems. This expression involves the operations minimization and addition. In fact, the expression is linear in the min-plus algebra, which is the dual of the max-plus algebra and which has minimization ($\oplus'$) and addition ($\otimes'$) as basic operations. So in the case the closed-loop system would be a min-max-plus system. For the MPC optimization problem of minimizing cost criterion (30)-(31) we can derive an analytic expression for the MPC control law as follows:

Lemma 6 *[60]*

Define

$$\tilde{\mathbf{u}}^\sharp(k) = \left[\ \mathbf{u}^T(k-1) \quad \mathbf{u}^T(k-1) \quad \cdots \quad \mathbf{u}^T(k-1)\ \right]^T, \tag{32}$$

$$\begin{aligned}\tilde{\mathbf{z}}(k) &= \tilde{\mathbf{C}} \otimes \mathbf{x}(k-1) \oplus \tilde{\mathbf{D}} \otimes \tilde{\mathbf{u}}^\sharp(k) \oplus \tilde{\mathbf{r}}(k),\\ &= \left[\ \mathbf{z}^T(k|k) \quad \mathbf{z}^T(k+1|k) \quad \ldots \quad \mathbf{z}^T(k+N_\mathrm{p}-1|k)\ \right]^T,\end{aligned} \tag{33}$$

and let for $\ell = 1, \ldots, n_u$

$$\mathbf{u}^*_\ell(k+j|k) = \begin{cases} \min\limits_i \min\limits_m (\mathbf{z}_m(k+i|k) - [\tilde{\mathbf{D}}_{ij}]_{m\ell}) & \text{for } j = N_\mathrm{p}-1 \\ \min\Big(\min\limits_i \min\limits_m (\mathbf{z}_m(k+i|k) - [\tilde{\mathbf{D}}_{ij}]_{m\ell}, \mathbf{u}^*_\ell(k+j+1|k)\Big) & \text{for } j = 1, \ldots, N_\mathrm{p}-2 \end{cases} \tag{34}$$

Then $\tilde{\mathbf{u}}^*(k)$ *is the optimal solution of the MPL-MPC problem.* ◇

From (34) we derive the min-plus expression[2]:

$$\tilde{\mathbf{u}}^* = (-\tilde{\mathbf{D}}^T) \otimes' \tilde{\mathbf{z}} \oplus' \mathbf{S} \otimes' \tilde{\mathbf{u}}^* \tag{35}$$

where

$$\mathbf{S} = \begin{bmatrix} \boldsymbol{\mathcal{E}}' & \mathbf{0} & \cdots & \boldsymbol{\mathcal{E}}' \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \boldsymbol{\mathcal{E}}' & \cdots & & \boldsymbol{\mathcal{E}}' \end{bmatrix}$$

where $\boldsymbol{\mathcal{E}}'_{ij} = \infty$ and $[\mathbf{0}]_{ij} = 0$ for $i = 1, \ldots, n_u$, $j = 1, \ldots, n_u$. This can be written in an explicit form as:

$$\tilde{\mathbf{u}}^* = \mathbf{S}^* \otimes' (-\tilde{\mathbf{D}}^T) \otimes' \tilde{\mathbf{z}}$$

where $\mathbf{S}^*$ is the min-plus Kleene star product (see Chapter 2 and [2])

$$\mathbf{S}^* = \mathbf{E}' \oplus' \mathbf{S} \oplus' \mathbf{S}^{\otimes' 2} \oplus' \ldots = \begin{bmatrix} \boldsymbol{\mathcal{E}}' & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \boldsymbol{\mathcal{E}}' & \cdots & & \boldsymbol{\mathcal{E}}' \end{bmatrix}$$

in which $\mathbf{E}'$ is the min-plus identity matrix with elements $\mathbf{E}'_{ij} = \varepsilon'$ for $i \neq j$ and diagonal elements $\mathbf{E}'_{ii} = 0$ for $i = 1, \ldots, n$.

In the general constrained case, the closed-loop system (consisting of the MPL process with the MPL-MPC controller) will not be an MPL system, but it will be piecewise affine in the state $\mathbf{x}(k-1)$ and reference $\mathbf{r}(k)$ and it can be described by max-min-plus-scaling functions (this follows directly from the results of [4, 27]). An closed-loop expression for the state cost criterion case is given in [47].

Stability

Stability in conventional system theory is concerned with boundedness of the states. In MPL systems however, k is an event counter and $\mathbf{x}_i(k)$ refers to the occurrence time of an event. So the sequence $\mathbf{x}_i(k), \mathbf{x}_i(k+1), \ldots$ should always be non-decreasing, and for $k \to \infty$ the event time $\mathbf{x}_i(k)$ will usually grow unbounded.

[2]Note that the min-plus expression (35) can also be derived using residuation theory (see Chapter 2 and [2, 17, 7]).

In manufacturing systems we deal with input, output, and state buffers. Assume that we have a reference signal with due dates and assume that finished parts are removed from the output buffer at the due dates (provided that they are present). Then we can adopt the notion of stability formulated by [50]: *A discrete-event system is called stable if all its buffer levels remain bounded.* However, in manufacturing systems there will be delays if the parts are not produced before the due date. These delays should remain bounded. Therefore, we add as an additional condition for stability that all delays between the reference signal and the actual output (i.e., due date) remain bounded as well. If there are no internal buffers that are not (indirectly) coupled to the output of the system (observability), then it is easy to verify that all the buffer levels are bounded if the dwelling times of the parts or batches in the system remain bounded. This implies that for an structurally controllable and structurally observable discrete event system with due date r(k) closed-loop stability is achieved if there exist finite constants M_{yr}, M_{xr} and M_{ur} such that

$$\max_{i} |\mathbf{y}_i(k) - \mathbf{r}_i(k)| \leq M_{yr} \quad , \quad \forall k \geqslant K \tag{36}$$

$$\max_{i,j} |\mathbf{x}_j(k) - \mathbf{r}_i(k)| \leq M_{xr} \quad , \quad \forall k \geqslant K \tag{37}$$

$$\max_{i,l} |\mathbf{u}_l(k) - \mathbf{r}_i(k)| \leq M_{ur} \quad , \quad \forall k \geqslant K \tag{38}$$

Let $\lambda_{\max}$ be the largest max-plus algebraic eigenvalue of $\mathbf{A}$. This eigenvalue $\lambda_{\max}$ gives a minimum for the average duration of a system cycle. In this chapter we consider a reference signal $\mathbf{r}(k)$, that the output should track. If the asymptotic slope of the reference signal $\mathbf{r}(k)$ is smaller than $\lambda_{\max}$, the system cannot complete tasks in time and $\mathbf{y}(k)-\mathbf{r}(k)$ will grow unbounded in time. Therefore in this chapter we choose

$$\mathbf{r}(k) = \mathbf{r}_0 + \rho\, k. \tag{39}$$

where $\mathbf{r}_0 \in \mathbb{R}$ is a vector of offsets. In [55] it has been shown that the condition $\rho > \lambda_{\max}$ is necessary for stability.

Definition 7 *Given an MPL system (8)-(9) and a reference (39) with*

$$\rho > \lambda_{\max}, \tag{40}$$

in which $\lambda_{\max}$ *denotes the largest max-plus-algebraic eigenvalue of A. The equilibrium point* $(\mathbf{x}_0, \mathbf{u}_0, \mathbf{r}_0)$ *is defined for a given* $\mathbf{r}_0$ *as the solution of*

$$(\mathbf{x}_0, \mathbf{u}_0) = \arg\max_{\mathbf{u}_0, \mathbf{x}_0} \sum_{i=1}^{n_u} [\mathbf{u}_0]_i \tag{41}$$

$$\textit{such that} \quad \mathbf{x}_0 = \mathbf{A} \otimes \mathbf{x}_0 \oplus \mathbf{B} \otimes \mathbf{u}_0 \tag{42}$$

$$\mathbf{r}_0 \geqslant \mathbf{C} \otimes \mathbf{x}_0 \tag{43}$$

Lemma 8 *[47, 58]*
Given an MPL system (8)-(9) and a reference (39) with $\rho > \lambda_{\max}$ *in which* $\lambda_{\max}$ *denotes the largest max-plus-algebraic eigenvalue of A. The equilibrium point* $(\mathbf{x}_0, \mathbf{u}_0, \mathbf{r}_0)$ *can be computed for a given* $\mathbf{r}_0$ *as follows*

$$\begin{aligned} \mathbf{u}_0 &= -((\mathbf{C} \otimes \mathbf{A}_\rho^* \otimes \mathbf{B})^T \otimes \mathbf{r}_0)\,, \\ \mathbf{x}_0 &= \mathbf{A}_\rho^* \otimes \mathbf{B} \otimes \mathbf{u}_0\,. \end{aligned} \tag{44}$$

where $[\mathbf{A}_\rho]_{ij} = [\mathbf{A}]_{ij} - \rho$ ◇

Consider a system (8)-(9) and a reference signal (39). To simplify the derivations we will restrict ourselves to SISO systems, so $\mathbf{B} \in \mathbb{R}_\varepsilon^{n\times 1}$, and $\mathbf{C} \in \mathbb{R}_\varepsilon^{1\times n}$. However, all derivations in this and the next section can easily be extended to MIMO systems. Let $\lambda_{\max}$ be the largest eigenvalue of the matrix $\mathbf{A}$, and consider a tracking rate $\rho > \lambda_{\max}$. We will now normalize this system. There exists a max-plus-algebraic invertible diagonal matrix $\mathbf{P}$ such that the matrix

$$\bar{\mathbf{A}} = (\mathbf{P}^{\otimes^{-1}} \otimes \mathbf{A} \otimes \mathbf{P}) - \rho \tag{45}$$

satisfies $\bar{\mathbf{A}}_{ij} < 0, \ \ \forall i,j$ [47]. Now define

$$\bar{\mathbf{B}} = (\mathbf{P}^{\otimes^{-1}} \otimes \mathbf{B}) + \mathbf{u}_0 \ \ , \tag{46}$$

$$\bar{\mathbf{C}} = (\mathbf{C} \otimes \mathbf{P}) - \mathbf{r}_0, \tag{47}$$

where $\mathbf{u}_0$ is given by (44). Define the normalized signals

$$\bar{\mathbf{x}}(k) = \left(\mathbf{P}^{\otimes^{-1}} \otimes \mathbf{x}(k)\right) - \rho\, k \ \ , \tag{48}$$

$$\bar{\mathbf{u}}(k) = \mathbf{u}(k) - \rho\, k - \mathbf{u}_0 \ \ , \tag{49}$$

$$\bar{\mathbf{y}}(k) = \mathbf{y}(k) - \rho\, k - \mathbf{r}_0 \ \ . \tag{50}$$

Then the MPL system (8)-(9) is equivalent to the normalized MPL system

$$\bar{\mathbf{x}}(k) = \bar{\mathbf{A}} \otimes \bar{\mathbf{x}}(k-1) \oplus \bar{\mathbf{B}} \otimes \bar{\mathbf{u}}(k) \tag{51}$$

$$\bar{\mathbf{y}}(k) = \bar{\mathbf{C}} \otimes \bar{\mathbf{x}}(k) \tag{52}$$

Corollary 9 *Consider the normalized system (51)-(52). By substitution of the system matrices of the normalized system in (44) we find*

$$\bar{\mathbf{r}}_0 = 0 \tag{53}$$

$$\bar{\mathbf{u}}_0 = -((\bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^* \otimes \bar{\mathbf{B}})^T \otimes r_0) = 0 \tag{54}$$

$$\bar{\mathbf{y}}_0 = \bar{\mathbf{C}} \otimes \bar{\mathbf{x}}_0 = \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^* \otimes \bar{\mathbf{B}} \otimes \bar{\mathbf{u}}_0 = 0 \tag{55}$$

From (55) we find $\bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^* \otimes \bar{\mathbf{B}} \leqslant 0$ *and so*

$$\bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes^i} \otimes \bar{\mathbf{B}} \leq 0 \ \ , \quad \forall i \geqslant 0 \tag{56}$$

$\diamond$

Consider the normalized MPL system (51)-(52). The MPC problem for this system is reformulated as follows:

$$\min_{\bar{\mathbf{u}}(k),\ldots,\bar{\mathbf{u}}(k+N_\mathrm{p}-1)} \sum_{j=0}^{N_\mathrm{p}-1} \left(\sum_{m=1}^{n_y} \max(\bar{\mathbf{y}}_m(k+j|k)\ ,\ 0) - \beta \sum_{\ell=1}^{n_u} \bar{\mathbf{u}}_\ell(k+j) \right) \tag{57}$$

subject to

$$\bar{\mathbf{u}}_\ell(k+j) - \bar{\mathbf{u}}_\ell(k+j-1) \geqslant -\rho, \ \text{ for } \ \ j = 0,\ldots,N_\mathrm{p}-1, \ell = 1\ldots,n_u, \tag{58}$$

Define the signals $\bar{\mathbf{u}}^\flat$ and $\bar{\mathbf{z}}$:

$$\bar{\mathbf{u}}^\flat(k+j) = (\bar{\mathbf{u}}(k-1) - \rho\,(j+1)) \oplus \mathbf{0}$$

$$\bar{\mathbf{z}}(k+j|k) = 0 \oplus \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes^{j+1}} \otimes \bar{\mathbf{x}}(k-1) \oplus \bigoplus_{i=0}^{j} \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes^{j-i}} \otimes \bar{\mathbf{B}} \otimes \bar{\mathbf{u}}^\flat(k+i)$$

for $j = 0,\ldots,N_\mathrm{p}-1$.

Proposition 10 *Assume $0 < \beta < 1/(n_u N_\mathrm{p})$ and define the matrices $\tilde{\bar{\mathbf{D}}}_\ell = \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes^\ell} \otimes \bar{\mathbf{B}}$ for $\ell = 0, \ldots, N_\mathrm{p} - 1$. Consider the maximization problem*

$$\max_{\bar{\mathbf{u}}(k),\ldots,\bar{\mathbf{u}}(k+N_\mathrm{p}-1)} \sum_{j=0}^{N_\mathrm{p}-1} \left(\sum_{\ell=1}^{n_u} \bar{\mathbf{u}}_\ell(k+j) \right) \tag{59}$$

subject to

$$[\tilde{\bar{\mathbf{D}}}_{i-j}]_{m,\ell} + \bar{\mathbf{u}}_\ell(k+j|k) \leq \bar{\mathbf{z}}_m(k+i|k) \ , \ \forall \ell, m, \ \forall i \geqslant j, \ i,j \in \{0,\ldots,N_\mathrm{p}-1\} \tag{60}$$

$$\bar{\mathbf{u}}_\ell(k+j) - \bar{\mathbf{u}}_\ell(k+j-1) \geqslant -\rho \ \ \forall \ell, \ \forall j \in \{0,\ldots,N_\mathrm{p}-1\} \tag{61}$$

Then, $\bar{\mathbf{u}}(k), \ldots, \bar{\mathbf{u}}(k+N_\mathrm{p}-1)$ is the optimal input sequence for the MPL-MPC problem (57)-(58) at event step k. The output for this optimal input sequence is given by

$$\bar{\mathbf{y}}(k+j|k) = \bar{\mathbf{z}}(k+j|k)$$

$\diamond$

Proof: Define

$$\tilde{\bar{\mathbf{u}}}(k) = \begin{bmatrix} \bar{\mathbf{u}}(k) \\ \bar{\mathbf{u}}(k+1) \\ \vdots \\ \bar{\mathbf{u}}(k+N_\mathrm{p}-1) \end{bmatrix} \ , \ \tilde{\bar{\mathbf{u}}}^*(k) = \begin{bmatrix} \bar{\mathbf{u}}^*(k) \\ \bar{\mathbf{u}}^*(k+1) \\ \vdots \\ \bar{\mathbf{u}}^*(k+N_\mathrm{p}-1) \end{bmatrix} \ , \ \tilde{\bar{\mathbf{u}}}^\sharp(k) = \begin{bmatrix} \bar{\mathbf{u}}^\sharp(k) \\ \bar{\mathbf{u}}^\sharp(k+1) \\ \vdots \\ \bar{\mathbf{u}}^\sharp(k+N_\mathrm{p}-1) \end{bmatrix}$$

where the sequence $\bar{\mathbf{u}}^*(k), \ldots, \bar{\mathbf{u}}^*(k+N_\mathrm{p}-1)$ is the optimal solution for MPL-MPC problem (57)-(58), and the sequence $\bar{\mathbf{u}}^\sharp(k), \ldots, \bar{\mathbf{u}}^\sharp(k+N_\mathrm{p}-1)$ is the optimal solution for optimization problem (59)-(61). First define

$$J_\mathrm{out}(\tilde{\bar{\mathbf{u}}}(k)) = \sum_{j=0}^{N_\mathrm{p}-1} \sum_{m=1}^{n_y} \max(\bar{\mathbf{y}}_m(k+j|k) \ , \ 0) \quad , \quad J_\mathrm{in}(\tilde{\bar{\mathbf{u}}}(k)) = \sum_{j=0}^{N_\mathrm{p}-1} \sum_{\ell=1}^{n_u} - \bar{\mathbf{u}}_\ell(k+j)$$

so that

$$J(\tilde{\bar{\mathbf{u}}}(k)) = J_\mathrm{out}(\tilde{\bar{\mathbf{u}}}(k)) + \beta \, J_\mathrm{in}(\tilde{\bar{\mathbf{u}}}(k))$$

We will prove that for any $\tilde{\bar{\mathbf{u}}}^\sharp(k) \neq \tilde{\bar{\mathbf{u}}}^*(k)$ we find $J(\tilde{\bar{\mathbf{u}}}^\sharp(k)) \geqslant J(\tilde{\bar{\mathbf{u}}}^*(k))$. The proof consists of two parts. First we define $\bar{\mathbf{u}}^\natural(k+j) = \bar{\mathbf{u}}^\sharp(k+j) \otimes \bar{\mathbf{u}}^*(k+j)$ and $\tilde{\bar{\mathbf{u}}}^\natural(k) = \tilde{\bar{\mathbf{u}}}^\sharp(k) \otimes \tilde{\bar{\mathbf{u}}}^*(k)$ and prove that $J(\tilde{\bar{\mathbf{u}}}^\natural(k)) \leqslant J(\tilde{\bar{\mathbf{u}}}^\sharp(k))$, and secondly we prove that $J(\tilde{\bar{\mathbf{u}}}^*(k)) \leqslant J(\tilde{\bar{\mathbf{u}}}^\natural(k))$.

Part 1:

Introduce the signal $\bar{\mathbf{u}}^\natural(k+j) = \bar{\mathbf{u}}^\sharp(k+j) \oplus \bar{\mathbf{u}}^*(k+j)$. First define

$$\begin{aligned} \bar{\mathbf{y}}^\natural(k+j|k) = {} & \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes^{j+1}} \otimes \bar{\mathbf{x}}(k-1) \\ & \oplus \bigoplus_{i=0}^{j} \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes^{j-i}} \otimes \bar{\mathbf{B}} \otimes \bar{\mathbf{u}}^\natural(k+i) \end{aligned}$$

Now we derive:

$$\begin{aligned}
\bar{\mathbf{y}}^{\natural}(k+j|k) \oplus \bar{\mathbf{z}}(k+j|k) &= \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes^{j+1}} \otimes \bar{\mathbf{x}}(k-1) \\
&\oplus \bigoplus_{i=0}^{j} \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes^{j-i}} \otimes \bar{\mathbf{B}} \otimes u^{\natural}(k+i) \oplus \bar{\mathbf{z}}(k+j|k) \\
&= \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes^{j+1}} \otimes \bar{\mathbf{x}}(k-1) \\
&\oplus \bigoplus_{i=0}^{j} \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes^{j-i}} \otimes \bar{\mathbf{B}} \otimes \bar{\mathbf{u}}^{\sharp}(k+i) \\
&\oplus \bigoplus_{i=0}^{j} \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes^{j-i}} \otimes \bar{\mathbf{B}} \otimes \bar{\mathbf{u}}^{*}(k+i) \oplus \bar{\mathbf{z}}(k+j|k) \\
&= \bar{\mathbf{y}}^{*}(k+j|k) \oplus \bar{\mathbf{z}}(k+j|k)
\end{aligned}$$

because

$$\bigoplus_{i=0}^{j} \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes^{j-i}} \otimes \bar{\mathbf{B}} \otimes \bar{\mathbf{u}}^{*}(k+i) \leq \bar{\mathbf{z}}(k+j|k)$$

and

$$\bar{\mathbf{y}}^{\sharp} = \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes^{j+1}} \otimes \bar{\mathbf{x}}(k-1) \oplus \bigoplus_{i=0}^{j} \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes^{j-i}} \otimes \bar{\mathbf{B}} \otimes \bar{\mathbf{u}}^{\sharp}(k+i)$$

This implies that $J(\tilde{\bar{\mathbf{u}}}^{*}(k)) = J(\tilde{\bar{\mathbf{u}}}^{s}harp(k))$. Note that $u^{\natural}(k+j) \geqslant u^{\sharp}(k+j)$ and so

$$\begin{aligned}
J_{\mathrm{in}}(\tilde{\bar{\mathbf{u}}}^{n}atural(k)) &= -\sum_{j=0}^{N_{\mathrm{p}}-1}\sum_{\ell=1}^{n_u} u^{\natural}_{\ell}(k+j) \\
&\leq -\sum_{j=0}^{N_{\mathrm{p}}-1}\sum_{\ell=1}^{n_u} u^{\sharp}_{\ell}(k+j) \\
&= J_{\mathrm{in}}(u^{\sharp}, k)
\end{aligned}$$

Part 2:

Next let us consider the vector $\tilde{\bar{\mathbf{u}}}^{\natural}(k)$ that satisfies (61) but does not satisfy (60). Let $\alpha > 0$ be such that

$$\max_{\ell=1,\ldots,n_u} \max_{j=1,\ldots,N_{\mathrm{p}}} (\bar{\mathbf{u}}^{\natural}_{\ell}(k+j) - \bar{\mathbf{u}}^{*}_{\ell}(k+j)) = \alpha$$

then there exists a $i', j' \in \{1, \ldots, N_{\mathrm{p}}-1\}$, $m' \in \{1, \ldots, n_y\}$ and $\ell' \in \{1, \ldots, n_u\}$ such that

$$\begin{aligned}
\bar{\mathbf{u}}^{\natural}_{\ell'}(k+j') - \bar{\mathbf{u}}^{*}_{\ell'}(k+j') &= \alpha \\
\bar{\mathbf{y}}^{*}_{m'}(k+i'|k) = [\tilde{\bar{\mathbf{D}}}_{i'-j'}]_{m',\ell'} + \bar{\mathbf{u}}^{*}_{\ell'}(k+j') &= \bar{\mathbf{z}}_{m'}(k+i'|k) \\
\bar{\mathbf{y}}^{\natural}_{m'}(k+i'|k) = [\tilde{\bar{\mathbf{D}}}_{i'-j'}]_{m',\ell'} + \bar{\mathbf{u}}^{\natural}_{\ell'}(k+j') &= \bar{\mathbf{z}}_{m'}(k+i'|k) + \alpha
\end{aligned}$$

and so

$$J_{\mathrm{out}}(\tilde{\bar{\mathbf{u}}}^{\natural}(k)) = \sum_{j=0}^{N_{\mathrm{p}}-1}\sum_{m=1}^{n_y} \max(\bar{\mathbf{y}}^{\natural}_{m}(k+j|k)\ ,\ 0) \tag{62}$$

$$\geqslant \sum_{j=0}^{N_{\mathrm{p}}-1}\sum_{m=1}^{n_y} \max(\bar{\mathbf{z}}_{m}(k+j|k)\ ,\ 0) + \alpha \tag{63}$$

$$= J_{\mathrm{out}}(\tilde{\bar{\mathbf{u}}}^{*}(k)) + \alpha \tag{64}$$

On the other hand

$$
\begin{aligned}
J_{\mathrm{in}}(\tilde{\bar{\mathbf{u}}}^{\natural}(k)) &= -\beta \sum_{j=0}^{N_{\mathrm{p}}-1} \sum_{\ell=1}^{n_u} \bar{\mathbf{u}}^{\natural}_{m}(k+j) \\
&\geqslant -\beta \sum_{j=0}^{N_{\mathrm{p}}-1} \sum_{\ell=1}^{n_u} \left(\bar{\mathbf{u}}^{*}_{m}(k+j) + \alpha\right) \\
&= -\beta J_{\mathrm{in}}(\tilde{\bar{\mathbf{u}}}^{*}(k)) - N_{\mathrm{p}} n_u \alpha
\end{aligned}
$$

This means that

$$
\begin{aligned}
J(\tilde{\bar{\mathbf{u}}}^{\natural}(k)) &= J_{\mathrm{out}}(\bar{\mathbf{u}}^{\natural}, k) + J_{\mathrm{in}}(\bar{\mathbf{u}}^{\natural}, k) \\
&\geqslant J_{\mathrm{out}}(\bar{\mathbf{u}}^{*}, k) + \alpha + J_{\mathrm{in}}(\bar{\mathbf{u}}^{*}, k) - \beta N_{\mathrm{p}} n_u \alpha \\
&= J(\tilde{\bar{\mathbf{u}}}^{*}(k)) + (1 - \beta N_{\mathrm{p}} n_u)\alpha
\end{aligned}
$$

With $\beta < 1/(N_{\mathrm{p}} n_u)$ we finally find that

$$
J(\tilde{\bar{\mathbf{u}}}^{\natural}(k)) \geqslant J(\tilde{\bar{\mathbf{u}}}^{*}(k))
$$

Combining the results of the parts 1 and 2 obtain

$$
J(\tilde{\bar{\mathbf{u}}}^{\sharp}(k)) \geqslant J(\tilde{\bar{\mathbf{u}}}^{\natural}(k)) \geqslant J(\tilde{\bar{\mathbf{u}}}^{*}(k))
$$

This proves that $\tilde{\bar{\mathbf{u}}}^{*}(k)$ is the optimal solution of the original MPC problem. Note that it is to show that if $\tilde{\bar{\mathbf{u}}}^{\natural}(k) \neq \tilde{\bar{\mathbf{u}}}^{*}(k)$ then we find a strict inequality $J(\tilde{\bar{\mathbf{u}}}^{\natural}(k)) > J(\tilde{\bar{\mathbf{u}}}^{*}(k))$ and if $\tilde{\bar{\mathbf{u}}}^{\sharp} \neq \tilde{\bar{\mathbf{u}}}^{\natural}$ we find the strict inequality $J(\tilde{\bar{\mathbf{u}}}^{\sharp}(k)) > J(\tilde{\bar{\mathbf{u}}}^{\natural}(k))$. This means that the optimum $\tilde{\bar{\mathbf{u}}}^{*}$ is unique. $\diamond$

The maximization problem (59)-(61) can be solved using linear programming algorithms. Note that because $\bar{\mathbf{z}}(k+j|k) \geqslant 0$ and $\tilde{\bar{\mathbf{D}}}_{\ell} = \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes^{\ell}} \otimes \bar{\mathbf{B}} \leq 0$ by (56), we find that due to the maximization of $\bar{\mathbf{u}}(k+j)$ in (59) we have $\bar{\mathbf{u}}(k+j) \geqslant 0$ for all j.

Next we look closer at the concept of stability for normalized systems. In a normalized MPL system, k is an event counter and $\bar{\mathbf{x}}_i(k)$ refers to the delay in occurrence time of an event. So similar to stability in conventional system theory, where boundedness of the states is required, in normalized MPL systems the state $\bar{\mathbf{x}}_i(k)$ should remain bounded. This implies that for an structurally controllable and structurally observable max-plus linear system with $\rho > \lambda_{\max}$ the closed-loop stability is achieved if there exist finite constants $\bar{y}_{\max}$, $\bar{x}_{\max}$, $\bar{u}_{\max}$, K such that for the output, state, and input of the corresponding normalized system we have

$$
\max_i |\bar{\mathbf{y}}_i(k)| \leq \bar{y}_{\max} \ , \ \ \forall k \geqslant K \tag{65}
$$

$$
\max_i |\bar{\mathbf{x}}_i(k)| \leq \bar{x}_{\max} \ , \ \ \forall k \geqslant K \tag{66}
$$

$$
\max_i |\bar{\mathbf{u}}_i(k)| \leq \bar{u}_{\max} \ , \ \ \forall k \geqslant K \tag{67}
$$

Let us now compare the stability conditions (65)-(67) with the notion of stability for discrete-event systems from [50], in which a discrete-event system is called stable if all its buffer levels remain bounded. Condition (65) means that the delay $\bar{\mathbf{y}}(k) = \mathbf{y}(k) - \mathbf{r}(k)$ remains bounded. Condition (66) implies that the number of parts in the output buffer will remain bounded. Finally, condition (67) together with (65) means that the time between the starting date $\mathbf{u}(k)$ and the output date $\mathbf{y}(k)$ (i.e., the throughput time) is bounded.

Before we can give our main result on stability in Theorem 13 we need to introduce two additional lemmas.

Lemma 11 *Let $\bar{\mathbf{x}} \in \mathbb{R}_\varepsilon^n$ and $\bar{\mathbf{A}} \in \mathbb{R}_\varepsilon^{n\times n}$ with $\bar{\mathbf{A}}_{ij} \leqslant 0$ for all i, j. Then we have*

$$\bar{\mathbf{A}}^{\otimes^{l+n}} \otimes \bar{\mathbf{x}} \leqslant \bar{\mathbf{A}}^{\otimes^{l+n-1}} \otimes \bar{\mathbf{x}} \oplus \bar{\mathbf{A}}^{\otimes^{l+n-2}} \otimes \bar{\mathbf{x}} \oplus \ldots \oplus \bar{\mathbf{A}}^{\otimes^{l}} \otimes \bar{\mathbf{x}} \tag{68}$$

for any integer $l \geqslant 1$. ◇

Proof: Note that if (68) holds for $l = 1$, it will hold for any integer $l \geqslant 1$ due to the monotonicity of max-plus-algebraic multiplication [2, 17].

We will first show that (68) holds for $l = 1$ and for the max-plus-algebraic unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ where $\mathbf{e}_j$ is defined as follows:

$$(\mathbf{e}_j)_i = \begin{cases} 0 & \text{if } i = j \\ \varepsilon & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, n$, i.e., we prove

$$(\bar{\mathbf{A}}^{\otimes^{n+1}} \otimes \mathbf{e}_j)_i \leqslant (\bar{\mathbf{A}}^{\otimes^{n}} \otimes \mathbf{e}_j)_i \oplus (\bar{\mathbf{A}}^{\otimes^{n-1}} \otimes \mathbf{e}_j)_i \oplus \ldots \oplus (\bar{\mathbf{A}} \otimes \mathbf{e}_j)_i \ . \tag{69}$$

Note that for $\mathbf{e}_j$ and any integer $\ell \geqslant 0$, we have $(\bar{\mathbf{A}}^{\otimes^{\ell}} \otimes \mathbf{e}_j)_i = (\bar{\mathbf{A}}^{\otimes^{\ell}})_{ij}$ for all i.

Now we use the fact that the max-plus-algebraic matrix power has the following graph-theoretic interpretation [2]: the value of $(\bar{\mathbf{A}}^{\otimes^{\ell}})_{ij}$ with ℓ a positive integer corresponds the maximum weight of a path of length ℓ from vertex j to vertex i in the precedence graph $\mathcal{G}(\bar{\mathbf{A}})$ of $\bar{\mathbf{A}}$, which is defined as follows: $\mathcal{G}(\bar{\mathbf{A}})$ has n vertices and an arc with weight $\bar{\mathbf{A}}_{ij}$ from vertex j to vertex i for every pair (i, j) such that $\bar{\mathbf{A}}_{ij} \neq \varepsilon$ (So $\bar{\mathbf{A}}_{ij} \neq \varepsilon$ indicates that there is no arc from vertex j to vertex i).

Let $i, j \in \{1, \ldots, n\}$. Now we consider two cases: if there is no path of any length from vertex j to vertex i, then we have $(\bar{\mathbf{A}}^{\otimes^{\ell}})_{ij} = \varepsilon$ for all ℓ and thus

$$\varepsilon = (\bar{\mathbf{A}}^{\otimes^{n+1}} \otimes \mathbf{e}_j)_i \leqslant (\bar{\mathbf{A}}^{\otimes^{n}} \otimes \mathbf{e}_j)_i \oplus \ldots \oplus (\bar{\mathbf{A}} \otimes \mathbf{e}_j)_i = \varepsilon \ . \tag{70}$$

So in this case (69) holds. Now we consider the case that there is at least one path from vertex j to vertex i in $\mathcal{G}(\bar{\mathbf{A}})$. Since $\mathcal{G}(\bar{\mathbf{A}})$ has n vertices, we obtain — after the removal of any loops in the path, if present — a path of length ℓ with $1 \leqslant \ell \leqslant n$. So the right-hand side of (69) is different from ε. Let us denote the value of the right-hand side of (69) in this case by $w_{\max}$. If we now consider a path P of length $n+1$ from vertex j to vertex i, then this path has to contain at least one loop, as well as loop-free path from vertex j to vertex i with a length between 1 and n. The maximal weight of the loop-free path will be less than or equal to $w_{\max}$, and due to the fact that the entries of $\bar{\mathbf{A}}$ are less than or equal to zero, the weight of the loops is also less than or equal to zero, which implies that the weight of P is also less than or equal to $w_{\max}$. So (69) also holds in this case.

So now we have proven that (68) holds for the max-plus-algebraic unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Since any vector $\bar{\mathbf{x}} \in \mathbb{R}_\varepsilon^n$ can be written as

$$\bar{\mathbf{x}} = \bigoplus_{i=1}^{n} \bar{\mathbf{x}}_i \otimes \mathbf{e}_i$$

and since max-plus-algebraic addition and multiplication are monotonous [2], (68) also holds. ◇

Lemma 12 *Let $\bar{\mathbf{x}} \in \mathbb{R}_\varepsilon^n$ and $\bar{\mathbf{A}} \in \mathbb{R}_\varepsilon^{n\times n}$ with $\bar{\mathbf{A}}_{ij} \leqslant 0$ for all i, j. Then we have*

$$\bar{\mathbf{A}}^{\otimes^l} \otimes \bar{\mathbf{x}} \leqslant \bar{\mathbf{A}}^{\otimes^{l-1}} \otimes \bar{\mathbf{x}} \oplus \bar{\mathbf{A}}^{\otimes^{l-2}} \otimes \bar{\mathbf{x}} \oplus \ldots \oplus \bar{\mathbf{A}} \otimes \bar{\mathbf{x}} \tag{71}$$

for any integer $l \geqslant n$. $\diamond$

Proof: From Lemma 11 it follows that

$$\bar{\mathbf{A}}^{\otimes^l} \otimes \bar{\mathbf{x}} \leqslant \bar{\mathbf{A}}^{\otimes^{l-1}} \otimes \bar{\mathbf{x}} \oplus \bar{\mathbf{A}}^{\otimes^{l-2}} \otimes \bar{\mathbf{x}} \oplus \ldots \oplus \bar{\mathbf{A}}^{\otimes^{l-n}} \otimes \bar{\mathbf{x}} \tag{72}$$

Using the implication $\mathbf{w} \leq \mathbf{v} \implies \mathbf{w} \leq \mathbf{v} \oplus \mathbf{z}$ it immediate follows that

$$\begin{aligned} \bar{\mathbf{A}}^{\otimes^l} \otimes \bar{\mathbf{x}} \leqslant & \Big(\bar{\mathbf{A}}^{\otimes^{l-1}} \otimes \bar{\mathbf{x}} \oplus \bar{\mathbf{A}}^{\otimes^{l-2}} \otimes \bar{\mathbf{x}} \oplus \ldots \oplus \bar{\mathbf{A}}^{\otimes^{l-n}} \otimes \bar{\mathbf{x}} \Big) \\ & \oplus \Big(\bar{\mathbf{A}}^{\otimes^{l-n-1}} \otimes \bar{\mathbf{x}} \oplus \bar{\mathbf{A}}^{\otimes^{l-n-2}} \otimes \bar{\mathbf{x}} \oplus \ldots \oplus \bar{\mathbf{A}} \otimes \bar{\mathbf{x}} \Big) \end{aligned} \tag{73}$$

$\diamond$

Theorem 13 *Let a normalized MPL system (51)-(52) be structurally controllable and structurally observable. For every event step k we compute the optimal input sequence by solving (57)-(58) and we apply only $\bar{\mathbf{u}}(k)$. Let $N_\mathrm{p} \geqslant n$ and $0 < \beta < 1/(N_\mathrm{p} n_u)$. Define the function*

$$V(k) = N_\mathrm{p} n_y \max_{j=0,\ldots,N_\mathrm{p}-1} \max_i \;\; \max(\bar{\mathbf{y}}_i(k+j|k), 0) \tag{74}$$

There holds:

$$V(k) \geqslant 0 \quad , \quad V(k+1) \leq V(k) \tag{75}$$

Furthermore, we have

$$V(k) \geqslant J(k) \tag{76}$$

Together with the fact that $J(k) \geqslant 0$, this means that the closed-loop system is stable. $\diamond$

Proof: First we prove (76):

$$\begin{aligned} V(k) &= N_\mathrm{p} n_y \max_{j=0,\ldots,N_\mathrm{p}-1} \max_i \;\; \max(\bar{\mathbf{y}}_i(k+j|k), 0) \\ &\geqslant \sum_{j=0}^{N_\mathrm{p}-1} \sum_{i=1}^{n_y} \max(\bar{\mathbf{y}}_i(k+j|k), 0) \\ &\geqslant \sum_{j=0}^{N_\mathrm{p}-1} \Big(\sum_{i=1}^{n_y} \max(\bar{\mathbf{y}}_i(k+j|k), 0) - \beta \sum_{\ell=1}^{n_u} \bar{\mathbf{u}}_\ell(k+j|k) \Big) \\ &= J(k) \end{aligned}$$

where we have used the fact that $\bar{\mathbf{u}}(k+j|k) \geqslant 0$ for all $j = 0, 1, \ldots, N_\mathrm{p} - 1$.
Next we prove (75). First of all, note that by (74) we have $V(k) \geqslant 0$. The next step is to prove $V(k+1) \leq V(k)$:
Consider

$$V(k+1) = N_\mathrm{p} n_y \max_{j=1,\ldots,N_\mathrm{p}} \max_i \max(\bar{\mathbf{y}}_i(k+j|k+1), 0)$$

We will first prove that $\bar{\mathbf{y}}(k+j|k+1) = \bar{\mathbf{y}}(k+j|k)$ for $j = 1, \ldots, N_\mathrm{p} - 1$.

With $\bar{\mathbf{x}}(k|k+1) = \bar{\mathbf{x}}(k|k)$ we can easily observe that $\bar{\mathbf{z}}(k+j|k+1) = \bar{\mathbf{z}}(k+j|k)$ and so according to Proposition 10 we have $\bar{\mathbf{y}}(k+j|k+1) = \bar{\mathbf{y}}(k+j|k)$. To prove that $V(k+1) \leq V(k)$ we have to prove:

$$\max_i \bar{\mathbf{y}}_i(k+N_\mathrm{p}|k+1) \leq \max_{j=0,\ldots,N_\mathrm{p}-1} \max_i \bar{\mathbf{y}}_i(k+j|k)$$

First note that at event step $k+1$ the signals $\bar{\mathbf{u}}^\flat$ and $\bar{\mathbf{z}}$ for $j \geqslant 1$ are given by

$$\begin{aligned}
\bar{\mathbf{u}}^\flat(k+j|k+1) &= (\bar{\mathbf{u}}(k) - \rho\, j) \oplus 0 \\
\bar{\mathbf{z}}(k+j|k+1) &= 0 \oplus \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes^j} \otimes \bar{\mathbf{x}}(k) \\
&\qquad \oplus \bigoplus_{i=1}^{j} \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes^{j-i}} \otimes \bar{\mathbf{B}} \otimes \bar{\mathbf{u}}^\flat(k+i|k).
\end{aligned}$$

Now define

$$\begin{aligned}
\bar{\mathbf{y}}^\natural(k+N_\mathrm{p}|k+1) &= \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes N_\mathrm{p}+1} \otimes \bar{\mathbf{x}}(k-1) \\
\bar{\mathbf{y}}^\sharp(k+N_\mathrm{p}|k+1) &= \bigoplus_{j=0}^{N_\mathrm{p}} \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes N_\mathrm{p}+1} \otimes \bar{\mathbf{B}} \otimes \bar{\mathbf{u}}(k|k) \\
&\qquad \oplus \bigoplus_{j=1}^{N_\mathrm{p}} \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes N_\mathrm{p}-j} \\
&\qquad \otimes \bar{\mathbf{B}} \otimes \bar{\mathbf{u}}^\flat(k+j|k+1)
\end{aligned}$$

Using this we derive

$$\begin{aligned}
&\mathbf{0} \oplus \bar{\mathbf{y}}^\natural(k+N_\mathrm{p}|k+1) \oplus \bar{\mathbf{y}}^\sharp(k+N_\mathrm{p}|k+1) = \\
&\quad = \mathbf{0} \oplus \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes N_\mathrm{p}+1} \otimes \bar{\mathbf{x}}(k-1) \oplus \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes N_\mathrm{p}+1} \otimes \bar{\mathbf{B}} \otimes \bar{\mathbf{u}}(k|k) \\
&\qquad \oplus \bigoplus_{j=1}^{N_\mathrm{p}} \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes N_\mathrm{p}-j} \otimes \bar{\mathbf{B}} \otimes \bar{\mathbf{u}}^\flat(k+j|k+1) \\
&\quad = \mathbf{0} \oplus \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes N_\mathrm{p}} \otimes \bar{\mathbf{x}}(k) \oplus \bigoplus_{j=1}^{N_\mathrm{p}} \bar{\mathbf{C}} \otimes \bar{\mathbf{A}}^{\otimes N_\mathrm{p}-j} \otimes \bar{\mathbf{B}} \otimes \bar{\mathbf{u}}^\flat(k+j|k+1) \\
&\quad = \bar{\mathbf{z}}(k+N_\mathrm{p}|k+1) \\
&\quad = \bar{\mathbf{y}}(k+N_\mathrm{p}|k+1)
\end{aligned}$$

where the last step is due to Proposition 10. From Lemma 12 we know that for all $\bar{\mathbf{x}} \in \mathbb{R}_\varepsilon^n$ we have

$$\bar{\mathbf{A}}^{\otimes^{N_\mathrm{p}+1}} \otimes \bar{\mathbf{x}} \leq \bar{\mathbf{A}}^{\otimes^{N_\mathrm{p}}} \otimes \bar{\mathbf{x}} \oplus \bar{\mathbf{A}}^{\otimes^{N_\mathrm{p}-1}} \otimes \bar{\mathbf{x}} \oplus \ldots \oplus \bar{\mathbf{A}} \otimes \bar{\mathbf{x}}$$

since $N_\mathrm{p} \geqslant n$, and so using observability we obtain

$$\begin{aligned}
\bar{\mathbf{C}}_i \otimes \bar{\mathbf{A}}^{\otimes^{N_\mathrm{p}+1}} \otimes \bar{\mathbf{x}}(k-1) &\leq \bar{\mathbf{C}}_i \otimes \bar{\mathbf{A}}^{\otimes^{N_\mathrm{p}}} \otimes \bar{\mathbf{x}}(k-1) \\
&\quad \oplus \bar{\mathbf{C}}_i \otimes \bar{\mathbf{A}}^{\otimes^{N_\mathrm{p}-1}} \otimes \bar{\mathbf{x}}(k-1) \oplus \ldots \oplus \bar{\mathbf{C}}_i \otimes \bar{\mathbf{A}} \otimes \bar{\mathbf{x}}(k-1)
\end{aligned}$$

where $\bar{\mathbf{C}}_i$ denotes the ith row of matrix $\bar{\mathbf{C}}$. This results in

$$\bar{\mathbf{y}}_i^\natural(k+N_\mathrm{p}|k+1) \leq \max_{j=0,\ldots,N_\mathrm{p}-1} \bar{\mathbf{y}}_i^\natural(k+j|k+1) \;\; \textit{for } j = 0, \ldots, N_\mathrm{p}$$

With $\bar{\mathbf{y}}_i(k+j|k) \le \bar{\mathbf{C}}_i \otimes \bar{\mathbf{A}}^{\otimes j+1} \otimes \bar{\mathbf{x}}(k-1)$ for $j = 0, \ldots, N_\mathrm{p} - 1$ we derive

$$\begin{aligned}\bar{\mathbf{y}}_i^\natural(k+N_\mathrm{p}|k+1) &\le \max_{j=0,\ldots,N_\mathrm{p}-1} \bar{\mathbf{y}}_i^\natural(k+j|k+1)\\ &\le \max_{j=0,\ldots,N_\mathrm{p}-1} \bar{\mathbf{y}}_i(k+j|k+1)\end{aligned}$$

for $j = 0, \ldots, N_\mathrm{p}$. Using $\bar{\mathbf{y}}(k+j|k) = \bar{\mathbf{z}}(k+j|k) \geqslant 0$ we find

$$0 \oplus \bar{\mathbf{y}}_i^\natural(k+N_\mathrm{p}|k+1) \le \max_{j=0,\ldots,N_\mathrm{p}-1} \bar{\mathbf{y}}_i(k+j|k)$$

Further we know that for $j = 0, \ldots, N_\mathrm{p} - 1$,

$$\bar{\mathbf{u}}^\flat(k+j|k+1) = (\bar{\mathbf{u}}(k) - \rho\, j) \oplus 0 \le \bar{\mathbf{u}}(k+j|k)$$

This means that

$$\begin{aligned}&\bar{\mathbf{C}}_i \otimes \bar{\mathbf{A}}^{\otimes N_\mathrm{p}-j} \otimes \bar{\mathbf{B}} \otimes \bar{\mathbf{u}}^\flat(k+j|k+1)\\ &\qquad \le \bar{\mathbf{C}}_i \otimes \bar{\mathbf{A}}^{\otimes N_\mathrm{p}-j} \otimes \bar{\mathbf{B}} \otimes \bar{\mathbf{u}}(k+j|k)\\ &\qquad \le \bar{\mathbf{y}}_i(k+j|k)\end{aligned}$$

for $j = 0, \ldots, N_\mathrm{p} - 1$. This results in

$$\bar{\mathbf{y}}_i^\sharp(k+N_\mathrm{p}|k+1) \le \max_{j=0,\ldots,N_\mathrm{p}-1} \bar{\mathbf{y}}_i(k+j|k)$$

and so it follows:

$$\begin{aligned}\bar{\mathbf{y}}_i(k+N_\mathrm{p}|k+1) &= 0 \oplus \bar{\mathbf{y}}_i^\natural(k+N_\mathrm{p}|k+1) \oplus \bar{\mathbf{y}}_i^\sharp(k+N_\mathrm{p}|k+1)\\ &\le \max_{j=0,\ldots,N_\mathrm{p}-1} \bar{\mathbf{y}}_i(k+j|k)\end{aligned}$$

We finally obtain:

$$\max_i \bar{\mathbf{y}}_i(k+N_\mathrm{p}|k+1) \le \max_{j=0,\ldots,N_\mathrm{p}-1} \max_i \bar{\mathbf{y}}_i(k+j|k)$$

We now have that $V(k)$ will be non-increasing, and so the function $J(k)$ will be bounded. This implies that there exists an upper bound for $\bar{\mathbf{y}}(k)$, and that $\bar{\mathbf{u}}(k)$ will have both an upper and lower bound. With the property that $\bar{\mathbf{y}}(k) - \bar{\mathbf{u}}(k) \geqslant \bar{\mathbf{C}} \otimes \bar{\mathbf{B}}$ we also prove that $\bar{\mathbf{y}}(k)$ has an lower bound. The system is structurally controllable, which means that if $\bar{\mathbf{u}}(k)$ has a lower bound, then $\bar{\mathbf{x}}(k)$ will have a lower bound. Due to the fact that $\lambda_{\max}(\bar{\mathbf{A}}) \le 0$, we find that if the initial state $\bar{\mathbf{x}}(0)$ has an upper bound and $\bar{\mathbf{u}}(k)$ has an upper bound, then $\bar{\mathbf{x}}(k)$ will have an upper bound. This proves that the closed-loop system is stable. ◇

3.2 Constrained case

The existence of a solution of MPL-MPC problem in the presence of constraint (15) at event step k can be verified by solving the system of (in)equalities (21)–(24), which describes the feasible set of the problem. Now, feasibility in the MPL-MPC problem is comparable to feasibility in conventional MPC. Infeasibility occurs when solving $\tilde{\mathbf{u}}(k)$ from (21)–(24) results in a solution set that is empty. In that case, some constraints could be relaxed. The constraints (21) and (23) should always be satisfied because of their physical meaning. The control horizon constraint (24) is used to reduce the number of variables in the optimization. By increasing N_c the degrees of freedom increases and the optimization may become feasible for a larger N_c. However, an increase of N_c may give a dramatic increase of computational burden and may also lead to instability in the case of modeling errors. So for constraint relaxation, we therefore concentrate on the constraint (22).

If for a certain step k the problem is not feasible, so if the set described by the constraints (21)–(24) is empty, then constraint (22) can be relaxed as follows. First, we choose a diagonal matrix $\mathbf{R} \in \mathbb{R}^{n_E \times n_E}$ with non-negative diagonal entries, where n_E is the number of rows of $\mathbf{E}(k)$. Now we introduce a vector $\boldsymbol{\nu}(k) \in \mathbb{R}^{n_E}$ of dummy variables and we solve the problem

$$\min_{\tilde{\mathbf{u}}(k),\tilde{\mathbf{y}}(k),\boldsymbol{\nu}(k)} \quad J_{\text{out}}(\tilde{\mathbf{y}}) + \beta J_{\text{in}}(\tilde{\mathbf{u}}) + \sum_{i=1}^{n_E} \boldsymbol{\nu}_i(k) \tag{77}$$

subject to (21), (23), (24) and

$$\mathbf{E}(k)\tilde{\mathbf{u}}(k) + \mathbf{F}(k)\tilde{\mathbf{y}}(k) + \mathbf{G}\tilde{\mathbf{r}}(k) \leqslant \mathbf{h}(k) + \mathbf{R}\,\boldsymbol{\nu}(k) \tag{78}$$

$$\boldsymbol{\nu}(k) \geqslant 0 \ . \tag{79}$$

The entries of diagonal matrix $\mathbf{R}$ give a measure on the violation degree of the corresponding constraints.

For $\mathbf{R} > 0$ optimization problem (77)-(79) is feasible since the constraints can always be met by making the components of $\boldsymbol{\nu}(k)$ sufficiently large. Also note that inclusion of the term $\boldsymbol{\nu}_1(k) + \cdots + \boldsymbol{\nu}_{n_E}(k)$ in the objective function makes the constraint violations w.r.t. the original infeasible problem as small as possible. Furthermore, if the original (infeasible) MPL-MPC problem satisfies the conditions of Theorem 4 (i.e., the mapping $\tilde{\mathbf{y}} \to \mathbf{F}(k)\tilde{\mathbf{y}}$ is a monotonically non-decreasing function of $\tilde{\mathbf{y}}$) then the problem (77) – (79) also satisfies these conditions so that Theorem 4 still applies. Moreover, the new objective function is also convex since the relaxation term is linear.

3.3 Timing

Max-plus-linear systems are different from conventional time-driven systems in the sense that the event counter k is not directly related to a specific time. So far we have assumed that at event step k the state $\mathbf{x}(k-1)$ is available (recall that $\mathbf{x}(k-1)$ contains the time instants at which the internal activities or processes of the system start for the $(k-1)$th cycle). Therefore, we will present a method to address the availability issue of the state at a certain time instant t. Since the components of $\mathbf{x}(k-1)$ correspond to event times, they are in general easy to measure. So we consider the case of full state information. Also note that measurements of occurrence times of events are in general not as susceptible to noise and measurement errors as measurements of continuous-time signals involving variables such as temperature, speed, pressure, etc. Let t be the time instant when an MPC problem has to be solved. We can define the initial cycle k as follows:

$$k = \arg\max \Big\{ \ell \mid \mathbf{x}_i(\ell-1) \leq t \ , \ \ \forall i \in \{1, 2, \ldots, n\} \Big\}$$

Hence, the state $\mathbf{x}(k-1)$ is completely known at time t and thus $\mathbf{u}(k-1)$ is also available (due to the fact that in practical applications the entries of the system matrices are non-negative or take the value ε). Note that at time t some components of the future[3] states and of the forthcoming inputs might be known (so $\mathbf{x}_i(k+\ell) \leq t$ and $\mathbf{u}_j(k+\ell) \leq t$ for some i, j and some $\ell \geqslant 0$). Due to causality, these states are completely determined by the known forthcoming inputs. During the optimization at time instant t the known values of the input have to be fixed by equality constraints, which fits perfectly in the

[3] Future in the event counter sense.

framework of a linear programming problem. With these new equality constraints we can perform the MPC optimization at time t.

3.4 Tuning

In this section we will give some guidelines to find suitable choices of the three tuning parameters (N_p, N_c, β) and to select an appropriate reference signal $\mathbf{r}(k)$. The selection of appropriate parameters has to lead to a stabilizing and effective control law. To facilitate the discussion we assume that we are dealing with a SISO system (so $n_y = n_u = 1$). Furthermore, we will assume irreducibility of the system[4]. In many applications, e.g., in manufacturing systems, this assumption is not restrictive [14].

The parameters N_p, N_c, and β are the three basic tuning parameters of the MPC algorithm. However, as we have already pointed out in the previous section, a closer look at the reference signal is necessary for stability reasons. As will be become clear in this section, the conventional MPC rules of thumb for tuning [53] of N_p, N_c and β can be applied to MPC for MPL systems as well, with minor changes only. Before we discuss the MPL-MPC tuning rules, we first need to consider some properties of the impulse response of an MPL system. The sequence $\{\mathbf{e}(k)\}_{k=0}^{\infty}$ with $\mathbf{e}(0) = \mathbf{0}$ and $\mathbf{e}(k) = \boldsymbol{\varepsilon}$ for $k \neq 0$ is the max-plus-algebraic unit impulse. The output sequence that results from applying a max-plus-algebraic unit impulse to an MPL system is called the impulse response of the system[5]. It is easy to verify that the impulse response of an MPL system with system matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ is given by $\{\mathbf{G}(k)\}_{k=0}^{\infty}$ with $\mathbf{G}(k) = \mathbf{C} \otimes \mathbf{A}^{\otimes^k} \otimes \mathbf{B}$.

Proposition 14 ([2, 14]) *Let $\{\mathbf{G}(k)\}_{k=0}^{\infty}$ be the impulse response of a SISO MPL system with an irreducible system matrix $\mathbf{A}$. Then there exist constants $c, k_0 \in \mathbb{N}\backslash\{0\}$, and $\rho_0 \in \mathbb{R}$ such that*

$$\mathbf{G}(k) = c\,\rho_0 + \mathbf{G}(k-c) \qquad \textit{for all } k \geqslant k_0. \tag{80}$$

$\diamond$

An impulse response that exhibits the behavior (80) is called *ultimately periodic* with *cycle period* c. The variable ρ_0 gives the average duration of a cycle and is equal to the max-plus-algebraic eigenvalue of system matrix $\mathbf{A}$. The length of the impulse response is defined as the minimal value k_0 for which (80) holds.

The state space representation of the input-output behavior of a given MPL system by a triple of system matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ is not unique. Just as in conventional system theory we define the minimal system order of an MPL system as the minimal dimension of the system matrix $\mathbf{A}$ over all possible state space realizations of the given system. In conventional system theory for linear discrete-time systems the minimal system order is

[4] An MPL system with system matrix $\mathbf{A} \in \mathbb{R}_\varepsilon^{n\times n}$ is said to be irreducible if $(\mathbf{A} \oplus \mathbf{A}^{\otimes^2} \oplus \ldots \oplus \mathbf{A}^{\otimes^{n-1}})_{ij} \neq \varepsilon$ for all i, j with $i \neq j$.

[5] If we consider a production system then we can give the following physical interpretation to the impulse response. At event counter $k = 0$ all the internal buffers of the system are empty. Then we start feeding raw material to the input buffer and we keep on feeding raw material at such a rate that the input buffer never becomes empty. The time instants at which finished products leave the system correspond to the terms of the impulse response.

given by the rank of the semi-infinite Hankel matrix $\mathbf{H}_{\infty,\infty}$ defined by

$$\mathbf{H}_{\infty,\infty} = \begin{bmatrix} \mathbf{G}(0) & \mathbf{G}(1) & \mathbf{G}(2) & \dots \\ \mathbf{G}(1) & \mathbf{G}(2) & \mathbf{G}(3) & \dots \\ \mathbf{G}(2) & \mathbf{G}(3) & \mathbf{G}(4) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} .$$

with $\mathbf{G}(k)$ the kth Markov parameter. In contrast to linear algebra the different notions of rank (like column rank, row rank, minor rank, ...) are in general not equivalent in the max-plus algebra. Nevertheless, a characterization of the minimal system order of an MPL system can be found in [22]. Unfortunately, computing the minimal system order of an MPL system is not a trivial task and it is often computationally very intensive. However, upper and lower bounds for the minimal system order of an MPL system can be determined as follows. The so-called max-plus-algebraic minor rank and Schein rank of $\mathbf{H}$ provide lower bounds [22, 23]. At present, there are no efficient (i.e., polynomial time) algorithms to compute the max-plus-algebraic minor rank or the Schein rank of a matrix. The max-plus-algebraic weak column rank of $\mathbf{H}$ provides an upper bound [22, 23]. Efficient methods to compute this rank are described in [17, 22].

Selection of the reference signal $\mathbf{r}(k)$

Let $\lambda_{\max}$ be the largest eigenvalue of the matrix A. Then the maximum production rate of the system is given by $1/\lambda_{\max}$. The slope of reference signal must therefore be such that the average production rate is lower than $1/\lambda_{\max}$. For a stable solution we need a reference signal $\mathbf{r}(k)$ for which there exist a $\rho > \lambda_{\max}$ and an $\mathbf{r}_0 \in \mathbb{R}$, such that $\mathbf{r}(k) \geqslant \mathbf{r}_0 + k\,\rho$ for all k.

Tuning of the parameter N_p

From Theorem 13 we know that for stability in the unconstrained case and with a reference signal (39) we need to choose $N_\mathrm{p} \geqslant n$. However, in the case of constraints or in the case of a different reference signal (e.g., a reference signal with due dates that are gathered in batches [55]), it is usually desirable that the time event set $\{0, \dots, N_\mathrm{p} - 1\}$ contains the crucial dynamics of the process to guarantee feasibility or to improve the performance.

Tuning of the parameter β

Theorem 13 gives an upper and lower bound for the parameter β:

$$0 < \beta < 1/(N_\mathrm{p} n_u)$$

Tuning of the parameter N_c

The real power of the MPC approach lies in the assumption made about future control actions. Instead of allowing them to be "free", the increments of $\mathbf{u}(k)$ are assumed to be zero:

$$\Delta^2\mathbf{u}(k + j - 1) = 0 \qquad \text{for } j = N_\mathrm{c}, \dots N_\mathrm{p} - 1.$$

The parameter N_c, called control horizon, can be chosen between 1 and N_p. Choosing a large N_c could be interesting when the constraints are stringent. On the other hand, one may expect that a small N_c will lead to a more robust control law in the case of modeling error. The choice $N_\mathrm{c} = 1$ often leads to an unstable or a degraded closed-loop behavior, because of a lack on degrees of freedom. In many cases, the optimal input signal will be asymptotically equal to $\mathbf{u}(k) = \mathbf{u}_0 + k\,\Delta\mathbf{u}_0$, where $\mathbf{u}_0$ and $\Delta\mathbf{u}_0$ are appropriate constants.

So then we need at least two degrees of freedom to be able to reach this asymptotic behavior.

4 Robust MPC

In this section we extend the noise-free deterministic model (8)–(9) to include uncertainty due to modeling errors or disturbances. Results for handling uncertainty of some specific classes of discrete-event systems are given in [12, 34, 49, 63] and the references therein. The literature on robust control for max-plus-linear systems is relatively sparse. Some of the contributions include closed-loop control based on residuation theory [37, 33, 41]. In this section we will consider three robust control methods for uncertain MPL systems. Two methods deal with a bounded uncertainty description, related to the interval uncertainty given by [33, 31], the third method deals with stochastic uncertainty. Note that there are few results in the literature on noise and modeling errors in an MPL context. Usually fast changes in the system matrices will be considered as noise and disturbances, whereas slow changes or permanent errors are considered as model mismatch. In this section both features will be treated in one single framework.

4.1 Noise and uncertainty model

The uncertainty caused by disturbances and errors in the estimation of physical variables, is gathered in an uncertainty vector $\mathbf{e}(k)$. We assume that the uncertainty vector $\mathbf{e}(k)$ captures the complete time-varying aspect of the system. In this chapter we consider two cases, related to the characterization of the perturbation:

- **bounded perturbation:** In this case we assume that the uncertainty/perturbation is bounded.
- **stochastic perturbation:** In this case we assume that the uncertainty/perturbation is a stochastic variable.

The system matrices of an MPL model usually consist of sums or maximizations of internal process times, transportation times, etc. (see, e.g., [2] or Section 1.2). Since the entries of $\mathbf{e}(k)$ directly correspond to the uncertainties in these duration times, it follows from Proposition 2 that the entries of the uncertain system matrices belong to $\mathcal{S}_{\text{mpns}}$. We obtain the system

$$\mathbf{x}(k) = \mathbf{A}(\mathbf{e}(k)) \otimes \mathbf{x}(k-1) \oplus \mathbf{B}(\mathbf{e}(k)) \otimes \mathbf{u}(k) \tag{81}$$

$$\mathbf{y}(k) = \mathbf{C}(\mathbf{e}(k)) \otimes \mathbf{x}(k) \tag{82}$$

where

$$\mathbf{A}(\mathbf{e}(k)) \in \mathcal{S}_{\text{mpns}}^{n\times n}(\mathbf{e}(k)), \quad \mathbf{B}(\mathbf{e}(k)) \in \mathcal{S}_{\text{mpns}}^{n\times n_u}(\mathbf{e}(k)), \quad \mathbf{C}(\mathbf{e}(k)) \in \mathcal{S}_{\text{mpns}}^{n_y\times n}(\mathbf{e}(k)) \ . \tag{83}$$

Prediction model

We collect the uncertainty over the interval $[k, k+N_\text{p}-1]$ in one vector

$$\tilde{\mathbf{e}}(k) = \begin{bmatrix} \mathbf{e}(k) \\ \vdots \\ \mathbf{e}(k+N_\text{p}-1) \end{bmatrix} \ .$$

In the bounded perturbation case we assume that $\tilde{\mathbf{e}}(k)$ is in a bounded polyhedral set $\mathcal{E}$. In the stochastic case we assume $\tilde{\mathbf{e}}(k)$ to be a random variable with probability density function $p(\tilde{\mathbf{e}}(k))$.

Now it is easy to verify that the prediction model, i.e., the prediction of the future outputs for the system (81)–(82) is given by

$$\tilde{\mathbf{y}}(k) = \tilde{\mathbf{C}}(\tilde{\mathbf{e}}(k)) \otimes \mathbf{x}(k-1) \oplus \tilde{\mathbf{D}}(\tilde{\mathbf{e}}(k)) \otimes \tilde{\mathbf{u}}(k) \ , \tag{84}$$

in which $\tilde{\mathbf{C}}(\tilde{\mathbf{e}}(k))$ and $\tilde{\mathbf{D}}(\tilde{\mathbf{e}}(k))$ are given by

$$\tilde{\mathbf{C}}(\tilde{\mathbf{e}}(k)) = \begin{bmatrix} \tilde{\mathbf{C}}_1(\tilde{\mathbf{e}}(k)) \\ \vdots \\ \tilde{\mathbf{C}}_{N_\mathrm{p}}(\tilde{\mathbf{e}}(k)) \end{bmatrix}, \quad \tilde{\mathbf{D}}(\tilde{\mathbf{e}}(k)) = \begin{bmatrix} \tilde{\mathbf{D}}_{11}(\tilde{\mathbf{e}}(k)) & \cdots & \tilde{\mathbf{D}}_{1N_\mathrm{p}}(\tilde{\mathbf{e}}(k)) \\ \vdots & \ddots & \vdots \\ \tilde{\mathbf{D}}_{N_\mathrm{p}1}(\tilde{\mathbf{e}}(k)) & \cdots & \tilde{\mathbf{D}}_{N_\mathrm{p}N_\mathrm{p}}(\tilde{\mathbf{e}}(k)) \end{bmatrix}$$

where

$$\tilde{\mathbf{C}}_m(\tilde{\mathbf{e}}(k)) = \mathbf{C}(k+m-1) \otimes \mathbf{A}(k+m-1) \otimes \ldots \otimes A(k)$$

and

$$\tilde{\mathbf{D}}_{mn}(\tilde{\mathbf{e}}(k)) = \begin{cases} \mathbf{C}(k{+}m{-}1) \otimes \mathbf{A}(k{+}m{-}1) \otimes \ldots \otimes \mathbf{A}(k{+}n) \otimes \mathbf{B}(k{+}n{-}1) & \text{if } m > n \\ \mathbf{C}(k{+}m{-}1) \otimes \mathbf{B}(k{+}m{-}1) & \text{if } m = n \\ \boldsymbol{\varepsilon} & \text{if } m < n \,. \end{cases}$$

Lemma 15 *The entries of* $\tilde{\mathbf{C}}(\tilde{\mathbf{e}}(k))$ *and* $\tilde{\mathbf{D}}(\tilde{\mathbf{e}}(k))$ *belong to* $\mathcal{S}_\mathrm{mpns}(\tilde{\mathbf{e}}(k))$*. For a given* $x(k-1)$ *and* $\tilde{\mathbf{u}}(k)$ *the entries of* $\tilde{\mathbf{y}}(k)$ *belong to* $\mathcal{S}_\mathrm{mpns}(\tilde{\mathbf{e}}(k))$. ◇

Proof: This is a direct consequence of the definition of $\tilde{\mathbf{C}}(\tilde{\mathbf{e}}(k))$, $\tilde{\mathbf{D}}(\tilde{\mathbf{e}}(k))$ and (84) in combination with (83) and Proposition 2. ◇

4.2 The bounded perturbation case

In this section we consider MPL system (81)-(82) where we assume that $\tilde{\mathbf{e}}(k)$ is in a bounded polyhedral set $\mathcal{E}$. Recall that in MPL-MPC we want to minimize the criterion

$$J(k) = J(\tilde{\mathbf{y}}(k), \tilde{\mathbf{u}}(k)) = J_\mathrm{out}(\tilde{\mathbf{y}}(k)) + \beta J_\mathrm{in}(\tilde{\mathbf{u}}(k))$$

with

$$\begin{aligned} J_\mathrm{out}(\tilde{\mathbf{y}}(k)) &= \sum_{j=0}^{N_\mathrm{p}-1} \sum_{i=1}^{n_y} \max(\mathbf{y}_i(k+j|k) - \mathbf{r}_i(k+j), 0) \quad , \\ J_\mathrm{in}(\tilde{\mathbf{u}}(k)) &= -\sum_{j=0}^{N_\mathrm{p}-1} \sum_{i=1}^{n_u} \mathbf{u}_i(k+j) \ . \end{aligned} \tag{85}$$

where J_out represents the tracking error and J_in is related to the input dates. We aim to find the optimal $(\tilde{\mathbf{u}}(k), \tilde{\mathbf{y}}(k))$ that minimizes $J(\tilde{\mathbf{y}}(k), \tilde{\mathbf{u}}(k))$, where $\tilde{\mathbf{y}}(k)$ and $\tilde{\mathbf{u}}(k)$ are related by (84). Note that, in contrast to the noise-free case, the relation between $\tilde{\mathbf{u}}(k)$ and $\tilde{\mathbf{y}}(k)$ is not unique anymore in the perturbed case because of the (bounded) perturbation $\tilde{\mathbf{e}}(k)$. Instead of considering general linear constraints (110) on the inputs and outputs as was done in [19], we will only consider linear constraints $\mathbf{E}(k)\tilde{\mathbf{u}}(k) \leq \mathbf{h}(k)$ on the input for the perturbed case. A typical example of such a constraint is an upper and lower bound for the input rate:

$$\mathbf{d}_\mathrm{min}(k+j) \leq \Delta\mathbf{u}(k+j) \leq \mathbf{d}_\mathrm{max}(k+j) \ .$$

The *worst-case MPC problem* at event step k is now defined as follows:

$$\min_{\tilde{\mathbf{u}}(k),\tilde{\mathbf{y}}(k)} \max_{\tilde{\mathbf{e}}(k)\in\mathcal{E}} J(\tilde{\mathbf{y}}(k),\tilde{\mathbf{u}}(k)) \tag{86}$$

$$\text{subject to } \tilde{\mathbf{y}}(k) = \tilde{\mathbf{C}}(\tilde{\mathbf{e}}(k)) \otimes x(k-1) \oplus \tilde{\mathbf{D}}(\tilde{\mathbf{e}}(k)) \otimes \tilde{\mathbf{u}}(k) \tag{87}$$

$$\Delta\mathbf{u}(k+j) \geqslant 0 \qquad \text{for } j = 0,\ldots,N_\mathrm{c}-1 \tag{88}$$

$$\Delta^2\mathbf{u}(k+j) = 0 \qquad \text{for } j = N_\mathrm{c},\ldots,N_\mathrm{p}-1 \tag{89}$$

$$\mathbf{E}(k)\tilde{\mathbf{u}}(k) \leq \mathbf{h}(k) \ . \tag{90}$$

We now will eliminate (87) by substituting it in the cost criterion and by maximizing the result over all possible $\tilde{\mathbf{e}}(k)$. For a fixed $\tilde{\mathbf{u}}(k)$ the worst-case $\tilde{\mathbf{e}}(k)$ will be denoted by $\tilde{\mathbf{e}}^{\#}(\tilde{\mathbf{u}}(k))$, or by $\tilde{\mathbf{e}}^{\#}(k)$ or $\tilde{\mathbf{e}}^{\#}$ for short if no confusion is possible. So for any $\tilde{\mathbf{u}}(k)$, we let[6]

$$\tilde{\mathbf{e}}^{\#}(k) = \arg\max_{\tilde{\mathbf{e}}(k)\in\mathcal{E}} J_\mathrm{out}(\tilde{\mathbf{y}}(\tilde{\mathbf{e}}(k),\tilde{\mathbf{u}}(k)))$$

$$J^{\#}_\mathrm{out}(\tilde{\mathbf{u}}(k)) = J_\mathrm{out}(\tilde{\mathbf{y}}(\tilde{\mathbf{e}}^{\#}(k),\tilde{\mathbf{u}}(k))) \ .$$

The *outer* worst-case MPC problem is now defined as follows:

$$\min_{\tilde{\mathbf{u}}(k)} J^{\#}_\mathrm{out}(\tilde{\mathbf{u}}(k)) + \beta J_\mathrm{in}(\tilde{\mathbf{u}}(k))$$

$$\text{subject to } \Delta\mathbf{u}(k+j) \geqslant 0 \qquad \text{for } j = 0,\ldots,N_\mathrm{c}-1 \tag{91}$$

$$\Delta^2\mathbf{u}(k+j) = 0 \qquad \text{for } j = N_\mathrm{c},\ldots,N_\mathrm{p}-1 \tag{92}$$

$$\mathbf{E}(k)\tilde{\mathbf{u}}(k) \leq \mathbf{h}(k) \ . \tag{93}$$

Now we make the following assumptions:

Assumption A1: J_out is a nondecreasing[7], convex function of $\tilde{\mathbf{y}}$.

Assumption A2: J_in is convex in $\tilde{\mathbf{u}}$.

These assumptions hold for several objective functions that are frequently encountered in a discrete-event systems context. As a consequence, they are not overly restrictive. Clearly, $J_{\mathrm{out},1}$ and $J_{\mathrm{in},1}$ satisfy Assumption A1 and Assumption A2.

Proposition 16 *[54]*
If Assumptions A1 and A2 hold, then the outer worst-case MPC problem is convex in $\tilde{\mathbf{u}}$.$\diamond$

So the outer worst-case MPC problem is a convex problem, which can be solved very efficiently using, e.g., an interior-point algorithm [48, 62].

Let us now consider the *inner* worst-case MPC problem:

$$\max_{\tilde{\mathbf{e}}(k)\in\mathcal{E}} J_\mathrm{out}(\tilde{\mathbf{y}}(\tilde{\mathbf{e}},\tilde{\mathbf{u}})) \tag{94}$$

$$\text{subject to } \tilde{\mathbf{y}}(\tilde{\mathbf{e}},\tilde{\mathbf{u}}) = \tilde{\mathbf{C}}(\tilde{\mathbf{e}}) \otimes \mathbf{x}(k-1) \oplus \tilde{\mathbf{D}}(\tilde{\mathbf{e}}) \otimes \tilde{\mathbf{u}} \ . \tag{95}$$

We will show how this problem can be solved efficiently. Recall that $\mathcal{E}$ is a bounded polyhedral set. The vertices of $\mathcal{E}$ form a lattice w.r.t. the partial order relation $\leq$. Let $\mathcal{E}^\mathrm{v}_{\tilde{e},\max}$ be the top points of this lattice, i.e., $\mathcal{E}^\mathrm{v}_{\tilde{e},\max}$ is the set of the vertex points $\tilde{\mathbf{e}}^\mathrm{v}_\max$ of $\mathcal{E}$ for which we have

$$\nexists \tilde{\mathbf{e}} \in \mathcal{E} \text{ with } \tilde{\mathbf{e}} \neq \tilde{\mathbf{e}}^\mathrm{v}_\max \text{ and } \tilde{\mathbf{e}}^\mathrm{v}_\max \leq \tilde{\mathbf{e}} \ .$$

Now consider the *reduced inner* worst-case MPC problem:

$$\max_{\tilde{\mathbf{e}}(k)\in\mathcal{E}^\mathrm{v}_{\tilde{e},\max}} J_\mathrm{out}\big(\tilde{\mathbf{C}}(\tilde{\mathbf{e}}) \otimes \mathbf{x}(k-1) \oplus \tilde{\mathbf{D}}(\tilde{\mathbf{e}}) \otimes \tilde{\mathbf{u}}\big) \ . \tag{96}$$

[6] Note that $J_\mathrm{in}(k)$ does not depend on $\tilde{\mathbf{e}}(k)$.

[7] The function $f : \mathbb{R}^n \to \mathbb{R}$ is nondecreasing if for any $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ with $\mathbf{x}_i \leq \mathbf{y}_i$ for $i = 1,\ldots,n$, we have $f(\mathbf{x}) \leq f(\mathbf{y})$.

Lemma 17 *[54]*
If Assumption A1 holds, then for a given $\mathbf{x}(k-1)$ *and* $\tilde{\mathbf{u}}(k)$ *the function* J_{out} *is a convex function of* $\tilde{\mathbf{e}}(k)$. $\diamond$

Proof: If the function h is defined by $h(\mathbf{x}) = f(g(\mathbf{x}))$ and if g is convex and f is convex and nondecreasing, then h is convex [51, Theorem 5.1]. Functions that belong to $\mathcal{S}_{\text{mpns}}$ are convex. Since for a given $\tilde{\mathbf{u}}$ we have $\tilde{\mathbf{y}}(\tilde{\mathbf{e}}, \tilde{\mathbf{u}}) \in \mathcal{S}_{\text{mpns}}$ by Lemma 15, $\tilde{\mathbf{y}}$ is convex as a function of $\tilde{\mathbf{e}}$. Furthermore, J_{out} is convex and nondecreasing as a function of $\tilde{\mathbf{y}}$ by Assumption A1. Hence, J_{out} is convex in $\tilde{\mathbf{e}}$. $\diamond$

Proposition 18 *[54]*
If Assumption A1 holds, then an optimal solution of the reduced inner worst-case MPC problem (96) is also an optimal solution of the (full) inner worst-case MPC problem (94)–(95). $\diamond$

The set $\mathcal{E}^{\text{v}}_{\tilde{e},\max}$ is independent of $\tilde{\mathbf{u}}$ and can thus be pre-computed off-line. Methods to compute all vertex points of a polyhedral set can be found in [38, 43][8]. Note that the computation can be made more efficient by already discarding the vertex points that cannot result in vertex points that will belong to $\mathcal{E}^{\text{v}}_{\tilde{e},\max}$ during the computation (cf. [18]). In combination with Proposition 18 this allows for an efficient solution of the inner worst-case MPC problem. Since the outer worst-case MPC problem is convex by Proposition 16, this implies that the overall worst-case MPC problem can be solved efficiently.

Robust closed-loop MPC

So far we looked at MPC methods in which the prediction was done using an open-loop method. (Note that MPC is a closed-loop method, because we use measurements of the output or state to update the state of the controller.) In this paragraph we consider the MPL version of the finite-horizon robust optimal control problem [6] for uncertain dynamic systems using the min-max paradigm. This means that we will optimize over feedback policies, rather than open-loop input sequences, and the incorporation of state and input constraints directly into the problem formulation [46]. In general, this results in increased feasibility and a better performance. We use a dynamic programming approach similar to the one used in [3, 20] for finite-horizon min-max control of uncertain linear systems with constraints. The main drawback will be a higher computational complexity.

Consider system (81)-(82) and introduce the short-hand notation

$$\mathbf{f}(\mathbf{x}(k-1), \mathbf{u}(k), \mathbf{e}(k)) = \mathbf{A}(\mathbf{e}(k)) \otimes \mathbf{x}(k-1) \oplus \mathbf{B}(\mathbf{e}(k)) \otimes \mathbf{u}(k)$$

It is easy to verify that $\mathbf{f}(\cdot) \in \mathcal{S}^n_{\text{mps}}$ and $\mathbf{f}(\cdot,\cdot,\mathbf{e}) \in \mathcal{S}^n_{\text{mpns}}$ for any fixed $\mathbf{e}$. Since $\mathbf{f}(\cdot,\mathbf{e})$ is a max expression of affine terms in $(\mathbf{x},\mathbf{u})$, each component of $\mathbf{f}(\cdot,\mathbf{e})$ is convex [52].

Consider performance index $J(k) = J_{\text{out}}(k) + \beta J_{\text{in}}(k)$, where J_{out} and J_{in} are defined in (85) and with stage cost

$$\ell_j(\mathbf{x},\mathbf{u},\mathbf{r},\mathbf{e},k) = \sum_{i=1}^{n_y} \max\{\mathbf{C}(\mathbf{e}(k+j)) \otimes \mathbf{x}_i(k+j|k) - \mathbf{r}_i(k+j), 0\} - \beta \sum_{i=1}^{n_u} \mathbf{u}_i(k+j), \quad (97)$$

[8] The paper [38] also provides more information on the complexity of computing the set of vertices of $\mathcal{E}$ and a (crude) upper bound for the number of elements of $\mathcal{E}$ (and thus also of $\mathcal{E}^{\text{v}}_{\tilde{e},\max}$).

where $0 \leq \beta$. We assume that the system is subject to input and state constraints over a finite horizon of length N_p:

$$\mathbf{E}(j)\mathbf{u}(k+j)+\mathbf{F}(j)\mathbf{x}(k+j)+\mathbf{G}(j)\mathbf{r}(k+j)+\mathbf{H}(j)\mathbf{e}(k+j) \leq \mathbf{h}(k+j), \quad j=0,2,\ldots,N_\mathrm{p} \tag{98}$$

where $\mathbf{E}(j) \in \mathbb{R}^{l\times n_u}, \mathbf{F}(j) \in \mathbb{R}^{l\times n}, \mathbf{G}(j) \in \mathbb{R}^{l\times n_y}, \mathbf{H}(j) \in \mathbb{R}^{l\times n_e}$ and $\mathbf{h}(j) \in \mathbb{R}^{l}$. The following assumption will be used:

A1: The matrices $\mathbf{F}(j)$ in (98) are non-negative for all $j=0,\ldots,N_\mathrm{p}$.

Instead of the control variable $\tilde{\mathbf{u}}(k)$ we will now introduce the decision variable as a control policy $\pi := (\boldsymbol{\mu}_1(\cdot), \boldsymbol{\mu}_2(\cdot), \ldots, \boldsymbol{\mu}_{N_\mathrm{p}}(\cdot))$, where each $\boldsymbol{\mu}_j : \mathbb{R}^n \times \mathbb{R}^{pN_\mathrm{p}} \to \mathbb{R}^m$ is a state feedback control law. Let $\boldsymbol{\phi}(j;\mathbf{x}(k-1),\pi(k),\tilde{\mathbf{e}}(k))$ denote the state solution of (81)-(82) at step j when the initial state is $\mathbf{x}(k-1)$, the control is determined by the policy π, i.e., $\mathbf{u}(k+j)=\boldsymbol{\mu}_{k+j}(\boldsymbol{\phi}(j;\mathbf{x}(k-1),\pi(k),\tilde{\mathbf{e}}(k)),\tilde{\mathbf{r}}(k))$. By definition, $\boldsymbol{\phi}(-1;\mathbf{x}(k-1),\pi(k),\tilde{\mathbf{e}}(k)) := \mathbf{x}(k-1)$. The cost is defined as:

$$V_{N_\mathrm{p}}(\mathbf{x}(k-1),\pi(k),\tilde{\mathbf{r}}(k),\tilde{\mathbf{e}}(k)) := \sum_{j=1}^{N_\mathrm{p}} \ell_j(\mathbf{x}(k+j),\mathbf{u}(k+j),\mathbf{r}(k+j),\mathbf{e}(k+j)), \tag{99}$$

where $\mathbf{x}(k+j)=\boldsymbol{\phi}(j;\mathbf{x}(k-1),\pi(k),\tilde{\mathbf{e}}(k))$, $\mathbf{u}(k+j) := \boldsymbol{\mu}_j(\mathbf{x}(k+j-1),\tilde{\mathbf{r}}(k))$, and ℓ_j is the stage cost.

For each initial condition $\mathbf{x}(k-1)$ and reference signal $\tilde{\mathbf{r}}(k)$ we define the set of feasible policies π:

$$\begin{aligned}\Pi_{N_\mathrm{p}}(\mathbf{x}(k-1),\tilde{\mathbf{r}}(k)) := \{\pi : \; & \mathbf{E}(j)\boldsymbol{\mu}_j(\boldsymbol{\phi}(j;\mathbf{x}(k-1),\pi(k),\tilde{\mathbf{e}}(k)),\tilde{\mathbf{r}}(k)) \\ &+\mathbf{F}(j)\boldsymbol{\phi}(j;\mathbf{x}(k-1),\pi(k),\tilde{\mathbf{e}}(k))+\mathbf{G}(j)\mathbf{r}(k+j)+\mathbf{H}(j)\mathbf{e}(k+j)\leq\mathbf{h}(j), \\ &\forall\tilde{\mathbf{e}}\in\mathcal{E}, j=0,\ldots,N_\mathrm{p}-1\},\end{aligned} \tag{100}$$

Also, let $X_{N_\mathrm{p}}(k)$ denote the set of initial states *and* reference signals for which a feasible policy exists, i.e., $X_{N_\mathrm{p}}(k) := \{(\mathbf{x}(k-1),\tilde{\mathbf{r}}(k)) : \Pi_{N_\mathrm{p}}(\mathbf{x}(k-1),\tilde{\mathbf{r}}(k)) \neq \emptyset\}$. The following min-max problem will be referred to as the *finite-horizon robust optimal control* problem:

$$V^0_{N_\mathrm{p}}(\mathbf{x}(k-1),\tilde{\mathbf{r}}(k)) := \inf_{\pi(k)\in\Pi_{N_\mathrm{p}}(\mathbf{x}(k-1),\tilde{\mathbf{r}}(k))} \max_{\tilde{\mathbf{e}}(k)\in\mathcal{E}} V_{N_\mathrm{p}}(\mathbf{x}(k-1),\pi(k),\tilde{\mathbf{r}}(k),\tilde{\mathbf{e}}(k)). \tag{101}$$

Let $\pi^0_{N_\mathrm{p}}(\mathbf{x}(k-1),\tilde{\mathbf{r}}(k)) =: (\boldsymbol{\mu}^0_1(x(k-1),\tilde{\mathbf{r}}(k)),\boldsymbol{\mu}^0_2(\cdot),\ldots,\boldsymbol{\mu}^0_{N_\mathrm{p}}(\cdot))$ denote a minimizer of the worst-case problem whenever the infimum is attained, i.e.,

$$\pi^0_{N_\mathrm{p}}(\mathbf{x}(k-1),\tilde{\mathbf{r}}(k)) \in \arg \min_{\pi(k)\in\Pi_{N_\mathrm{p}}(\mathbf{x}(k-1),\tilde{\mathbf{r}}(k))} \max_{\tilde{\mathbf{e}}(k)\in\mathcal{E}} V_{N_\mathrm{p}}(\mathbf{x},\pi,\tilde{\mathbf{r}},\tilde{\mathbf{e}}) .$$

In [46] it has been shown that the *finite-horizon robust optimal control* can be solved using dynamic programming [6, 5, 39]. The sequences $\{V^0_s(\cdot),\boldsymbol{\kappa}_s(\cdot),X_s\}_{s=1}^{N_\mathrm{p}}$ and $\{\boldsymbol{\mu}^0_s(\cdot)\}_{s=1}^{N_\mathrm{p}}$ can be computed iteratively, without gridding, performing the following steps:

- Given X_{s-1}, first compute Z_s

$$\begin{aligned}Z_s : = \{&(\mathbf{x}(k-1),\tilde{\mathbf{r}}(k),\mathbf{u}(k)) : \mathbf{E}(N_\mathrm{p}-s+1)\mathbf{u}(k) \\ &+\mathbf{F}(N_\mathrm{p}-s+1)\mathbf{f}(\mathbf{x}(k-1),\mathbf{u}(k),\mathbf{e}(k)) \\ &+\mathbf{G}(N_\mathrm{p}-s+1)\mathbf{r}(N_\mathrm{p}-s+1)(k)+\mathbf{H}(N_\mathrm{p}-s+1)\mathbf{e}(k) \leq \mathbf{h}(N_\mathrm{p}-s+1), \\ &(\mathbf{f}(\mathbf{x}(k-1),\mathbf{u}(k),\mathbf{e}(k)),\tilde{\mathbf{r}}(k)) \in X_{s-1}(k), \forall \mathbf{e}\in\mathcal{E}\},\end{aligned} \tag{102}$$

  followed by a projection operation

$$X_s(k) = \mathrm{Proj}_{n+n_yN_\mathrm{p}} Z_s(k).$$

- Given $V^0_{s-1}(\cdot)$, the function $J_s(\cdot) \in \mathcal{S}_{\mathrm{mps}}$ is computed:

$$\begin{aligned} &J_s(\mathbf{x}(k{-}1), \tilde{\mathbf{r}}(k), \mathbf{u}(k)) := \\ &\qquad \max_{\mathbf{e}\in\mathcal{E}}\{\ell_{N_\mathrm{p}-s+1}(\mathbf{f}(\mathbf{x}(k{-}1), \mathbf{u}(k), \mathbf{e}(k)), \mathbf{u}(k), \mathbf{r}_{N_\mathrm{p}-s+1}(k), \mathbf{e}(k)) \\ &\qquad + V^0_{s-1}(\mathbf{f}(\mathbf{x}(k{-}1), \mathbf{u}(k), \mathbf{e}(k)), \tilde{\mathbf{r}}(k))\}, \\ &\qquad\qquad \forall (\mathbf{x}(k{-}1), \tilde{\mathbf{r}}(k), \mathbf{u}(k)) \in Z_s(k), \end{aligned} \tag{103}$$

- Given $J_s(\cdot)$ and Z_s, compute $V^0_s(\cdot)$ and $\boldsymbol{\kappa}_s(\cdot)$

$$V^0_s(\mathbf{x}(k{-}1), \tilde{\mathbf{r}}(k)) = \min_{u(k)}\{J_s(\mathbf{x}(k{-}1), \tilde{\mathbf{r}}(k), \mathbf{u}(k)) : (\mathbf{x}(k{-}1), \tilde{\mathbf{r}}(k), \mathbf{u}(k)) \in Z_s(k)\},$$
$$\forall (\mathbf{x}(k{-}1), \tilde{\mathbf{r}}(k)) \in X_s(k), \tag{104}$$
$$\boldsymbol{\kappa}_s(\mathbf{x}, \tilde{\mathbf{r}}) = \arg\min_{\mathbf{u}}\{J_s(\mathbf{x}, \tilde{\mathbf{r}}, \mathbf{u}) : (\mathbf{x}, \tilde{\mathbf{r}}, \mathbf{u}) \in Z_s\}\ \forall (\mathbf{x}, \tilde{\mathbf{r}}) \in X_s, \tag{105}$$

  The (set-valued) control law $\boldsymbol{\kappa}(\mathbf{x}, \tilde{\mathbf{r}})$ is a polyhedron for a given $(\mathbf{x}, \tilde{\mathbf{r}}) \in X$. Moreover, it is always possible to select a continuous and PWA control law $\boldsymbol{\mu}^0_{N_\mathrm{p}-s+1}(\cdot)$ such that $\boldsymbol{\mu}^0_{N_\mathrm{p}-s+1}(\mathbf{x}, \tilde{\mathbf{r}}) \in \boldsymbol{\kappa}(\mathbf{x}, \tilde{\mathbf{r}})$ for all $(\mathbf{x}, \tilde{\mathbf{r}}) \in X$.

We have shown that we can compute an optimal control policy over a prediction horizon of N_p steps by solving N_p parametric LP problems. The key assumptions that allow us to guarantee convexity of the partial return functions and their domains at each dynamic programming iteration, were that the stage cost be a max-plus-non-negative-scaling expression in the state *and* that the matrices associated with the state constraints all have non-negative entries.

4.3 The stochastic perturbation case

In this section we consider MPL system (81)-(82) where we assume that the uncertainty has stochastic properties. Hence, $\mathbf{e}(k)$ is a stochastic variable.

The *stochastic MPL-MPC problem* for event step k can be defined as:

$$\min_{\tilde{\mathbf{u}}(k)} J_\mathrm{out}(k) + \beta J_\mathrm{in}(k)$$

subject to

$$\mathbf{x}(k{+}j) = \mathbf{A}\otimes\mathbf{x}(k{+}j{-}1)\oplus\mathbf{B}\otimes\mathbf{u}(k{+}j) \qquad \text{for } j = 0, \ldots, N_\mathrm{p}-1 \tag{106}$$
$$\mathbf{y}(k+j) = \mathbf{C}\otimes\mathbf{x}(k+j) \qquad \text{for } j = 0, \ldots, N_\mathrm{p}-1 \tag{107}$$
$$\Delta\mathbf{u}(k+j) \geqslant 0 \qquad \text{for } j = 0, \ldots, N_\mathrm{p}-1 \tag{108}$$
$$\Delta^2\mathbf{u}(k+\ell) = 0 \qquad \text{for } j = 0, \ldots, N_\mathrm{p}-1 \tag{109}$$
$$\mathbf{E}(k)\tilde{\mathbf{u}}(k) + \mathbf{F}(k)\mathbb{E}[\tilde{\mathbf{y}}(k)] + \mathbf{G}\tilde{\mathbf{r}}(k) \leq \mathbf{h}(k) \tag{110}$$

where (110) represents the linear constraints on the inputs and the outputs and $\mathbb{E}[\cdot]$ denotes the expectation. In this section J_out and J_in are chosen as follows:

$$J_\mathrm{out}(k) = \sum_i \mathbb{E}[\tilde{\boldsymbol{\eta}}_i(k)] \tag{111}$$

$$J_\mathrm{in}(k) = -\sum_j \tilde{\mathbf{u}}_j(k) \tag{112}$$

where the "tardiness" error is given by

$$\tilde{\boldsymbol{\eta}}(k) = ((\tilde{\mathbf{y}}(k) - \tilde{\mathbf{r}}(k)) \oplus \mathbf{0}),$$

or

$$\tilde{\boldsymbol{\eta}}_i(k) = \max(\ \tilde{\mathbf{y}}_i(k) - \tilde{\mathbf{r}}_i(k)\ ,\ 0\ )\ . \tag{113}$$

With (84) we obtain

$$J_{\text{out}}(k) \quad = \quad \sum_i \mathbb{E}\left[\max\left(\ \left\{[\tilde{\mathbf{C}}(k)]_i \otimes \mathbf{x}(k) \oplus [\tilde{\mathbf{D}}(k)]_i \otimes \tilde{\mathbf{u}}(k)\right\} - \tilde{\mathbf{r}}_i(k)\ ,\ 0\ \right)\right] \tag{114}$$

where $[\tilde{\mathbf{C}}(k)]_i$ and $[\tilde{\mathbf{D}}(k)]_i$ denote the ith row of $\tilde{\mathbf{C}}(k)$ and $\tilde{\mathbf{D}}(k)$, respectively.

Convexity of stochastic MPL-MPC

In order to compute the optimal MPC input signal, we need the expectation of the signals $\tilde{\boldsymbol{\eta}}(k)$ and $\tilde{\mathbf{y}}(k)$. In this section we present a method to compute $\mathbb{E}[\tilde{\boldsymbol{\eta}}_i(k)]$ and $\mathbb{E}[\tilde{\mathbf{y}}(k)]$ and we show that these expectations are convex. As a consequence, the MPL-MPC problem is convex.

Lemma 19 *[56]*
Define the vector $\mathbf{z}(k)$ *as*

$$\mathbf{z}(k) = \begin{bmatrix} -\tilde{\mathbf{r}}(k) \\ \mathbf{x}(k-1) \\ \tilde{\mathbf{u}}(k) \\ \tilde{\mathbf{e}}(k) \end{bmatrix}$$

Then, the future tardiness error $\tilde{\boldsymbol{\eta}}(k)$ *and the future output signal* $\tilde{\mathbf{y}}(k)$ *belong to* $\mathcal{S}_{\text{mpns}}(\mathbf{z}(k))$*.*
$\diamond$

Proof: Lemma 15 shows that the entries of $\tilde{\mathbf{C}}(\tilde{\mathbf{e}}(k))$ and $\tilde{\mathbf{D}}(\tilde{\mathbf{e}}(k))$ belong to $\mathcal{S}_{\text{mpns}}(\tilde{\mathbf{e}}(k))$. Then, using (113), (84) and again Proposition 2 we find that both $\tilde{\boldsymbol{\eta}}(k)$ and $\tilde{\mathbf{y}}(k)$ belong to $\mathcal{S}_{\text{mpns}}(\mathbf{z}(k))$. $\diamond$

Let $\mathbf{v}(k) \in \mathcal{S}_{\text{mpns}}(\mathbf{z}(k))$, where $\mathbf{z}(k)$ is as defined in Lemma 19. In the sequel of this section we will derive how to compute the expectation $\mathbb{E}[\mathbf{v}(k)]$, and show that $\mathbb{E}[\mathbf{v}(k)]$ has some nice convexity properties. Define $\mathbf{w}(k) = \begin{bmatrix} -\tilde{\mathbf{r}}^T(k) & \mathbf{x}^T(k-1) & \tilde{\mathbf{u}}^T(k) \end{bmatrix}^T$ to be the non-stochastic part of $\mathbf{z}(k)$. Then, because of Lemma 19 and the definition of max-plus-non-negative-scaling functions, there exist scalars α_j and non-negative vectors $\boldsymbol{\beta}_j$ and $\boldsymbol{\gamma}_j$, such that

$$\mathbf{v}(k) = \max_{j=1,\ldots,n_v}\left(\alpha_j + \boldsymbol{\beta}_j^T\mathbf{w}(k) + \boldsymbol{\gamma}_j^T\tilde{\mathbf{e}}(k)\right)$$

Define the sets $\Phi_j(\mathbf{w}(k))$, $j = 1,\ldots,n_v$ such that for all $\tilde{\mathbf{e}}(k) \in \Phi_j(\mathbf{w}(k))$ there holds:

$$\mathbf{v}(k) = \alpha_j + \boldsymbol{\beta}_j^T\mathbf{w}(k) + \boldsymbol{\gamma}_j^T\tilde{\mathbf{e}}(k)$$

and

$$\bigcup_{j=1}^{n_v}\Phi_j(\mathbf{w}(k)) = \mathbb{R}^{n_{\tilde{e}}}$$

Denote the probability density function of $\tilde{\mathbf{e}}$ by p. Then

$$\begin{aligned}
\hat{\mathbf{v}}(k) &= \mathbb{E}[\mathbf{v}(k)] && (115)\\
&= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathbf{v}(k)\,p(\tilde{\mathbf{e}})\,d\tilde{\mathbf{e}} && (116)\\
&= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\max_{j=1,\ldots,n_v}\left(\alpha_j + \boldsymbol{\beta}_j^T\mathbf{w}(k) + \boldsymbol{\gamma}_j^T\tilde{\mathbf{e}}\right)p(\tilde{\mathbf{e}})\,d\tilde{\mathbf{e}} && (117)\\
&= \sum_{j=1}^{n_v}\int_{\tilde{\mathbf{e}}\in\Phi_j(w(k))}\!\!\!\cdots\int\left(\alpha_j + \boldsymbol{\beta}_j^T\mathbf{w}(k) + \boldsymbol{\gamma}_j^T\tilde{\mathbf{e}}\right)p(\tilde{\mathbf{e}})\,d\tilde{\mathbf{e}} && (118)
\end{aligned}$$

where $d\tilde{\mathbf{e}} = d\tilde{\mathbf{e}}_1\, d\tilde{\mathbf{e}}_2 \ldots d\tilde{\mathbf{e}}_{n_{\tilde{e}}}$.

The following proposition shows that $\hat{\mathbf{v}}(k)$ is convex in the vector $\mathbf{w}(k)$.

Proposition 20 *[56]*
The function $\hat{\mathbf{v}}(k)$ as defined in (115) is convex in $\mathbf{w}(k)$ and a subgradient $\mathbf{g}_{\mathbf{v}}(k)$ is given by

$$\mathbf{g}_{\mathbf{v}}(k) = \sum_{\ell=1}^{n_v} \boldsymbol{\beta}_\ell^T \int_{\tilde{\mathbf{e}} \in \Phi_\ell(\mathbf{w}(k))} \cdots \int p(\tilde{\mathbf{e}})\, d\tilde{\mathbf{e}} \tag{119}$$

$\diamond$

Now consider the MPC problem (20) – (24). First note that because of Lemma 20, $\mathbb{E}[\tilde{\boldsymbol{\eta}}_i(k)]$ and $\mathbb{E}[\tilde{\mathbf{y}}(k)]$ are convex in $\mathbf{w}(k)$. This means that $J_{\text{out}}(k)$ and $J(k)$ are convex in $\tilde{\mathbf{u}}(k)$.

Property 21 *If the linear constraints are monotonically nondecreasing as a function of $\mathbb{E}[\tilde{\mathbf{y}}(k)]$ (in other words, if $[\mathbf{F}]_{ij} \geqslant 0$ for all i, j), the constraint (22) becomes convex in $\tilde{\mathbf{u}}(k)$.*

So, if the linear constraints are monotonically nondecreasing, the MPL-MPC problem turns out to be a convex problem in $\tilde{\mathbf{u}}(k)$, and both a subgradient of the constraints and a subgradient of the cost criterion can easily be derived using Lemma 20. Note that convex optimization problems can be solved using reliable and efficient optimization algorithms, based on interior point methods [48, 62].

Piecewise affine and piecewise polynomial probability density functions

So far, we did not make any assumption on the characterization of probability function $p(\tilde{\mathbf{e}})$. For the computation of the cost criterion and the constraints we need the values of $\mathbb{E}[\tilde{\mathbf{y}}(k)]$ and $\mathbb{E}[\tilde{\boldsymbol{\eta}}(k)]$. If we choose for example a Gaussian distribution, they can be calculated from (118) using numerical integration. Numerical integration is usually time-consuming and cumbersome, but can be avoided by choosing piecewise affine or piecewise polynomial probability density functions (possibly as an approximation of the real probability density function).

Let $p(\tilde{\mathbf{e}})$ be piecewise affine functions, so consider sets P_ℓ , $\ell = 1, \ldots, n_p$, such that for $\tilde{\mathbf{e}} \in P_\ell$ the probability density function is given by:

$$p(\tilde{\mathbf{e}}) = \mu_\ell + \zeta_\ell^T \tilde{\mathbf{e}}$$

Consider a signal $\mathbf{v}(k) \in \mathcal{S}_{\text{mpns}}(\mathbf{z}(k))$ and let $\mathbf{w}(k)$ be its non-stochastic part. Let $E_{j\ell}(\mathbf{w}(k)) = \Phi_j(\mathbf{w}(k)) \cap P_\ell$ for $j = 1, \ldots, n_v$, $\ell = 1, \ldots, n_p$, then $\hat{\mathbf{v}}(k)$ is given by

$$\hat{\mathbf{v}}(k) = \sum_{\ell=1}^{n_p} \sum_{j=1}^{n_v} \int_{\tilde{\mathbf{e}} \in E_{j\ell}(\mathbf{w}(k))} \cdots \int \left(\alpha_j + \boldsymbol{\beta}_j^T \mathbf{w}(k) + \boldsymbol{\gamma}_j^T \tilde{\mathbf{e}} \right) (\mu_\ell + \zeta_\ell^T \tilde{\mathbf{e}})\, d\tilde{\mathbf{e}}$$

This is an integral of a quadratic function in $\tilde{\mathbf{e}}$ and can be computed analytically for all regions $E_{j\ell}$. In general, for piecewise polynomial probability density functions, the integral will be a polynomial function in $\tilde{\mathbf{e}}$, and can be computed analytically for all regions $E_{j\ell}$ [10, 30].

If piecewise affine or polynomial probability density functions are used as an approximation of "true" non-polynomial probability functions, the quality of the approximation can be improved by increasing the number of sets n_p.

Remark 22 Approximate methods Note that in the algorithm to solve the robust MPC problem with stochastic uncertainty, an optimization problem has to be solved at each time step, which is, in general, a highly complex and computationally hard problem. Two approaches have been developed to reduce the computational burden. The first approach [61] considers a method based on variability expansion. In particular, it is shown that the computational load is reduced if one decreases the level of 'randomness' in the system. Another method [21] uses an approximation approach that is based on the pth raw moments of a random variable. This method results in a much lower computational complexity and computation time while still guaranteeing a good performance. $\diamond$

5 Conclusions and future work

This chapter has focused on the application of the popular Model Predictive Control (MPC) framework to Max-Plus-Linear (MPL) systems. One of the main advantages of the MPL-MPC approach is that it allows to include general linear inequality constraints on the inputs, states, and outputs of the system. We have provided a review of the main principles underlying MPL-MPC, and outlined some of the theoretical, computational, and implementation aspects. We focused on the main ingredients of the MPC framework, i.e., prediction model, cost criterion, and constraints, and we formulated the standard MPC problem for MPL systems. We reviewed some algorithms to solve the MPL-MPC problem and discussed the key advantages and disadvantages. Besides some basic results for the stability of the closed-loop controlled system, an analytic expression of the controller has been given. We have also addressed the robust formulation of MPL-MPC, in which the uncertainty caused by disturbances and errors in the estimation of physical variables is gathered in an uncertainty variable that parameterizes the system matrices A, B, and C. We have distinguished between the case that the uncertainty is bounded and the case that the uncertainty has stochastic properties.

In future work we will further study the problem of closed-loop stability and focus on the relaxation of the conditions on the controller tuning parameters. Finally we will focus on extending the derived properties to switching max-plus-linear systems [57, 59], i.e., discrete-event systems that can switch between different modes of operation. In each mode the system is then described by an MPL state equation and an MPL output equation, with different system matrices for each mode.

Acknowledgments

Research partially funded by the Dutch Technology Foundation STW project "Model-predictive railway traffic management – A framework for closed-loop control of large-scale railway systems" and by the European Union Seventh Framework Programme [FP7/2007-2013] under grant agreement no. 257462 Network of Excellence "Highly-complex and networked control systems (HYCON2)".

References

[1] S. Amari, I. Demongodin, J.J. Loiseau, and C. Martinez. Max-plus control design for temporal constraint meeting in timed event graph. *IEEE Transactions on Automatic Control*, 57(2):462–467, 2012.

[2] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat. *Synchronization and Linearity.* John Wiley & Sons, New York, Text can be downloaded from *http://www-rocq.inria.fr/metalau/cohen/SED/book-online.html*, 1992.

[3] A. Bemporad, F. Borrelli, and M. Morari. Min-max control of constrained uncertain discrete-time linear systems. *IEEE Transactions on Automatic Control*, 48(9):1600–1606, September 2003.

[4] A. Bemporad, W.P.M.H. Heemels, and B. De Schutter. On hybrid systems and closed-loop MPC systems. In *Proceedings of the 40th IEEE Conference on Decision and Control (CDC01)*, pages 1645–1650, Orlando, Florida, December 2001.

[5] D. Bertsekas. *Dynamic Programming and Optimal Control — Volumes I & II*. Athena Scientific, Nashua, New Hampshire, 2001.

[6] D.P. Bertsekas and I.B. Rhodes. Sufficiently informative functions and the minimax feedback control of uncertain dynamic systems. *IEEE Transactions on Automatic Control*, 18(2):117–124, April 1971.

[7] T.S. Blyth. *Lattices and Ordered Algebraic Structures*. Springer-Verlag, Berlin, Germany, 2005.

[8] J.L. Boimond and J.L. Ferrier. Internal model control and max-algebra: Controller design. *IEEE Transactions on Automatic Control*, 41(3):457–461, March 1996.

[9] J.F. Bonnans, J.C. Gilbert, C. Lemaréchal, and C.A. Sagastizábal. *Numerical optimization: Theoretical and practical aspects*. Springer-Verlag, Berlin, Germany, 2006.

[10] B. Bueler, A. Enge, and K. Fukuda. Exact volume computation for polytopes: A practical study. In *G. Kalai and G. M. Ziegler, editors, Polytopes - Combinatorics and Computation, DMV-Seminar 29*, pages 131–154, Birkhäuser, March 2000.

[11] E.F. Camacho and C. Bordons. *Model Predictive Control in the Process Industry, Advances in Industrial Control*. Springer, London, UK, 1995.

[12] J. Cardoso, R. Valette, and D. Dubois. Possibilistic Petri nets. *IEEE Transactions on Systems, Man and Cybernetics, Part B: Cybernetics*, 29(5):573–582, 1999.

[13] D.D. Cofer and V.K. Garg. Supervisory control of real-time discrete-event systems using lattice theory. *IEEE Transactions on Automatic Control*, 41(2):199–209, February 1996.

[14] G. Cohen, D. Dubois, J.P. Quadrat, and M. Viot. A linear-system-theoretic view of discrete-event processes and its use for performance evaluation in manufacturing. *IEEE Transactions on Automatic Control*, 30(3):210–220, March 1985.

[15] G. Cohen, P. Moller, J.P. Quadrat, and M. Viot. Algebraic tools for the performance evaluation of discrete event systems. *IEEE Proceedings: Special issue on Discrete Event Systems*, 77(1):39–58, January 1989.

[16] B. Cottenceau, L. Hardouin, J. Boimond, and J. Ferrier. Model reference control for timed event graphs in dioid. *Automatica*, 37(8):1451–1458, 2001.

[17] R.A. Cuninghame-Green. *Minimax Algebra*, volume 166 of *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag, Berlin, Germany, 1979.

[18] B. De Schutter and B. De Moor. The extended linear complementarity problem. *Mathematical Programming*, 71(3):289–325, December 1995.

[19] B. De Schutter and T. van den Boom. Model predictive control for max-plus-linear discrete event systems. *Automatica*, 37(7):1049–1056, July 2001.

[20] M. Diehl and J. Björnberg. Robust dynamic programming for min-max model predictive control of constrained uncertain systems. *IEEE Transactions on Automatic Control*, 49(12):2253–2257, 2004.

[21] S. Farahani, T.J.J. van den Boom, H. van der Weide, and B. De Schutter. An approximation approach for model predictive control of stochastic max-plus linear systems. In *Proceedings of the Workshop on Discrete Event Systems (WODES) 2010*, pages 386–391, Berlin, Germany, August 30 – September 1, 2010.

[22] S. Gaubert. *Théorie des Systèmes Linéaires dans les Dioïdes.* PhD thesis, Ecole Nationale Supérieure des Mines de Paris, France, July 1992.

[23] S. Gaubert. On rational series in one variable over certain dioids. Technical Report 2162, INRIA, Le Chesnay, France, January 1994.

[24] M.J. Gazarik and B.E.W. Kamen. Reachability and observability of linear systems over max-plus. *Kybernetika*, 35(1):2–12, 1999.

[25] H. Goto, K. Takeyasu, S. Masuda, and T. Amemiya. A gain scheduled model predictive control for linear-parameter-varying max-plus-linear systems. In *Proceedings of the American Control Conference*, Denver, Colorado, June 2003.

[26] A. Guezzi, Ph. Declerck, and J.-L. Boimond. From monotone inequalities to model predictive control. In *Proceedings of the Conference on Emerging Technologies and Factory Automation (ETFA 2008)*, pages 1310–1317, Hamburg, Germany, September 2008.

[27] W. Heemels, B. De Schutter, and A. Bemporad. Equivalence of hybrid dynamical models. *Automatica*, 37(7):1085–1091, July 2001.

[28] B. Heidergott, G.J. Olsder, and J. van der Woude. *Max Plus at Work.* Princeton University Press, Princeton, 2006.

[29] R. Katz. Max-plus (a,b)-invariant spaces and control of timed discrete-event systems. *IEEE Transactions on Automatic Control*, 52(2):229–241, 2007.

[30] J.B. Lasserre. Integration on a convex polytope. *Proceedings of the 1998 American Mathematical Society*, 126(8):2433–2441, August 1998.

[31] E. Le Corronc, B. Cottenceau, and L. Hardouin. Control of uncertain (max,+)-linear systems in order to decrease uncertainty. In *Proceedings of the Workshop on Discrete Event Systems*, pages 410–415, Berlin, Germany, August 30–September 1, 2010.

[32] M. Lhommeau, L. Hardouin, and B. Cottenceau. Disturbance decoupling of timed event graphs by output feedback controller. In *Proceedings of the 6th International Workshop on Discrete Event Systems (WODES)*, Zaragoza, Spain, October 2002.

[33] M. Lhommeau, L. Hardouin, B. Cottenceau, and L. Jaulin. Interval analysis in dioid: application to robust controller design for timed-event graphs. *Automatica*, 40(11):1923–1930, November 2004.

[34] F. Liu. Robust and adaptive supervisory control of discrete event systems. *IEEE Transactions on Automatic Control*, 38(12):1848–1852, December 1993.

[35] J.M. Maciejowski. *Predictive Control with Constraints.* Prentice Hall, Pearson Education Limited, Harlow, UK, 2002.

[36] C.A. Maia, C.R. Andrade, and L. Hardouin. On the control of max-plus linear system subject to state restriction. *Automatica*, 47(5):988–992, 2011.

[37] C.A. Maia, L. Hardouin, R. Santos-Mendes, and B. Cottenceau. Optimal closed-loop of timed event graphs in dioids. *IEEE Transactions on Automatic Control*, 48(12):2284–2286, December 2003.

[38] T.H. Mattheiss and D.S. Rubin. A survey and comparison of methods for finding all vertices of convex polyhedral sets. *Mathematics of Operations Research*, 5(2):167–185, May 1980.

[39] D.Q. Mayne. Control of constrained dynamic systems. *European Journal of Control*, 7(2-3):87–99, 2001.

[40] D.Q. Mayne, J.B. Rawlings, C.V. Rao, and P.O.M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, 2000.

[41] E. Menguy, J. Boimond, L. Hardouin, and J. Ferrier. Just-in-time control of timed event graphs: Update of reference input, presence of uncontrollable input. *IEEE Transactions on Automatic Control*, 45(9):2155–2159, 2000.

[42] E. Menguy, J.L. Boimond, and L. Hardouin. Adaptive control for linear systems in max-algebra. In *Proceedings of the International Workshop on Discrete Event Systems (WODES'98)*, pages 481–488, Cagliari, Italy, August 1998.

[43] T.S. Motzkin, H. Raiffa, G.L. Thompson, and R.M. Thrall. The double description method. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games*, number 28 in *Annals of Mathematics Studies*, pages 51–73. Princeton University Press, Princeton, New Jersey, 1953.

[44] I. Necoara, B. De Schutter, T.J.J. van den Boom, and H. Hellendoorn. Stable model predictive control for constrained max-plus-linear systems. *Discrete Event Dynamic Systems: Theory and Applications*, 17(3):329–354, September 2007.

[45] I. Necoara, B. De Schutter, T.J.J. van den Boom, and H. Hellendoorn. Robust control of constrained max-plus-linear systems. *International Journal of Robust and Nonlinear Control*, 19(2):218–242, January 2009.

[46] I. Necoara, E.C. Kerrigan, B. De Schutter, and T.J.J. van den Boom. Finite-horizon min-max control of max-plus-linear systems. *IEEE Transactions on Automatic Control*, 52(6):1088–1093, June 2007.

[47] I. Necoara, T.J.J. van den Boom, B. De Schutter, and J. Hellendoorn. Stabilization of max-plus-linear systems using model predictive control: The unconstrained case. *Automatica*, 44(4):971–981, April 2008.

[48] Y. Nesterov and A. Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming.* SIAM, Philadephia, Pennsylvania, USA, 1994.

[49] S.J. Park and J.T. Lim. Fault-tolerant robust supervisor for discrete event systems with model uncertainty and its application to a workcell. *IEEE Transactions on Robotics and Automation*, 15(2):386–391, April 1999.

[50] K.M. Passino and K.L. Burgess. *Stability Analysis of Discrete Event Systems.* John Wiley & Sons, New York, USA, 1998.

[51] R.T. Rockafellar. *Convex Analysis.* Princeton University Press, Princeton, New Jersey, 1970.

[52] R.T. Rockafellar and R.J.B. Wets. *Variational Analysis.* Springer-Verlag, Berlin, Germany, 2004.

[53] A.R.M. Soeterboek. *Predictive Control - A Unified Approach.* Prentice Hall, Englewood Cliffs, New Jersey, 1992.

[54] T.J.J. van den Boom and B. De Schutter. Model predictive control for perturbed max-plus-linear systems. *Systems & Control Letters*, 45(1):21–33, January 2002.

[55] T.J.J. van den Boom and B. De Schutter. Properties of MPC for max-plus-linear systems. *European Journal of Control*, 8(5):53–62, 2002.

[56] T.J.J. van den Boom and B. De Schutter. Model predictive control for perturbed max-plus-linear systems: A stochastic approach. *International Journal of Control*, 77(3):302–309, February 2004.

[57] T.J.J. van den Boom and B. De Schutter. Modelling and control of discrete event systems using switching max-plus-linear systems. *Control Engineering Practice*, 14(10):1199–1211, October 2006.

[58] T.J.J. van den Boom and B. De Schutter. MPC for max-plus-linear systems with output weighting: steady-state behavior and guaranteed stability. In *Proceedings of the Conference on Decision and Control 2009*, pages 2286–2291, Shanghai, China, December 2009.

[59] T.J.J. van den Boom and B. De Schutter. Modeling and control of switching max-plus-linear systems with random and deterministic switching. *Discrete Event Dynamic Systems*, 12(3):293–332, September 2012.

[60] T.J.J. van den Boom, B. De Schutter, and I. Necoara. On MPC for max-plus-linear systems: Analytic solution and stability. In *Proceedings of the joint Conference on Decision and Control/European Control Conference 2005*, pages 7816–7821, Sevilla, Spain, December 2005.

[61] T.J.J. van den Boom, B. Heidergott, and B. De Schutter. Complexity reduction in MPC for stochastic max-plus-linear systems by variability expansion. *Automatica*, 43(6):1058–1063, June 2007.

[62] S.J. Wright. *Primal-Dual Interior Point Methods.* SIAM, Philadephia, Pennsylvania, 1997.

[63] S. Young and V.K. Garg. Model uncertainty in discrete event systems. *SIAM Journal on Control and Optimization*, 33(1):208–226, January 1995.

[64] Y. Zou and S. Li. Max-plus-linear model-based predictive control for constrained hybrid systems: linear programming solution. *Journal of Control Theory and Applications*, 5(1):71–76, 2007.