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Optimal Nonlinear Solutions for Reverse Stackelberg Games with Incomplete Information

Zhou Su, Simone Baldi and Bart De Schutter

Abstract— **The reverse Stackelberg game provides a suitable decision-making framework for hierarchical control problems like network pricing and toll design. We propose a novel numerical solution approach for systematic computation of optimal nonlinear leader functions, also known as incentives, for reverse Stackelberg games with incomplete information and general, nonconcave utility functions. In particular, we apply basis function approximation to the class of nonlinear leader functions, and treat the incentive design problem as a standard semi-infinite programming problem. A worked example is provided to illustrate the proposed solution approach and to demonstrate its efficiency.**

I. INTRODUCTION

The Stackelberg game, a hierarchical leader-follower game first introduced in the 1930s in economic context [1], has received growing recognition in the systems and control field since the 1970s [2], [3]. In a Stackelberg game, the leader makes her decision first; then the followers, informed of the leader's decision, make their decisions accordingly. The reverse Stackelberg game, in which the leader proposes a function mapping from the followers' decision spaces to the leader's decision space, instead of making a direct decision, can be viewed as a more general case of the original Stackelberg game. The reverse Stackelberg games have been successfully applied to many hierarchical decision making problems like nonlinear network pricing [4], optimal routing [5], and toll design [6].

We follow the "type" notation proposed in [7] for games with incomplete information, where at least one player possesses certain important attributes, the actual value of which is only known to himself. These attributes can be, e.g., the production cost or risk attitude. The type of a player is then characterized by a vector of these attributes. The actual type of a player is only known to himself, and his opponents only know the type space and the type distribution, i.e. all possible alternatives of each attribute and the probability of each possible combination. Moreover, we consider no type for the leader in reverse Stackelberg games with incomplete information. Compared with the situation of complete information, the leader's lack of information regarding the followers produces a less desirable results for her [4].

Solving the Stackelberg game is equivalent to solving a bilevel programming problem [8], and even the simplest linear bilevel programming problem has been proved is NP-hard [9]. The more general reverse Stackelberg games are even more difficult to solve, especially when a wide class of leader functions are considered and the players have general, nonconcave utility functions. Most papers that discuss nonlinear leader functions often focus on deriving analytic solutions for problem-specific utility functions [4], [10], [11]. A systematic approach to compute optimal nonlinear leader functions for reverse Stackelberg games with general utility functions is proposed in [12], but under the restrictive assumption of complete information. Therefore, in this paper, we focus on numerical solution approaches for the more realistic Stackelberg game with incomplete information, considering nonlinear leader functions and general utility functions.

The key contribution of this paper is a systematic solution approach based on basis functions and semi-infinite programming for reverse Stackelberg games with incomplete information, considering nonlinear leader functions and general, nonconcave utility functions.

The paper is organized as follows. We describe the reverse Stackelberg game and its formulation in Section II, and propose the solution approach based on basis functions and semi-infinite programming in Section III. A numerical example is provided in Section IV to illustrate the solution approach. Finally we conclude this study and list some directions of future work in Section V.

II. PRELIMINARIES

We consider a two-person reverse Stackelberg game with the player set $\{\mathrm{L}, \mathrm{F}\}$, where L denotes the leader and F denotes the follower. The leader's decision is $d_{\mathrm{L}} \in D_{\mathrm{L}} \subset \mathbb{R}^{n_{\mathrm{L}}}$ and the follower's decision is $d_{\mathrm{F}} \in D_{\mathrm{F}} \subset \mathbb{R}^{n_{\mathrm{F}}}$, where the decision spaces D_{L} and D_{F} are both continuous and compact. The follower's type is denoted by $t \in T$, with the type space T a discrete and compact set. The follower's type is only known to himself, but the type distribution $P : T \rightarrow [0, 1]$ is known to both players. The leader's utility function is $U_{\mathrm{L}} : D_{\mathrm{L}} \times D_{\mathrm{F}} \rightarrow \mathbb{R}$, and the follower's utility function is $U_{\mathrm{F}} : D_{\mathrm{L}} \times D_{\mathrm{F}} \times T \rightarrow \mathbb{R}$. Let $\underline{U}_{\mathrm{F},t}$ be the reservation utility of the agent, which specifies the minimum utility the follower requires to participate in the game.

In a reverse Stackelberg game, the leader moves first by announcing a leader function $\gamma_{\mathrm{L}} : D_{\mathrm{F}} \rightarrow D_{\mathrm{L}}$. The set of admissible leader functions is denoted by Γ_{L}. The follower then decides his best response $d_{\mathrm{F}}^{\mathrm{BR}}$ to the announced leader function. If the best response gives a utility strictly lower

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than his reservation utility[1], the follower will quit and the game terminates. Otherwise, the follower executes $d_{\mathrm{F}}^{\mathrm{BR}}$ and the game ends by the leader executing the promised decision $\gamma_{\mathrm{L}}(d_{\mathrm{F}}^{\mathrm{BR}})$.

A. Game Formulation

Under incomplete information, the leader's objective is to maximizes her expected utility over all possible follower's types, which is achieved by announcing a leader function that maximizes her expected utility, considering all possible responses from the follower. As proposed in [4], [12], we decompose the problem of designing the optimal leader function into two sequential optimization problems: the leader's global optimization problem, which yields the global optimum (if it exists), and the incentive design problem, which induces the follower to adopt the global optimum, under the assumption of full rationality[2].

Let $t \in T$ denote the follower's type. Furthermore, define $d_{\mathrm{L},t}$ and $d_{\mathrm{F},t}$ as the decision variable of the leader and the follower regarding the follower of type t, respectively. Let $\underline{U}_{\mathrm{F},t}$ denote the reservation utility of a follower of type t. The optimization problem to find the global optimum, which is called the desired point (also called team solution in literature [4]), that maximize the leader's expected utility can be formulated as:

To find

$$\{(d_{\mathrm{L},t}^*,\, d_{\mathrm{F},t}^*)\}_{t\in T} \in \operatorname*{arg\,max}_{\{(d_{\mathrm{L},t},\, d_{\mathrm{F},t})\in D_{\mathrm{L}}\times D_{\mathrm{F}}\}_{t\in T}} \sum_{t\in T} P(t) U_{\mathrm{L}}(d_{\mathrm{L},t},\, d_{\mathrm{F},t}) \tag{1}$$

subject to:

$$U_{\mathrm{F}}(d_{\mathrm{L},t}^*,\, d_{\mathrm{F},t}^*,\, t) \geq \underline{U}_{\mathrm{F},t} \quad \forall t \in T \tag{2}$$

$$U_{\mathrm{F}}(d_{\mathrm{L},t}^*,\, d_{\mathrm{F},t}^*,\, t) \geq U_{\mathrm{F}}(d_{\mathrm{L},\hat{t}}^*,\, d_{\mathrm{F},\hat{t}}^*,\, t) \quad \forall t,\, \hat{t} \in T. \tag{3}$$

Constraint (2) is the *participation constraint*, which garantees the participation of the follower, and constraint (3) is the *incentive compatibility* constraint, which ensures that the follower has no incentive to pretend to be of any type other than his true type.

Assume that the leader's global optimum $\{(d_{\mathrm{L},t}^*,\, d_{\mathrm{F},t}^*)\}_{t\in T}$ exists, which can be found by global optimization techniques like multi-start; then the incentive design problem is to find a leader function $\gamma_{\mathrm{L}} \in \Gamma_{\mathrm{L}}$ that induces the follower to adopt the team solution, i.e.

To find

$$\gamma_{\mathrm{L}} \in \Gamma_{\mathrm{L}} \tag{4}$$

subject to:

$$d_{\mathrm{F},t}^* \in \operatorname*{arg\,max}_{d_{\mathrm{F}}\in D_{\mathrm{F}}} U_{\mathrm{F}}(\gamma_{\mathrm{L}}(d_{\mathrm{F}}),\, d_{\mathrm{F}},\, t) \quad \forall t \in T \tag{5}$$

$$\gamma_{\mathrm{L}}(d_{\mathrm{F},t}^*) = d_{\mathrm{L},t}^* \quad \forall t \in T. \tag{6}$$

[1]Reservation utility is the lowest utility that a player will accept to participate in a game.

[2]In game theory, a player is said to have full rationality if he always acts in a way to maximize his utility.

Constraint (5) ensures that the follower has no incentive to deviate from the leader's global optimum, regardless of his type. Constraint (6) ensures that the optimal leader function passes through the desired point for any type of follower. The team solution $\{(d_{\mathrm{L},t}^*,\, d_{\mathrm{F},t}^*)\}_{t\in T}$ is called *incentive controllable*, if the feasibility program (4)-(6) has a solution. A leader function is called game-optimal, if it is a solution to (4)-(6) for the leader's global optimum[3].

In summary, an optimal leader function should pass through the desired point for any type. Moreover, it should not intersect with the 0-level curve of the function

$$g_{\inf}(d_{\mathrm{L}},\, d_{\mathrm{F}},\, t) := U_{\mathrm{F}}(d_{\mathrm{L}},\, d_{\mathrm{F}},\, t) - U_{\mathrm{F}}(d_{\mathrm{L},t}^*,\, d_{\mathrm{F},t}^*,\, t) \tag{7}$$

and it should remain inside the sublevel set

$$\Lambda := \{(d_{\mathrm{L}},\, d_{\mathrm{F}}) \in D_{\mathrm{L}} \times D_{\mathrm{F}} | g_{\inf}(d_{\mathrm{L}},\, d_{\mathrm{F}},\, t) \leq 0\} \tag{8}$$

for any $t \in T$.

III. Solution Approach Based on Basis Functions and Semi-Infinite Programming

The computation of the team solution (1)-(3) is a tractable standard optimization problem, as the number of decision variables and constraints are both finite. Assume the leader's global optimum $\{(d_{\mathrm{L},t}^*,\, d_{\mathrm{F},t}^*)\}_{t\in T}$ exists and is unique [4]. An analytic solution for a general nonlinear leader function to the incentive design problem (4)-(6) is difficult to obtain, especially for general, nonconcave utility functions. The difficulties in solving the incentive design problem (4)-(6) lie in the fact that the decision space Γ_{L} is a function space of infinite dimensions, and that the equilibrium constraint (5) is also numerically challenging for general nonconvex utility functions, as it involves solving a global optimization problem.

A. Basis Function Approach

Basis functions are universal approximators that can approximate any given function with arbitrary accuracy when the set of selected basis functions is rich enough [13]. A finite set of basis functions $\mathcal{B} = \{b_i : \mathbb{R}^{n_{\mathrm{F}}} \to \mathbb{R}^{n_{\mathrm{L}}}\}_{i=1}^n$ is used to approximate the leader function γ_{L}. Each of these basis functions is further denoted by $b_i(\cdot;\xi_i)$, $i = 1, \ldots, n$, to emphasize its dependence on the parameter vector ξ_i, which contains information regarding the location and the shape of each basis function (e.g. the center and the width of a radial basis function). The leader function γ_{L} can be represented by a linear combination of the selected basis functions[5]

$$\gamma_{\mathrm{L}}(\cdot) = \sum_{i=1}^{n} \alpha_i \odot b_i(\cdot;\xi_i) \tag{9}$$

with weights $\alpha_i \in \mathbb{R}^{n_{\mathrm{L}}}$ and parameter vectors $\xi_i \in \Xi$. As for the basis functions, we denote the leader function represented

[3]When the leader has multiple global optima, a leader function is called game-optimal if it is the solution to (4)-(6) for at least one global optimum.

[4]If there are multiple global optima, we can repeat (4)-(6) for each global optimum, and choose the γ_{L} that gives the highest expected follower utility over his type space.

[5]The operator $\odot$ represents the elementwise (Schur) product.

by basis function approximation (9) as $\gamma_\mathrm{L}(\cdot;\, \alpha,\, \xi)$, to highlight its dependence on the parameters and weight of each basis function, with $\alpha = [\alpha_1^\mathrm{T} \cdots \alpha_n^\mathrm{T}]^\mathrm{T}$ and $\xi = [\xi_1^\mathrm{T} \cdots \xi_n^\mathrm{T}]^\mathrm{T}$. Then we can approximate the incentive design problem (4)-(6) by the following feasibility program:
To find:

$$(\alpha,\, \xi) \in \mathbb{R}^{n_\mathrm{L} \times n} \times \Xi^n \tag{10}$$

subject to:

$$g_{\inf}(\gamma_\mathrm{L}(d_\mathrm{F}),\, d_\mathrm{F},\, t) = U_\mathrm{F}\left(\sum_{i=1}^{n} \alpha_i \odot b_i(d_\mathrm{F}; \xi_i),\, d_\mathrm{F},\, t\right)$$

$$-U_\mathrm{F}\left(d^*_{\mathrm{L},t},\, d^*_{\mathrm{F},t},\, t\right) \le 0 \quad \forall d_\mathrm{F} \in D_\mathrm{F}, \forall t \in T \tag{11}$$

$$\sum_{i=1}^{n} \alpha_i \odot b_i(d^*_{\mathrm{F},t}; \xi_i) = d^*_{\mathrm{L},t} \quad \forall t \in T \tag{12}$$

$$\sum_{i=1}^{n} \alpha_i \odot b_i(d_\mathrm{F}; \xi_i) \in D_\mathrm{L} \quad \forall d_\mathrm{F} \in D_\mathrm{F}. \tag{13}$$

Constraint (11) and (12) corresponds to constraint (5) and (6), respectively. Constraint (13) is to guarantee that the resulting leader function indeed maps the follower's decision space to the leader's decision space.
Constraint (11) and (13) are complicating, as they must be satisfied on a continuous domain D_F. Constraint (13) can be replaced by a finite linear constraint if a stricter rule is applied to the selection of basis functions. Instead of $\mathcal{B} = \{b_i : \mathbb{R}^{n_\mathrm{F}} \to \mathbb{R}^{n_\mathrm{L}}\}_{i=1}^n$, we can choose $\tilde{\mathcal{B}} = \{b_i : D_\mathrm{F} \to D_\mathrm{L}\}_{i=1}^n$ as the set of selected basis functions. Then constraint (13) can be replaced by the following two linear constraints:

$$\alpha \in [0, 1]^{n_\mathrm{L}} \tag{14}$$

$$\sum_{i=1}^{n} \alpha_i = 1 \tag{15}$$

In this way γ_L as a convex combination of the basis functions in $\tilde{\mathcal{B}}$ also maps D_F to D_L, and constraint (13) is satisfied by construction. The feasibility program for the incentive design problem thus becomes (10)-(12),(14),(15), with the new set of selected basis function $\tilde{\mathcal{B}}$.

B. Semi-Infinite Programming

The incentive design problem (10)-(12),(14),(15) is still intractable, as there still remains one complicating constraint (11), which must be satisfied on a continuous domain D_F. Mathematical programming problems with a finite number of decision variables but an infinite number of constraints are called Semi-Infinite Programming (SIP) problems [14]. Standard SIP problems can be represented by the following general form[6]

$$\min_{x \in X} f(x) \tag{16}$$

[6]For clarity we omit all finite constraints in Section III-B, as they can be easily added to the resulting finite programming problem.

subject to:

$$g_i(x,\, y) \le 0 \quad \forall y \in Y_i, \forall i \in \{1, \dots, p\} \tag{17}$$

where X and Y_i are continuous, compact subsets of $\mathbb{R}^{n_x}$ and $\mathbb{R}^{n_y}$, respectively, and the functions $f : \mathbb{R}^{n_x} \to \mathbb{R}$ and $g_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$ are real-valued and continuous on their respective domains, for all i. For clarity we call x the decision variable and y the index variable. Furthermore, we call the continuous set Y_i the *index set* of each infinite constraint g_i. The intractable feasibility program (10)-(12),(14),(15) can then be transformed to a tractable standard SIP problem (16),(17) with $|T|$ infinite constraints, by treating the parameters of the basis function $(\alpha,\, \xi)$ as the decision variable x, and d_F as the index variable y of the infinite constraints.
A comprehensive survey on numerical methods for semi-infinite programming problems is given in [14]. The major challenge in solving a semi-infinite programming problem lies in the fact that to check the feasibility of a point $\bar{x} \in X$, the following lower-level optimization problem

$$\max_{y \in Y_i} g_i(\bar{x},\, y) \tag{18}$$

must be solved to global optimality for each $i \in \{1, \dots, p\}$ [15]. Let y_i^* denote the global optimum for the i-th lower-level problem (18); then $\bar{x}$ is feasible as long as:

$$\max_{i \in \{1, \dots, p\}} g_i(\bar{x}, y_i^*) \le 0.$$

The difficulty of solving a semi-infinite programming problem depends on whether the lower-level problems are convex. As the convexity of the lower-level problem is so crucial in solving a semi-infinite programming problem, we now provide several sufficient conditions to check the convexity of the lower-level problem for the feasibility program (10)-(12),(14),(15).

Theorem 1: Let U_F and γ_L (in the form of (9)) be continuous and twice differentiable on their respective domains; then the lower-level problem of the feasibility program (10)-(12),(14),(15) is convex if any of the following conditions is satisfied:

(1) $U_\mathrm{F}(\cdot,\, \cdot,\, t)$ is linear in d_L and d_F, and non-decreasing in d_L for all $t \in T$ and γ_L is concave;
(2) $U_\mathrm{F}(\cdot,\, \cdot,\, t)$ is linear in d_L and d_F, and non-increasing in d_L for all $t \in T$ and γ_L is convex;
(3) $U_\mathrm{F}(\cdot,\, \cdot,\, t)$ is concave in d_L and d_F, and non-decreasing in d_L for all $t \in T$, and γ_L is non-decreasing and concave;
(4) $U_\mathrm{F}(\cdot,\, \cdot,\, t)$ is concave in d_L and d_F, and non-increasing in d_L for all $t \in T$, and γ_L is non-decreasing and convex;

The proof is given in the appendix. Remark: Theorem 1 can also be applied to select proper basis functions when U_F is concave. As both convexity and monotonicity are preserved by convex combination, Theorem 1 also holds if we replace γ_L by "each basis function" in condition (1)-(4).
The importance of the convexity of the lower-level problems is that it allows for the usage of equivalent reformulation methods. When the lower-level problems are convex, the

semi-infinite programming problem (16)-(17) can be equivalently transformed to a tractable finite programming problem through bilevel reformulations [15], like the Mathematical Program with Complementary Constraints (MPCC) reformulation [16], and the reformulation based on lower-level Wolfe duality [17]. However, such equivalent reformulation methods cannot be directly applied when at least one lower-level problem is nonconvex. Many numerical methods have been developed for semi-infinite programming problems with general, nonconvex lower-level problems. We refer the interested readers to [14], [15] for a comprehensive survey. Moreover, many numerical solvers have also been developed for semi-infinite programming problems, like *fseminf* in the Matlab Optimization Toolbox, and the AMPL-coded NSIPS solver [18], [19] available in the NEOS server [20].

IV. Numerical Example

In this section we present a numerical example to illustrate the procedure of a systematic computation of the optimal non-linear leader function for reverse Stackelberg games with incomplete information and general, nonconcave utility functions.

A. Settings

Let the leader and the follower's decision spaces be $D_{\mathrm{L}} = [-5,\, 5]$ and $D_{\mathrm{F}} = [-2,\, 2]$, respectively. We denote $\underline{d}_k$ and $\overline{d}_k$ the lower and upper bounds of D_k for $k \in \{\mathrm{L},\, \mathrm{F}\}$, respectively. The follower's type space is given by $T = \{t_1,\, t_2\}$ where $t_1 = 1$ and $t_2 = 5$, and the type distribution is $P(t_1) = 0.75$ and $P(t_2) = 0.25$. The Rosenbrock function [21], a popular valley-shaped non-convex testing function for optimization algorithms, is selected as the utility functions[7] for both players. In particular, we let the utility functions of the leader and the follower to be:

$$U_{\mathrm{L}} = -(1+d_{\mathrm{F}})^2 - 100(d_{\mathrm{L}} + d_{\mathrm{F}}^2)^2 \tag{19}$$

$$U_{\mathrm{F}} = -(1-d_{\mathrm{F}})^2 - 100(t d_{\mathrm{L}} - d_{\mathrm{F}}^2)^2. \tag{20}$$

The type t can be viewed as a parameter that influences the shape of the follower's utility u_{F}.

Two radial basis function families, the Gaussian radial basis functions and the inverse multiquadric functions, are selected to approximate the leader function. The Gaussian radial basis functions are defined by:

$$\phi(r) = \exp(-\frac{r^2}{\Delta^2}) \tag{21}$$

and the inverse multiquadric functions are defined by:

$$\phi(r) = \frac{1}{\sqrt{r^2 + \Delta^2}} \tag{22}$$

Then each basis function can be represented by:

$$b_i(d_{\mathrm{F}}) = \frac{\overline{d}_{\mathrm{L}} - \underline{d}_{\mathrm{L}}}{\overline{\phi} - \underline{\phi}}\phi(r) + \underline{d}_{\mathrm{L}} \tag{23}$$

where $r = \|d_{\mathrm{F}} - c_i\|_2$ is the Euclidean distance to the center of the $i-$th radial basis function, and $\overline{\phi}$ and $\underline{\phi}$ are the upper and lower bounds of the selected radial basis function. The centers $\{c_i\}_{i=1}^n$ are equidistantly placed on D_{F}, and the width is fixed to $\Delta = \dfrac{\overline{d}_{\mathrm{F}} - \underline{d}_{\mathrm{F}}}{n-1}$.

Since both the centers and widths are fixed for the radial basis functions, the parameter vector ξ is empty as we only optimize the weights α. Since we have the freedom to choose a well-behaved objective function to the feasibility program (10)-(12),(14),(15), we add the following quadratic objective:

$$\min \sum_{i=1}^{n} \alpha_i^2. \tag{24}$$

The leader's global optimization problem (1)-(3) is solved by the nonlinear programming solver SNOPT from Tomlab 8.0 with multi-start, and the standard semi-infinite programming problem (24)(10)-(12),(14),(15) for the incentive design problem is solved by *fseminf* from the Matlab Optimization Toolbox. All simulations are performed on a desktop computer with an Intel i5-3470 Quad core and 16 GB of RAM, running Matlab R2015b on a 64-bit version of SUSE Linux Enterprise Desktop 11.

As the semi-infinite programming solver *fseminf* uses a discretization method, which does not guarantee feasibility of each iteration, we will measure the violation of the infinite constraint (11) after a leader function is obtained. A fine uniform grid $\tilde{D}_{\mathrm{L}} \times \tilde{D}_{\mathrm{F}}$ (with 101×101 grid points) is generated for $D_{\mathrm{L}} \times D_{\mathrm{F}}$ for post-evaluation of the infinite constraint (12). The following measure is used to evaluate the constraint violation:

$$v_t = \frac{\max\limits_{\tilde{d}_{\mathrm{F}} \in \tilde{D}_{\mathrm{F}}} g_{\mathrm{inf}}(\gamma_{\mathrm{L}}(\tilde{d}_{\mathrm{F}}),\, \tilde{d}_{\mathrm{F}},\, t)}{\max\limits_{(\tilde{d}_{\mathrm{L}},\, \tilde{d}_{\mathrm{F}}) \in \tilde{D}_{\mathrm{L}} \times \tilde{D}_{\mathrm{F}}} g_{\mathrm{inf}}(\tilde{d}_{\mathrm{L}},\, \tilde{d}_{\mathrm{F}},\, t)} \quad \forall t \in T. \tag{25}$$

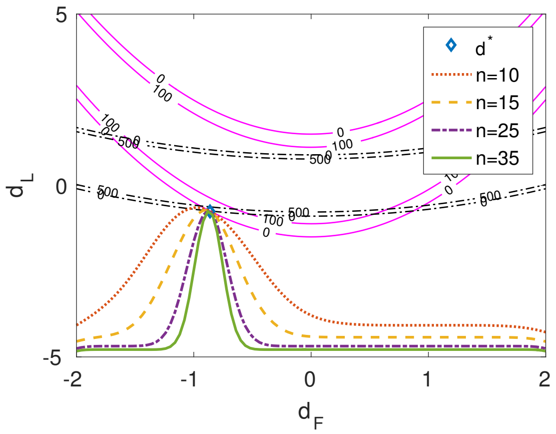
The denominator represents the maximal violation of a given constraint on the whole evaluation grid, and the numerator calculates the maximal constraint violation when the resulting γ_{L} is implemented. In this way we can have a quantitative measure on the performance of each leader function.

B. Discussions of Results

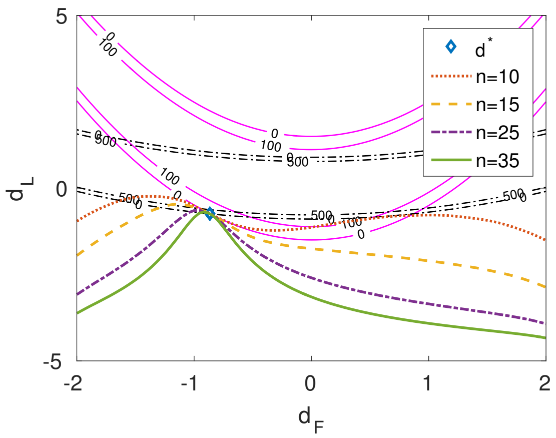
The team solution computed by SNOPT is $(-0.7450,\, -0.8634)$ for t_1 and $(-0.7445,\, -0.8637)$ for t_2. The leader functions obtained from different numbers of Gaussian radial basis functions and inverse multiquadric functions are shown in Figure 1. An optimal leader function should pass through the leader's desired points and does not intersect with the 0-level curves of g_{inf}, so that the follower cannot obtain a strictly higher utility if he deviates from the leader's desired points, regardless of his type. As we can see, all the leader functions are continuous and lie in the leader's decision space D_{L}, and all of them pass through the desired points[8]. Thus constraint (12) is satisfied

[7]The signs are reversed as the Rosenbrock function is designed for minimization problems.

[8]Note that the global optimum in general includes two different points for different types, but they are very close to each other in this example.

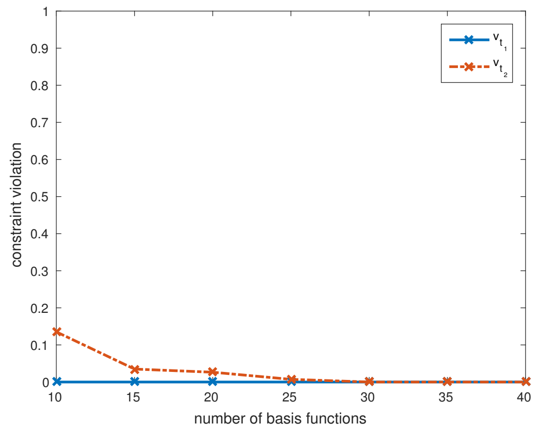


(a) Gaussian radial basis functions

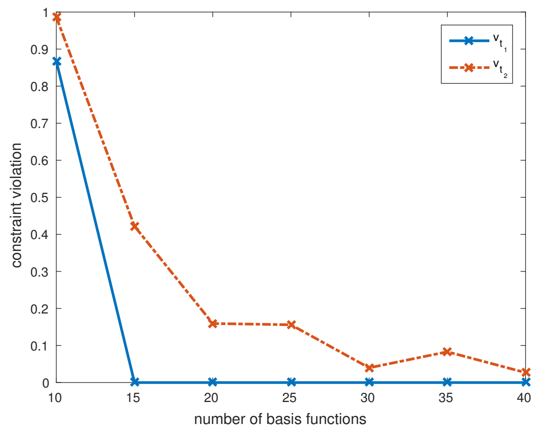


(b) Inverse multiquadric basis functions

Fig. 1: Leader functions obtained from the indirect approach based on basis function approximation and semi-infinite programming. The solid and dashed contour lines are the 0-level curve of g_{inf} for t_1 and t_2, respectively.



(a) Gaussian radial basis functions



(b) Inverse multiquadric basis functions

Fig. 2: Measure of constraint violation of the infinite constraint g_{inf} for both types.

for all of them. However, not all resulting leader functions satisfy the infinite constraint (11). For example, as shown in Figure 1b, the leader functions obtained using 10 and 15 inverse multiquadric basis functions both intersect with the 0-level curve of g_{inf} for t_2. Gaussian radial basis functions demonstrate a better performance in comparison, as shown in Figure 1a, as even the leader function resulting from only 10 basis functions has no obvious intersection with either 0-level curve.

The performance of the two basis function families, quantified by the constraint violation (25), is visualized in Figure 2. Both basis function families show an improvement of performance as the number of basis functions increases. Gaussian radial basis functions obviously perform better than inverse multiquadric functions, as the infinite constraint for t_1 is never violated (v_{t_1} remains 0 in Figure 2a), and the maximum violation of g_{inf} for t_1 is only 0.14, compared to 1 for the inverse multiquadric case. Moreover, the performance using 10 Gaussian basis functions is better than the performance using 25 inverse multiquadratic functions, and with 30 Gaussian radial basis functions we can already find a "perfect" leader function with no constraint violations. From Figure 2 we can conclude that a selection of 30 basis functions is already sufficient to obtain a well-performing leader function, as the maximal constraint violation is no more than 0.1 for both Gaussian radial basis functions and inverse multiquadric basis functions.

The mean CPU time to solve the incentive design problem using *fseminf* with different numbers of Gaussian and inverse multiquadric radial basis functions is shown in Figure 3. We can see that neither choice of basis functions is computationally very demanding, as the largest problems ($n = 40$) can be computed within 1.6 seconds, and within 1 second we can already obtain a satisfactory leader function ($n = 30$).

V. Conclusions and Future work

A structured numerical solution approach has been developed for the class of nonlinear leader functions for reverse Stackelberg games with incomplete information and general utility functions. Basis functions are used to approximate the

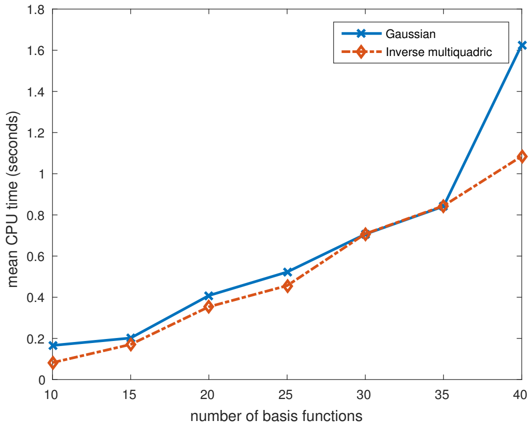


Fig. 3: Mean CPU time for the compuation of the incentive design problem using Gaussian radial basis functions and inverse multiquadratic basis functions. The mean is taken over 10 runs.

nonlinear leader function, transforming the incentive design problems into a standard semi-infinite programming problem, which has been extensively studied in literature. The worked example shows that the proposed solution approach provides satisfactory results in relative short CPU time using typical basis functions like Gaussian radial basis functions, and standard semi-infinite programming solvers.

In future research, a comprehensive computational experiment considering various nonconcave utility functions can be performed to investigate more families of basis functions. In addition, continuous type space for the follower will be considered. In this case, even the computation of the leader's global optimum becomes intractable. Another direction will be extending the proposed solution approach to dynamic reverse Stackelberg games.

APPENDIX

Proof of Theorem 1

Proof: The key to determine the convexity of the lower-level problem is to determine the concavity of the functions $g_{\inf}(\gamma_{\rm L}(\cdot), \cdot, t)$ for all $t \in T$. The second-order derivative of $g_{\inf}$ w.r.t $d_{\rm F}$ is given by:

$$\frac{\partial^2 g_{\inf}}{\partial d_{\rm F}^2} = \underbrace{\frac{\partial^2 U_{\rm F}}{\partial d_{\rm L}^2}\left(\frac{{\rm d}\gamma_{\rm L}}{{\rm d}d_{\rm F}}\right)^2}_{\text{term 1}} + \underbrace{\frac{\partial U_{\rm F}}{\partial d_{\rm L}}\frac{{\rm d}^2\gamma_{\rm L}}{{\rm d}d_{\rm F}^2}}_{\text{term 2}} + \underbrace{2\frac{\partial^2 U_{\rm F}}{\partial d_{\rm L} d_{\rm F}}\frac{{\rm d}\gamma_{\rm L}}{{\rm d}d_{\rm F}}}_{\text{term 3}} + \underbrace{\frac{\partial^2 U_{\rm F}}{\partial d_{\rm F}^2}}_{\text{term 4}}. \tag{26}$$

First we prove condition (1) and (2). When $U_{\rm F}(\cdot, \cdot, t)$ is linear, all its second-order derivatives become 0, so only term 2 remains in (26). If $U_{\rm F}$ is non-decreasing in $d_{\rm L}$ and $\gamma_{\rm L}$ is concave, then $\frac{\partial U_{\rm F}}{\partial d_{\rm L}} \geq 0$ and $\frac{{\rm d}^2\gamma_{\rm L}}{{\rm d}d_{\rm F}^2} \leq 0$; thus $\frac{\partial^2 g_{\inf}}{\partial d_{\rm F}^2} \leq 0$, and therefore $g_{\inf}$ is concave in $d_{\rm F}$. So now condition (1) is proved; condition (2) can be proved following similar arguments.

Now we prove condition (3) and (4). When $U_{\rm F}(\cdot, \cdot, t)$ is concave, then its Hessian is negative semi-definite, so term 1 and term 4 are both less than 0. Since $\gamma_{\rm L}$ is non-decreasing, $\frac{{\rm d}\gamma_{\rm L}}{{\rm d}d_{\rm L}} \geq 0$, so term 3 is also less than 0. Moreover, term 2 is non-positive if $U_{\rm F}$ is non-decreasing and $\gamma_{\rm L}$ is concave, so condition (3) is proved; or $U_{\rm F}$ is non-increasing and $\gamma_{\rm L}$ is convex, so condition (4) is proved. ■

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