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Abstract

First we determine necessary and for some cases also sufficient conditions for a polynomial to be the characteristic polynomial of a matrix with elements in $\mathbb{R}_{\max}$. Then we indicate how to construct a matrix such that its characteristic polynomial is equal to a given monic polynomial in $\mathbb{S}_{\max}$, the extension of $\mathbb{R}_{\max}$. Next we use these results to develop a procedure to find the minimal state space realization of a single input single output (SISO) discrete event system, given its Markov parameters.

1 Introduction

1.1 Overview

There exists a wide range of frameworks to model and to analyze discrete event systems: Petri nets, generalized semi-Markov processes, formal languages, perturbation analysis, computer simulation and so on. In this paper we concentrate on discrete event systems that can be described with the max algebra. We address the minimal state space realization problem for max-algebraic single input single output (SISO) systems. We show that the characteristic equation in the max algebra plays an important role in the solution of this problem. Therefore we first make a study of the characteristic equation of a matrix in the max algebra. Next we use these results to propose a procedure to find a minimal state space description of a max-linear time-invariant SISO discrete event system.

In the first section we introduce the notations and some of the definitions and properties that will be used throughout the remainder of this report. In the second section we give some necessary and sufficient conditions for the coefficients of a polynomial such that it is the characteristic polynomial of a matrix in the max algebra. Then we indicate how to construct a matrix for which the characteristic polynomial is equal to a given polynomial. These results will then be used to determine a lower bound for the minimal order of the state space description of a SISO system in the max algebra. This will enable us to find the minimal realization of a SISO discrete event system, given its Markov parameters. Finally we shall illustrate this procedure with a few examples.

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1.2 Notations

One of the mathematical tools used in this report is the max algebra. In this introduction we only explain the notations we use to represent the max-algebraic operations. A complete introduction to the max algebra can be found in [1].
In this report we use the following notations: $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$. $\varepsilon = -\infty$ is the neutral element for $\oplus$ in $\mathbb{R}_{\max} = (\mathbb{R} \cup \{\varepsilon\}, \oplus, \otimes)$. To avoid confusion we always write the $\otimes$ sign explicitly. The inverse element of $a \neq \varepsilon$ for $\otimes$ in $\mathbb{R}_{\max}$ is denoted by $a^{\otimes^{-1}}$. The division is defined as follows:

$$\frac{a}{b} = a \otimes b^{\otimes^{-1}} \quad \text{if } b \neq \varepsilon \ .$$

If A is an m by n matrix then the element on the i-th row and on the j-th column is denoted by a_{ij}. E_n is the n by n identity matrix in $\mathbb{R}_{\max}$: $e_{ij} = 0$ if $i = j$ and $e_{ij} = \varepsilon$ if $i \neq j$. The operations $\oplus$ and $\otimes$ are extended to matrices in the usual way. $A^{\otimes^k} = \underbrace{A \otimes A \otimes \ldots \otimes A}_{k \text{ times}}$.
We also use the extension of the max algebra $\mathbb{S}_{\max}$ that was introduced in [1, 5]. $\mathbb{S}_{\max}$ is a kind of symmetrization of $\mathbb{R}_{\max}$. We shall restrict ourselves to the most important features of $\mathbb{S}_{\max}$. For a more formal derivation the interested reader is referred to [5].
There are three kinds of elements in $\mathbb{S}_{\max}$: the positive elements ($\mathbb{S}^{\oplus}_{\max}$, this corresponds to $\mathbb{R}_{\max}$), the negative elements ($\mathbb{S}^{\ominus}_{\max}$) and the balanced elements ($\mathbb{S}^{\bullet}_{\max}$). The positive and the negative elements are called signed ($\mathbb{S}^{\vee}_{\max} = \mathbb{S}^{\oplus}_{\max} \cup \mathbb{S}^{\ominus}_{\max}$). The $\ominus$ operation in $\mathbb{S}_{\max}$ is defined as follows:

$$\begin{aligned} a \ominus b &= a && \text{if } a > b\,, \\ a \ominus b &= \ominus b && \text{if } a < b\,, \\ a \ominus a &= a^{\bullet} \ . \end{aligned}$$

If $a \in \mathbb{S}_{\max}$ then it can be written as $a = a^+ \ominus a^-$ where a^+ is the positive part of a and a^- is the negative part of a; $|a| = a^+ \oplus a^-$ is the absolute value of a. There are three possible cases: if $a \in \mathbb{S}^{\oplus}_{\max}$ then $a^+ = a$ and $a^- = \varepsilon$, if $a \in \mathbb{S}^{\ominus}_{\max}$ then $a^+ = \varepsilon$ and $a^- = \ominus a$ and if $a \in \mathbb{S}^{\bullet}_{\max}$ then $a^+ = a^- = |a|$.

Example 1.1 Let $a = 3^{\bullet} \in \mathbb{S}^{\bullet}_{\max}$, then $a^+ = 3$, $a^- = 3$ and $|a| = 3$.
For $b = \ominus 2 \in \mathbb{S}^{\ominus}_{\max}$ we have $b^+ = \varepsilon$, $b^- = 2$ and $|b| = 2$.

This symmetrization allows us to 'solve' equations that have no solutions in $\mathbb{R}_{\max}$. Unfortunately we then have to introduce balances (∇) instead of equalities. Informally an $\ominus$ sign in a balance means that the element should be at the other side: so $3 \ominus 3 \ \nabla\ 2$ means $3 \ \nabla\ 2 \oplus 3$. If both sides of a balance are signed (positive or negative) we can replace the balance by an equality.
To select submatrices of a matrix we use the following notation:
$A([i_1, i_2, \ldots, i_k], [j_1, j_2, \ldots, j_l])$ is the matrix resulting from A by eliminating all rows except for rows $i_1, i_2, \ldots, i_k$ and all columns except for columns $j_1, j_2, \ldots, j_l$. $A(i, :)$ is the i-th row of A and $A(:, j)$ is the j-th column of A. $[1 : n]$ stands for $[1, 2, \ldots, n]$.

1.3 Some definitions and theorems

Definition 1.2 (Determinant) *Consider a matrix $A \in \mathbb{S}^{n \times n}_{\max}$. The determinant of A is* *defined as*

$$\det A = \bigoplus_{\sigma \in \mathcal{P}_n} \operatorname{sgn}(\sigma) \otimes \bigotimes_{i=1}^{n} a_{i\sigma(i)}$$

where $\mathcal{P}_n$ is the set of all permutations of $\{1, \ldots, n\}$, and $\operatorname{sgn}(\sigma) = 0$ if the permutation σ is even and $\operatorname{sgn}(\sigma) = \ominus 0$ if the permutation is odd.

Theorem 1.3 *Let $A \in \mathbb{S}_{\max}^{n \times n}$. The homogeneous linear balance $A \otimes x \ \nabla \ \varepsilon$ has a non-trivial signed solution if and only if $\det A \ \nabla \ \varepsilon$.*

Proof: See [5]. The proof given there is constructive so it can be used to find a solution.

Definition 1.4 (Characteristic equation) *Let $A \in \mathbb{S}_{\max}^{n \times n}$. The characteristic equation of A is defined as $\det(A \ominus \lambda \otimes E_n) \ \nabla \ \varepsilon$.*

This leads to

$$\lambda^{\otimes^n} \oplus \bigoplus_{p=1}^{n} a_p \otimes \lambda^{\otimes^{n-p}} \ \nabla \ \varepsilon$$

which will be called a *monic* balance, since the coefficient of $\lambda^{\otimes^n}$ equals 0 (i.e. the identity element for $\otimes$).
If we define $\alpha_p = a_p^+$ and $\beta_p = a_p^-$ and if we move all terms with negative coefficients to the right hand side we get

$$\lambda^{\otimes^n} \oplus \bigoplus_{i=1}^{n} \alpha_i \otimes \lambda^{\otimes^{n-i}} \ \nabla \ \bigoplus_{j=1}^{n} \beta_j \otimes \lambda^{\otimes^{n-j}}$$

with $\alpha_p, \beta_p \in \mathbb{R}_{\max}$. In [8] Olsder defines a variant of this equation using the dominant instead of the determinant. This leads to signed coefficients: $a_p^{\text{Olsder}} \in \mathbb{S}_{\max}^{\vee}$ or $\alpha_p^{\text{Olsder}} \otimes \beta_p^{\text{Olsder}} = \varepsilon$.

Theorem 1.5 (Cayley-Hamilton) *In $\mathbb{S}_{\max}$ every square matrix satisfies its characteristic equation.*

Proof: See [6] and [8].

2 The characteristic equation of a positive matrix

A positive matrix is a matrix the elements of which lie in $\mathbb{R}_{\max}$. In this section we derive necessary conditions for a polynomial in $\mathbb{S}_{\max}$ to be generated by a matrix with elements in $\mathbb{R}_{\max}$.

2.1 The characteristic equation

In this subsection we derive a formula for the coefficients of the characteristic equation.

Property 2.1 *Consider $A \in \mathbb{S}_{\max}^{n \times n}$ and $k \in \mathbb{S}_{\max}$, then $\det(k \otimes A) = k^{\otimes^n} \otimes \det A$.*

Proof:
$$\begin{aligned}\det(k\otimes A) &= \bigoplus_{\sigma\in\mathcal{P}_n} \operatorname{sgn}(\sigma)\otimes\bigotimes_{i=1}^{n}(k\otimes a_{i\sigma(i)})\\ &= \bigoplus_{\sigma\in\mathcal{P}_n} \operatorname{sgn}(\sigma)\otimes\left(k^{\otimes^n}\otimes\bigotimes_{i=1}^{n}(k\otimes a_{i\sigma(i)})\right)\\ &= k^{\otimes^n}\otimes\bigoplus_{\sigma\in\mathcal{P}_n} \operatorname{sgn}(\sigma)\otimes\bigotimes_{i=1}^{n}(k\otimes a_{i\sigma(i)})\\ &= k^{\otimes^n}\otimes\det A\end{aligned}$$

We know that $\det(A\ominus\lambda\otimes E_n)\;\nabla\;\varepsilon$ represents the characteristic equation of A. But because of property 2.1 we have that $\det(A\ominus\lambda\otimes E_n)=(\ominus 0)^{\otimes^n}\otimes\det(\lambda\otimes E_n\ominus A)$ and since $a\;\nabla\;\varepsilon$ if and only if $\ominus a\;\nabla\;\varepsilon$ the characteristic equation can also be represented by $\det(\lambda\otimes E_n\ominus A)\;\nabla\;\varepsilon$.

Definition 2.2 (Principal submatrix) *Let $A\in\mathbb{S}_{\max}^{n\times n}$ and let $\{i_1,i_2,\ldots,i_k\}$ be a combination of k elements out of $\{1,\ldots,n\}$. Then the matrix $A([i_1,i_2,\ldots,i_k],[i_1,i_2,\ldots,i_k])$ is a k by k principal submatrix of A. It can be obtained from A by deleting $n-k$ rows and columns.*

Every square n by n matrix A has $\begin{pmatrix} n\\ k\end{pmatrix}$ principal submatrices of size $k\times k$.

We represent the max-algebraic sum of the determinants of all k by k submatrices of A as $E_k(A)$:

$$E_k(A)=\bigoplus_{\varphi\in\mathcal{C}_n^k}\det A([i_1,i_2,\ldots,i_k],[i_1,i_2,\ldots,i_k])$$

where $\mathcal{C}_n^k$ is the set of all combinations of k numbers out of $\{1,\ldots,n\}$ and $\varphi=\{i_1,i_2,\ldots,i_k\}$.

Property 2.3 *If we represent the characteristic equation of $A\in\mathbb{S}_{\max}^{n\times n}$ as $\lambda^{\otimes^n}\oplus\bigoplus_{p=1}^{n}a_p\otimes\lambda^{\otimes^{n-p}}\;\nabla\;\varepsilon$ then $a_p=(\ominus 0)^{\otimes^p}\otimes E_p(A)$.*

Proof:

$$\det(\lambda\otimes E_n\ominus A)\;=\;\det\begin{bmatrix}\lambda\ominus a_{11} & \ominus a_{12} & \ldots & \ominus a_{1n}\\ \ominus a_{21} & \lambda\ominus a_{22} & \ldots & \ominus a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \ominus a_{n1} & \ominus a_{n2} & \ldots & \lambda\ominus a_{nn}\end{bmatrix}.$$

Since $\det(u_1,\ldots,u_i\oplus v_i,\ldots,u_n)=\det(u_1,\ldots,u_i,\ldots,u_n)\oplus\det(u_1,\ldots,v_i,\ldots,u_n)$ we can split this determinant up as

$$\det\begin{bmatrix}\lambda & \varepsilon & \ldots & \varepsilon\\ \varepsilon & \lambda & \ldots & \varepsilon\\ \vdots & \vdots & \ddots & \vdots\\ \varepsilon & \varepsilon & \ldots & \lambda\end{bmatrix}\oplus\det\begin{bmatrix}\ominus a_{11} & \varepsilon & \ldots & \varepsilon\\ \ominus a_{21} & \lambda & \ldots & \varepsilon\\ \vdots & \vdots & \ddots & \vdots\\ \ominus a_{n1} & \varepsilon & \ldots & \lambda\end{bmatrix}\oplus\det\begin{bmatrix}\lambda & \ominus a_{12} & \ldots & \varepsilon\\ \varepsilon & \ominus a_{22} & \ldots & \varepsilon\\ \vdots & \vdots & \ddots & \vdots\\ \varepsilon & \ominus a_{n2} & \ldots & \lambda\end{bmatrix}\oplus\ldots\oplus$$

$$\det\begin{bmatrix}\lambda & \varepsilon & \ldots & \ominus a_{1n}\\ \varepsilon & \lambda & \ldots & \ominus a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \varepsilon & \varepsilon & \ldots & \ominus a_{nn}\end{bmatrix}\oplus\ldots\oplus\bigoplus_{\varphi\in\mathcal{C}_n^p}\det B_\varphi\oplus\ldots\oplus\det\begin{bmatrix}\ominus a_{11} & \ominus a_{12} & \ldots & \ominus a_{1n}\\ \ominus a_{21} & \ominus a_{22} & \ldots & \ominus a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \ominus a_{n1} & \ominus a_{n2} & \ldots & \ominus a_{nn}\end{bmatrix}$$

with $\quad B_\varphi(:,i) = \ominus A(:,i) \qquad \text{if } i \in \varphi \ ,$
$\qquad\quad B_\varphi(:,i) = \lambda \otimes E_n(:,i) \quad \text{if } i \notin \varphi \ .$

If $\varphi = \{i_1, i_2, \ldots, i_p\}$, we have that

$$\det B_\varphi = \lambda^{\otimes^{n-p}} \otimes (\ominus 0)^{\otimes^p} \otimes \det A([i_1, i_2, \ldots, i_p], [i_1, i_2, \ldots, i_p])$$

since B_φ has $n-p$ diagonal entries that are equal to λ and p columns that contain elements of $\ominus A$. We have used property 2.1 to put the $\ominus$ signs before the determinant. So we find that

$$\det(\lambda \otimes E_n \ominus A) = \bigoplus_{p=0}^{n} (\ominus 0)^{\otimes^p} \otimes E_p(A) \otimes \lambda^{\otimes^{n-p}}$$

with $E_p(A)$ the max-algebraic sum of the determinants of all possible p by p principal submatrices of A.

This results in:
$$\begin{aligned} a_0 &= 0 \ , \\ a_1 &= \ominus \mathrm{tr}\,(A) = \ominus \bigoplus_{i=1}^{n} a_{ii} \ , \\ a_n &= (\ominus 0)^{\otimes^n} \otimes \det A \ . \end{aligned}$$

Example 2.4 Consider $A = \begin{bmatrix} 0 & 5 & 9 \\ 5 & 20 & 10 \\ 9 & 10 & 18 \end{bmatrix}$.

The characteristic equation of A is $\lambda^{\otimes^3} \ \ominus \ 20 \otimes \lambda^{\otimes^2} \ \oplus \ 38 \otimes \lambda \ \oplus \ 38^\bullet \ \nabla \ \varepsilon$.

2.2 Properties of the characteristic equation

Proposition 2.5 *In $\mathbb{S}_{\max}$ every monic n-th degree linear balance is the characteristic equation of an $n \times n$ matrix.*

Proof: Suppose that the linear balance has the following form

$$\lambda^{\otimes^n} \ \oplus \ a_1 \otimes \lambda^{\otimes^{n-1}} \ \oplus \ \ldots \ \oplus \ a_{n-1} \otimes \lambda \ \oplus \ a_n \ \nabla \ \varepsilon \ .$$

We shall prove that this is the characteristic equation of the matrix

$$A = \begin{bmatrix} \varepsilon & 0 & \varepsilon & \ldots & \varepsilon \\ \varepsilon & \varepsilon & 0 & \ldots & \varepsilon \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \varepsilon & \ldots & 0 \\ \ominus a_n & \ominus a_{n-1} & \ominus a_{n-2} & \ldots & \ominus a_1 \end{bmatrix} .$$

We use the formula of property 2.3 to calculate the coefficients of $\lambda^{\otimes^{n-p}}$ in the characteristic equation of A. If we take the p by p principal submatrices of A we see that each of them has an ε-column – and thus a determinant equal to ε – except for $B_p = A([n{-}p{+}1 : n], [n{-}p{+}1 : n])$.

So $a_p = (\ominus 0)^{\otimes^p} \otimes E_p(A) = (\ominus 0)^{\otimes^p} \otimes \det B_p$. B_p is obtained by deleting the first $n-p$ rows and columns of A, so

$$B_p = \begin{bmatrix} \varepsilon & 0 & \varepsilon & \dots & \varepsilon \\ \varepsilon & \varepsilon & 0 & \dots & \varepsilon \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \varepsilon & \dots & 0 \\ \ominus a_p & \ominus a_{p-1} & \ominus a_{p-2} & \dots & \ominus a_1 \end{bmatrix} .$$

If we develop the determinant of B_p to the first column we get $\det B_p = (\ominus 0)^{\otimes^{p-1}} \otimes (\ominus a_p) = (\ominus 0)^{\otimes^p} \otimes a_p$ so the coefficient of $\lambda^{\otimes^{n-p}}$ equals $(\ominus 0)^{\otimes^p} \otimes (\ominus 0)^{\otimes^p} \otimes a_p = (\ominus 0)^{\otimes^{2p}} \otimes a_p = a_p$. Thus the linear balance is indeed the characteristic equation of A.

In the next section we shall see that not every monic polynomial corresponds to the characteristic polynomial of a positive matrix.

2.3 Properties of the characteristic polynomial of positive matrices

Property 2.6 *If $A \in \mathbb{R}_{\max}^{n \times n}$ then $a_1 \in \mathbb{S}_{\max}^{\ominus}$.*

Proof: We know that $a_1 = \ominus \mathrm{tr}\,(A) = \ominus \bigoplus_{i=1}^{n} a_{ii}$ with $a_{ii} \in \mathbb{R}_{\max}$ so $a_1 \in \mathbb{S}_{\max}^{\ominus}$.

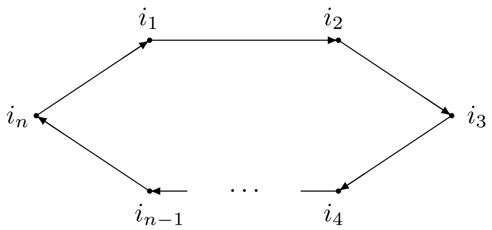
To prove the following property we first need a lemma involving permutations. The parity of a permutation can be determined in various ways. We use:

Property 2.7 *The parity of a permutation is equal to the parity of the number of its elementary cycles of even length.*

First consider a circular permutation σ_c of n elements:

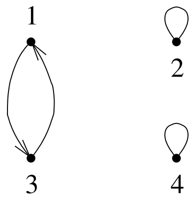
$$\sigma_c(i_1) = i_2,\, \sigma_c(i_2) = i_3,\, \dots, \sigma_c(i_{n-1}) = i_n,\, \sigma_c(i_n) = i_1 \quad .$$

The graph of this permutation is



This permutation has a cycle of length n.
If n is even, then $\sigma_c \in \mathcal{P}_n$ is odd because there is 1 cycle of even length.
If n is odd, then $\sigma_c \in \mathcal{P}_n$ is even because there are 0 cycles of even length.
If a permutation of n numbers is not circular we can decompose it into r elementary cycles C_i of length l_i, with $r > 1$ and $\sum_{i=1}^{r} l_i = n$. Each cycle will be a circular permutation.

Example 2.8 Let $\sigma \in \mathcal{P}_4$ be defined as $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 1$ and $\sigma(4) = 4$. The graph of this permutation is



This permutation can be decomposed into three elementary cycles, each of which is a circular permutation of its vertices. There is one cycle of even length $(1 \to 3 \to 1)$. So σ is an odd permutation.

Lemma 2.9 *If $\sigma_{2k,\mathrm{even}}$ $(k > 0)$ is an even permutation of $2k$ elements, then it can be decomposed into two even permutations of an odd number of elements or two odd permutations of an even number of elements:*

$$\sigma_{2k,\mathrm{even}} = \sigma_{2l+1,\mathrm{even}} \cup \sigma_{2k-2l-1,\mathrm{even}} \quad or \quad \sigma_{2m,\mathrm{odd}} \cup \sigma_{2k-2m,\mathrm{odd}} \ .$$

If $\sigma_{2k+1,\mathrm{odd}}$ $(k > 0)$ is an odd permutation of $2k+1$ elements, then it can be decomposed into an even permutation of an odd number of elements and an odd permutation of an even number of elements:

$$\sigma_{2k+1,\mathrm{odd}} = \sigma_{2p+1,\mathrm{even}} \cup \sigma_{2k-2p,\mathrm{odd}}. \ .$$

Proof:

First consider $\sigma_{2k,\mathrm{even}}$. This is an even permutation of an even number of elements so it is not circular and it can be decomposed into elementary cycles. Suppose that there are c_{even} cycles of even length each having $n_{\mathrm{even},i}$ elements and c_{odd} cycles of odd length each having $n_{\mathrm{odd},j}$ elements. Let $n_{\mathrm{tot,even}} = \sum_{i=1}^{c_{\mathrm{even}}} n_{\mathrm{even},i}$ and $n_{\mathrm{tot,odd}} = \sum_{j=1}^{c_{\mathrm{odd}}} n_{\mathrm{odd},j}$. Since the parity of $\sigma_{2k,\mathrm{even}}$ is even, c_{even} should also be even. $n_{\mathrm{tot,even}}$ is always even. The total number of elements $n_{\mathrm{tot}} = 2k$ is even, so we have that $n_{\mathrm{tot,odd}}$ is also even and hence that c_{odd} is even. There are two cases: $c_{\mathrm{even}} = 0$ and $c_{\mathrm{even}} \neq 0$.

If $c_{\mathrm{even}} = 0$ then $c_{\mathrm{odd}} \neq 0$ because $2k \neq 0$. Take one cycle of odd length $2l+1$. This corresponds to an even permutation of $2l+1$ elements: $\sigma_{2l+1,\mathrm{even}}$. The other cycles form a permutation with 0 cycles of even length, so it is an even permutation of the remaining $2k-2l-1$ elements: $\sigma_{2k-2l-1,\mathrm{even}}$.

If $c_{\mathrm{even}} \neq 0$ we take one cycle of even length $2m$. This corresponds to $\sigma_{2m,\mathrm{odd}}$. The remaining cycles constitute a permutation with an odd number $(c_{\mathrm{even}} - 1)$ of cycles of even length: $\sigma_{2k-2m,\mathrm{odd}}$.

So we have proven that $\sigma_{2k,\mathrm{even}}$ can be decomposed as $\sigma_{2l+1,\mathrm{even}} \cup \sigma_{2k-2l-1,\mathrm{even}}$ or $\sigma_{2m,\mathrm{odd}} \cup \sigma_{2k-2m,\mathrm{odd}}$.

Now consider $\sigma_{2k+1,\mathrm{odd}}$. This is an odd permutation of an odd number of elements so it is not circular and it can be decomposed into elementary cycles. Since the parity of $\sigma_{2k+1,\mathrm{odd}}$ is odd, c_{even} should also be odd. $n_{\mathrm{tot,even}}$ is always even, and since the total number of elements $n_{\mathrm{tot}} = 2k+1$ is odd we have that $n_{\mathrm{tot,odd}}$ is odd and hence that c_{odd} is odd. This means that $c_{\mathrm{odd}} \neq 0$. So let us take one cycle of odd length $2p+1$. This corresponds to an even

permutation of $2p+1$ elements: $\sigma_{2p+1,\text{even}}$.
The other cycles will then correspond to a permutation of with an odd number (c_{even}) of cycles of even length, so it is an odd permutation of $2k-2p$ elements: $\sigma_{2k-2p,\text{odd}}$.
So $\sigma_{2k+1,\text{odd}} = \sigma_{2p+1,\text{even}} \cup \sigma_{2k-2p,\text{odd}}$.

Now we give some properties of $a_p = (\ominus 0)^{\otimes^p} \otimes E_p(A) = a_p^+ \ominus a_p^-$. First we suppose that we don't simplify $\ominus$. This means that for $a = 3 \ominus 4$ we have $a^+ = 3$ and $a^- = 4$. Later we shall see how we have to adapt the properties to take simplification into account, because then we shall have that $a = 3 \ominus 4$ results in $a = \ominus 4$ or $a^+ = \varepsilon$ and $a^- = 4$.

Property 2.10 *Let $A \in \mathbb{R}_{\max}^{n\times n}$ and let $a_p = (\ominus 0)^{\otimes^p} \otimes E_p(A) = a_p^+ \ominus a_p^-$ (without simplifying $\ominus$). Then $\forall p \in \{2,\ldots,n\}$: $a_p^+ \leqslant \bigoplus_{r=1}^{\lfloor \frac{p}{2} \rfloor} a_r^- \otimes a_{p-r}^-$, where $\lfloor x \rfloor$ stands for the largest integer number less than or equal to x.*

Proof: We know that

$$
\begin{aligned}
a_p &= (\ominus 0)^{\otimes^p} \otimes E_p(A) \\
&= (\ominus 0)^{\otimes^p} \bigoplus_{\varphi \in \mathcal{C}_n^p} \bigoplus_{\sigma \in \mathcal{P}_p} \operatorname{sgn}(\sigma) \otimes a_{i_1 i_{\sigma(1)}} \otimes a_{i_2 i_{\sigma(2)}} \otimes \ldots \otimes a_{i_p i_{\sigma(p)}}
\end{aligned}
$$

with $\varphi = \{i_1, i_2, \ldots, i_p\}$.
If we extract the positive and the negative part of a_p (without simplifying $\ominus$), we find for $k > 0$:

$$a_{2k}^+ = \bigoplus_{\varphi \in \mathcal{C}_n^{2k}} \bigoplus_{\sigma \in \mathcal{P}_{2k,\text{even}}} a_{i_1 i_{\sigma(1)}} \otimes a_{i_2 i_{\sigma(2)}} \otimes \ldots \otimes a_{i_{2k} i_{\sigma(2k)}} \tag{1}$$

$$a_{2k}^- = \bigoplus_{\varphi \in \mathcal{C}_n^{2k}} \bigoplus_{\sigma \in \mathcal{P}_{2k,\text{odd}}} a_{i_1 i_{\sigma(1)}} \otimes a_{i_2 i_{\sigma(2)}} \otimes \ldots \otimes a_{i_{2k} i_{\sigma(2k)}} \tag{2}$$

$$a_{2k+1}^+ = \bigoplus_{\varphi \in \mathcal{C}_n^{2k+1}} \bigoplus_{\sigma \in \mathcal{P}_{2k+1,\text{odd}}} a_{i_1 i_{\sigma(1)}} \otimes a_{i_2 i_{\sigma(2)}} \otimes \ldots \otimes a_{i_{2k+1} i_{\sigma(2k+1)}} \tag{3}$$

$$a_{2k+1}^- = \bigoplus_{\varphi \in \mathcal{C}_n^{2k+1}} \bigoplus_{\sigma \in \mathcal{P}_{2k+1,\text{even}}} a_{i_1 i_{\sigma(1)}} \otimes a_{i_2 i_{\sigma(2)}} \otimes \ldots \otimes a_{i_{2k+1} i_{\sigma(2k+1)}} \ . \tag{4}$$

Let us first consider a_{2k}^+. The terms of a_{2k}^+ are generated by even permutations of $2k$ elements. According to lemma 2.9 such a permutation can be decomposed into two even permutations of odd lengths or two odd permutations of even lengths. So if we consider all possible concatenations of two even permutations of odd lengths (corresponding to $a_{2l+1}^- \otimes a_{2k-2l-1}^-$) or two odd permutations of even length ($a_{2m}^- \otimes a_{2k-2m}^-$), we are sure to have included all terms of a_{2k}^+. In other words $a_{2k}^+ \leqslant \bigoplus_{r=1}^{2k-1} a_r^- \otimes a_{2k-r}^-$. Since $(a_r^- \otimes a_{2k-r}^-) \oplus (a_{2k-r}^- \otimes a_r^-) = a_r^- \otimes a_{2k-r}^-$ we find $a_{2k}^+ \leqslant \bigoplus_{r=1}^{k} a_r^- \otimes a_{2k-r}^-$.

Now consider a_{2k+1}^{+}, the terms of which are generated by odd permutations of $2k+1$ elements. Lemma 2.9 also tells us that such a permutation can be decomposed into an odd permutation of an even number of elements and an even permutation of an odd number of elements. Using the same reasoning as for a_{2k}^{+}, we find that $a_{2k+1}^{+} \leqslant \bigoplus_{r=1}^{k} a_r^{-} \otimes a_{2k+1-r}^{-}$.

Combining the two inequalities leads to $a_p^{+} \leqslant \bigoplus_{r=1}^{\lfloor \frac{p}{2} \rfloor} a_r^{-} \otimes a_{p-r}^{-}$.

We don't have a similar expression for a_p^{-} because then some of the generating permutations are circular, and these cannot be decomposed into more than one elementary cycle.

Normally we simplify $\ominus$, by setting $a_p^{-} = \varepsilon$ if $a_p^{-} < a_p^{+}$ and $a_p^{+} = \varepsilon$ if $a_p^{-} > a_p^{+}$.
Therefore we shall from now on represent the characteristic equation of $A \in \mathbb{R}_{\max}^{n \times n}$ as

$$\lambda^{\otimes^n} \oplus \bigoplus_{i=2}^{n} \alpha_i \otimes \lambda^{\otimes^{n-i}} \;\nabla\; \beta_1 \otimes \lambda^{\otimes^{n-1}} \oplus \bigoplus_{j=2}^{n} \beta_j \otimes \lambda^{\otimes^{n-j}}$$

with $\begin{cases} \alpha_p = a_p^{+}, & \beta_p = \varepsilon & \text{if } a_p^{+} > a_p^{-} \;, \\ \alpha_p = \varepsilon, & \beta_p = a_p^{-} & \text{if } a_p^{+} < a_p^{-} \;, \\ \alpha_p = a_p^{+}, & \beta_p = a_p^{-} & \text{if } a_p^{+} = a_p^{-} \;. \end{cases}$

So there are three possible cases: $\alpha_p = \varepsilon$, $\beta_p = \varepsilon$ or $\alpha_p = \beta_p$. We already have omitted α_1 because property 2.6 leads to $\alpha_1 = a_1^{+} = \varepsilon$.
We have that $\alpha_p \leqslant a_p^{+}$, $\beta_p \leqslant a_p^{-}$ and $|a_p| = \alpha_p \oplus \beta_p$.

Property 2.11 $\forall i \in \{2, \ldots, n\}$: $\alpha_i \leqslant \bigoplus_{r=1}^{\lfloor \frac{i}{2} \rfloor} (\alpha_r \oplus \beta_r) \otimes (\alpha_{i-r} \oplus \beta_{i-r})$, *where* $\lfloor x \rfloor$ *stands for the largest integer number less than or equal to* x.

Proof: Using the fact that $a_i^{-} \leqslant |a_i|$ property 2.10 leads to

$$\begin{aligned} a_i^{+} &\leqslant \bigoplus_{r=1}^{\lfloor \frac{i}{2} \rfloor} |a_r^{-}| \otimes |a_{i-r}^{-}| \\ &\leqslant \bigoplus_{r=1}^{\lfloor \frac{i}{2} \rfloor} (\alpha_r \oplus \beta_r) \otimes (\alpha_{i-r} \oplus \beta_{i-r}) \;. \end{aligned}$$

We also know that $\alpha_i \leqslant a_i^{+}$. So

$$\alpha_i \leqslant \bigoplus_{r=1}^{\lfloor \frac{i}{2} \rfloor} (\alpha_r \oplus \beta_r) \otimes (\alpha_{i-r} \oplus \beta_{i-r}) \;.$$

We even have a more stringent property:

Property 2.12 $\forall i \in \{2, \ldots, n\}$ *at least one of the following statements is true:*

$$
\begin{array}{lll}
1) & \alpha_i & \leqslant \displaystyle\bigoplus_{r=1}^{\lfloor \frac{i}{2} \rfloor} \beta_r \otimes \beta_{i-r} \\
2) & \alpha_i & < \displaystyle\bigoplus_{r=2}^{\lfloor \frac{i}{2} \rfloor} \alpha_r \otimes \alpha_{i-r} \\
3) & \alpha_i & < \displaystyle\bigoplus_{r=2}^{i-1} \alpha_r \otimes \beta_{i-r}
\end{array}
$$

where $\lfloor x \rfloor$ *stands for the largest integer number less than or equal to* x*.*

Proof: Take an arbitrary $i \in \{2, \ldots, n\}$. Then according to property 2.10 there exists an $s \leqslant \left\lfloor \dfrac{i}{2} \right\rfloor$ such that

$$\alpha_i \leqslant a_i^+ \leqslant a_s^- \otimes a_{i-s}^- \ .$$

We have that either $a_s^- = \beta_s$ or $a_s^- < \alpha_s$ and the same goes for a_{i-s}^-.
This means that at least one of the following inequalities holds:

$$
\begin{array}{ll}
1) & \alpha_i \leqslant \beta_s \otimes \beta_{i-s} \leqslant \displaystyle\bigoplus_{r=1}^{\lfloor \frac{i}{2} \rfloor} \beta_r \otimes \beta_{i-r} \\
2) & \alpha_i < \alpha_s \otimes \alpha_{i-s} \leqslant \displaystyle\bigoplus_{r=2}^{\lfloor \frac{i}{2} \rfloor} \alpha_r \otimes \alpha_{i-r} \\
3) & \alpha_i < \beta_s \otimes \alpha_{i-s} \ \oplus \ \alpha_s \otimes \beta_{i-s} \leqslant \displaystyle\bigoplus_{r=2}^{i-1} \alpha_r \otimes \beta_{i-r} \ .
\end{array}
$$

In the last two max-algebraic sums we can start from $r = 2$ because $\alpha_1 = \varepsilon$.

Property 2.12 gives necessary conditions for the coefficients of an $\mathbb{S}_{\max}$ polynomial such that it is the characteristic polynomial of a positive matrix.

3 Necessary and sufficient conditions for a polynomial to be the characteristic polynomial of a positive matrix

We now give some necessary and sufficient conditions for the coefficients of the characteristic equation of a positive matrix. These conditions will play an important role when one wants to determine the minimal order of a SISO system in the max algebra, as will be shown in section 4.
We shall prove that the conditions are sufficient by giving for each set of conditions a matrix the characteristic equation of which will satisfy the conditions.
So if we have a monic polynomial in $\mathbb{S}_{\max}$ the results of this section will allow us to

1. check whether the given polynomial can be the characteristic polynomial of a positive matrix and
2. construct a matrix such that its characteristic polynomial is equal to the given polynomial.

For the lower dimensional cases we can give an analytic description of the matrix we are looking for. For higher dimensional cases we shall first state a conjecture and then develop a heuristic algorithm that will (in most cases) find a solution.
In this section we shall encounter matrices with the following structure:

$$A = \begin{bmatrix} \kappa_{0,1} & \kappa_{0,2} & \kappa_{0,3} & \cdots & \kappa_{0,n-1} & \kappa_{0,n} \\ 0 & \kappa_{1,2} & \kappa_{1,3} & \cdots & \kappa_{1,n-1} & \kappa_{1,n} \\ \varepsilon & 0 & \kappa_{2,3} & \cdots & \kappa_{2,n-1} & \kappa_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \varepsilon & \varepsilon & \varepsilon & \cdots & \kappa_{n-2,n-1} & \kappa_{n-2,n} \\ \varepsilon & \varepsilon & \varepsilon & \cdots & 0 & \kappa_{n-1,n} \end{bmatrix}$$

The coefficients of the characteristic equation of A (without simplification of $\ominus$) are given by:

$$a_k^+ \;=\; \bigoplus_{\phi \in \mathcal{C}_n^p \,,\, p \text{ even}} \; \bigoplus_{\sum_{r=1}^{p} (j_r - i_r) = k \,,\, i_r < j_r \leqslant i_{r+1}} \kappa_{i_1,j_1} \otimes \kappa_{i_2,j_2} \otimes \ldots \otimes \kappa_{i_p,j_p} \tag{5}$$

$$a_k^- \;=\; \bigoplus_{\phi \in \mathcal{C}_n^p \,,\, p \text{ odd}} \; \bigoplus_{\sum_{r=1}^{p} (j_r - i_r) = k \,,\, i_r < j_r \leqslant i_{r+1}} \kappa_{i_1,j_1} \otimes \kappa_{i_2,j_2} \otimes \ldots \otimes \kappa_{i_p,j_p} \tag{6}$$

where $\phi = \{i_1, i_2, \ldots, i_p\}$.

Proof: We know that $a_k = (\ominus 0)^{\otimes^k} \otimes E_k(A)$ with

$$E_k(A) = \bigoplus_{\phi \in \mathcal{C}_n^k} \det A([i_1, i_2, \ldots, i_k], [i_1, i_2, \ldots, i_k])$$

and by reordering if necessary: $i_1 < i_2 < \ldots < i_k$.
Now let

$$\begin{aligned} B(i_1, i_2, \ldots, i_k) &= A([i_1, i_2, \ldots, i_k], [i_1, i_2, \ldots, i_k]) \\ &= \begin{bmatrix} \kappa_{i_1-1,i_1} & \kappa_{i_1-1,i_2} & \kappa_{i_1-1,i_3} & \cdots & \kappa_{i_1-1,i_k} \\ \gamma_{i_2,i_1} & \kappa_{i_2-1,i_2} & \kappa_{i_2-1,i_3} & \cdots & \kappa_{i_2-1,i_k} \\ \varepsilon & \gamma_{i_3,i_2} & \kappa_{i_3-1,i_3} & \cdots & \kappa_{i_3-1,i_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \varepsilon & \cdots & \kappa_{i_k-1,i_k} \end{bmatrix} \end{aligned}$$

with $\begin{cases} \gamma_{i_{r+1},i_r} & = 0 \quad \text{if} \quad i_{r+1} = i_r + 1 \;, \\ & = \varepsilon \quad \text{if} \quad i_{r+1} \neq i_r + 1 \;. \end{cases}$

We shall prove by induction that

$$\det B(i_1, i_2, \ldots, i_K) =$$

$$(\ominus 0)^{\otimes^K} \otimes \left[\bigoplus_{\phi \in \mathcal{C}_n^p \,,\, p \text{ even}} \; \bigoplus_{\sum_{r=1}^{p} (j_r - i_r) = K \,,\, i_r < j_r \leqslant i_{r+1}} \kappa_{i_1,j_1} \otimes \ldots \otimes \kappa_{i_p,j_p} \;\ominus \right.$$

$$\left. \bigoplus_{\phi \in \mathcal{C}_n^p \,,\, p \text{ odd}} \; \bigoplus_{\sum_{r=1}^{p} (j_r - i_r) = K \,,\, i_r < j_r \leqslant i_{r+1}} \kappa_{i_1,j_1} \otimes \ldots \otimes \kappa_{i_p,j_p} \right] \;. \tag{7}$$

Note that K the number of columns of $B(i_1, i_2, \dots, i_K)$.

$\underline{K=1}$

$\det B(i_1) = \det[\kappa_{i_1-1,i_1}] = \kappa_{i_1-1,i_1}$ with $i_1 - (i_1 - 1) = 1 = K$ and $i_1 - 1 < i_1$ so equation (7) is satisfied.

$\underline{K=k}$

Suppose that equation (7) holds for $K = 1, 2, \dots k-1$.
By developing the determinant to the first column we get

$$\begin{aligned}
\det B(i_1, i_2, \dots, i_k) &= \kappa_{i_1-1,i_1} \otimes \det B(i_2, \dots, i_k) \ominus \\
&\quad \gamma_{i_2,i_1} \otimes \{\kappa_{i_1-1,i_2} \otimes \det B(i_3, \dots, i_k) \ominus \\
&\quad\ \ \gamma_{i_3,i_2} \otimes [\kappa_{i_1-1,i_3} \otimes \det B(i_4, \dots, i_k) \ \ominus\ \dots]\} \\
&= \kappa_{i_1-1,i_1} \otimes \det B(i_2, \dots, i_k) \\
&\quad \ominus \gamma_{i_2,i_1} \otimes \kappa_{i_1-1,i_2} \otimes \det B(i_3, \dots, i_k) \\
&\quad \oplus \gamma_{i_3,i_2} \otimes \gamma_{i_2,i_1} \otimes \kappa_{i_1-1,i_3} \otimes \det B(i_4, \dots, i_k) \ \oplus\ \dots\ \oplus \\
&\quad (\ominus 0)^{\otimes^{k-2}} \otimes \gamma_{i_{k-1},i_{k-2}} \otimes \dots \otimes \gamma_{i_2,i_1} \otimes \kappa_{i_1-1,i_{k-1}} \otimes \det B(i_k) \\
&\quad (\ominus 0)^{\otimes^{k-1}} \otimes \gamma_{i_k,i_{k-1}} \otimes \dots \otimes \gamma_{i_2,i_1} \otimes \kappa_{i_1-1,i_k} \ .
\end{aligned}$$

Consider the l-th term

$$(\ominus 0)^{\otimes^{l-1}} \otimes \gamma_{i_{k-1},i_{k-2}} \otimes \dots \otimes \gamma_{i_2,i_1} \otimes \kappa_{i_1-1,i_l} \otimes \det B(i_{l+1}, \dots, i_k) \ .$$

We have that $K = k - (l+1) + 1 = k - l$ for $B(i_{l+1}, \dots, i_k)$ so

$$\det B(i_{l+1}, \dots, i_k) =$$

$$(\ominus 0)^{\otimes^{k-l}} \otimes \left[\bigoplus_{\phi \in \mathcal{C}_n^p\,,\, p \text{ even}} \ \bigoplus_{\sum_{r=1}^p (j_r - i_r) = k-l\,,\, i_r < j_r \leqslant i_{r+1}} \kappa_{i_1,j_1} \otimes \dots \otimes \kappa_{i_p,j_p} \ \ominus \right.$$

$$\left. \bigoplus_{\phi \in \mathcal{C}_n^p\,,\, p \text{ odd}} \ \bigoplus_{\sum_{r=1}^p (j_r - i_r) = k-l\,,\, i_r < j_r \leqslant i_{r+1}} \kappa_{i_1,j_1} \otimes \dots \otimes \kappa_{i_p,j_p} \right] \ .$$

$$\text{Now}\quad \gamma_{i_{k-1},i_{k-2}} \otimes \dots \otimes \gamma_{i_2,i_1} \begin{array}{ll} = 0 & \text{if} \ \left.\begin{array}{rcl} i_l &=& i_{l-1}+1 \\ & \vdots & \\ i_3 &=& i_2+1 \\ i_2 &=& i_1+1 \end{array}\right\} \text{ or } \ i_l = i_1 + l - 1 \ , \\ = \varepsilon & \text{otherwise} \ . \end{array}$$

So if $i_l = i_1 + l - 1$ the l-th term becomes

$$(\ominus 0)^{\otimes^{l-1}} \otimes \kappa_{i_1-1,i_1+l-1} \otimes (\ominus 0)^{\otimes^{k-l}} \otimes$$

$$\left[\bigoplus_{p \text{ even}} \ \bigoplus_{\sum (j_r - i_r) = k-l} \ \bigotimes_{r=1}^{p} \kappa_{i_r,j_r} \ \ominus \ \bigoplus_{p \text{ odd}} \ \bigoplus_{\sum (j_r - i_r) = k-l} \ \bigotimes_{r=1}^{p} \kappa_{i_r,j_r} \right]$$

or

$$(\ominus 0)^{\otimes^{k-1}} \otimes \left[\bigoplus_{p \text{ odd}} \bigoplus_{\sum(j_r - i_r)=k} \bigotimes_{r=1}^{p} \kappa_{i_r,j_r} \ominus \bigoplus_{p \text{ even}} \bigoplus_{\sum(j_r - i_r)=k} \bigotimes_{r=1}^{p} \kappa_{i_r,j_r} \right] .$$

So we finally find that

$$\begin{aligned} a_k &= (\ominus 0)^{\otimes^k} \bigoplus_{i_1<\ldots<i_k} \det B(i_1,\ldots,i_k) \\ &= \bigoplus_{p \text{ even}} \bigoplus_{\sum(j_r - i_r)=k} \bigotimes_{r=1}^{p} \kappa_{i_r,j_r} \ominus \bigoplus_{p \text{ odd}} \bigoplus_{\sum(j_r - i_r)=k} \bigotimes_{r=1}^{p} \kappa_{i_r,j_r} . \end{aligned}$$

Extracting the positive and the negative parts leads to equations (5) and (6).

In the next subsections we shall case by case determine necessary and sufficient conditions for

$$\lambda^{\otimes^n} \oplus \bigoplus_{i=1}^{n} \alpha_i \otimes \lambda^{\otimes^{n-i}} \nabla \bigoplus_{j=1}^{n} \beta_j \otimes \lambda^{\otimes^{n-j}} \tag{8}$$

to be the characteristic equation of a positive matrix and indicate how such a matrix can be found. In all cases we have $\alpha_1 = \varepsilon$ as a necessary condition.

We also define $$\begin{aligned} \kappa_{i,j} &= \frac{\alpha_j}{\beta_i} && \text{if} \quad \beta_i \neq \varepsilon \ , \\ &= \varepsilon && \text{if} \quad \beta_i = \varepsilon \ . \end{aligned}$$

3.1 The 1×1 case

The only necessary and also sufficient condition is $\alpha_1 = \varepsilon$.
The matrix $[\beta_1]$ has $\lambda \nabla \beta_1$ as its characteristic equation.

3.2 The 2×2 case

The necessary and also sufficient conditions are $\begin{cases} \alpha_1 = \varepsilon \\ \alpha_2 \leqslant \beta_1 \otimes \beta_1 \ . \end{cases}$

The matrix $\begin{bmatrix} \beta_1 & \beta_2 \\ 0 & \kappa_{1,2} \end{bmatrix}$ has $\lambda^{\otimes^2} \oplus \alpha_2 \nabla \beta_1 \otimes \lambda \oplus \beta_2$ as its characteristic equation.

Proof:
From property 2.10 we know that $\alpha_2 \leqslant \beta_1 \otimes \beta_1$. This means that if $\beta_1 = \varepsilon$ then also $\alpha_2 = \varepsilon$.

First we prove that $\beta_1 \otimes \kappa_{1,2} = \alpha_2$:

$$\begin{aligned} \beta_1 \otimes \kappa_{1,2} &= \beta_1 \otimes \frac{\alpha_2}{\beta_1} &&= \alpha_2 && \text{if } \beta_1 \neq \varepsilon \ , \\ &= \varepsilon \otimes \varepsilon &&= \varepsilon && \text{if } \beta_1 = \varepsilon \text{ and thus also } \alpha_2 = \varepsilon \ . \end{aligned}$$

We always have that $\kappa_{1,2} \leqslant \beta_1$ because

$$
\begin{aligned}
\kappa_{1,2} &= \frac{\alpha_2}{\beta_1} &\leqslant \beta_1 &\quad \text{if } \beta_1 \neq \varepsilon \ , \\
&= \varepsilon &\leqslant \beta_1 &\quad \text{if } \beta_1 = \varepsilon \ .
\end{aligned}
$$

Using formulas (5) and (6) we find

$$
\begin{aligned}
a_1 &= \ominus\beta_1 \ominus \kappa_{1,2} \\
&= \ominus\beta_1 \\
a_2 &= \beta_1 \otimes \kappa_{1,2} \ominus \beta_2 \\
&= \alpha_2 \ominus \beta_2 \ .
\end{aligned}
$$

3.3 The 3×3 case

The necessary and also sufficient conditions are $\begin{cases} \alpha_1 = \varepsilon \\ \alpha_2 \leqslant \beta_1 \otimes \beta_1 \\ \alpha_3 \leqslant \beta_1 \otimes \beta_2 \text{ or } \alpha_3 < \beta_1 \otimes \alpha_2 \ . \end{cases}$

The matrix $\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \\ 0 & \kappa_{1,2} & \kappa_{1,3} \\ \varepsilon & 0 & \varepsilon \end{bmatrix}$ has $\lambda^{\otimes^3} \ \oplus \ \alpha_2 \otimes \lambda \ \oplus \ \alpha_3 \ \nabla \ \beta_1 \otimes \lambda^{\otimes^2} \ \oplus \ \beta_2 \otimes \lambda \ \oplus \ \beta_3$ as its characteristic equation.

Proof:
We already know that $\kappa_{1,2} \leqslant \beta_1$ and that $\beta_1 \otimes \kappa_{1,2} = \alpha_2$. Analogously we can prove that $\beta_1 \otimes \kappa_{1,3} = \alpha_3$ since if $\beta_1 = \varepsilon$ we also have $\alpha_3 = \varepsilon$.

Now we prove that $\kappa_{1,3} \leqslant \beta_2$ if $\beta_2 \geqslant \alpha_2$ and that $\kappa_{3,1} < \alpha_3$ if $\beta_2 < \alpha_2$:
If $\beta_2 \geqslant \alpha_2$ the necessary condition for α_3 becomes $\alpha_3 \leqslant \beta_2 \otimes \beta_1$. So

$$
\begin{aligned}
\kappa_{1,3} &= \frac{\alpha_3}{\beta_1} &\leqslant \beta_2 &\quad \text{if } \beta_1 \neq \varepsilon \ , \\
&= \varepsilon &\leqslant \beta_2 &\quad \text{if } \beta_1 = \varepsilon \ .
\end{aligned}
$$

If $\beta_2 < \alpha_2$ we have that $\alpha_3 < \alpha_2 \otimes \beta_1$ so $\beta_1 \neq \varepsilon$ and

$$
\kappa_{1,3} = \frac{\alpha_3}{\beta_1} < \alpha_2 \ .
$$

So we always have that $\alpha_2 \ominus \beta_2 \ominus \kappa_{1,3} = \alpha_2 \ominus \beta_2$.

We find

$$
\begin{aligned}
a_1 &= \ominus\beta_1 \ominus \kappa_{1,2} \\
&= \ominus\beta_1 \\
a_2 &= \beta_1 \otimes \kappa_{1,2} \ominus \beta_2 \ominus \kappa_{1,3} \\
&= \alpha_2 \ominus \beta_2 \\
a_3 &= \beta_1 \otimes \kappa_{1,3} \ominus \beta_3 \\
&= \alpha_3 \ominus \beta_3 \ .
\end{aligned}
$$

3.4 The 4×4 case

First we distinguish three possible cases:

Case A: $\alpha_4 \leqslant \beta_1 \otimes \beta_3$ or $\alpha_4 < \beta_1 \otimes \alpha_3$

Case B: $\alpha_4 > \beta_1 \otimes \beta_3$ and $\alpha_4 \geqslant \beta_1 \otimes \alpha_3$ and $\alpha_4 \leqslant \beta_2 \otimes \beta_2$ and
$(\ \beta_1 = \varepsilon$ or $\alpha_2 = \varepsilon$ or $\beta_4 = \alpha_4\ )$

Case C: $\alpha_4 > \beta_1 \otimes \beta_3$ and $\alpha_4 \geqslant \beta_1 \otimes \alpha_3$ and $\alpha_4 \leqslant \beta_2 \otimes \beta_2$ and
$\alpha_2 = \beta_2 \neq \varepsilon$ and $\beta_4 = \varepsilon$.

If the coefficients don't fall into (exactly) one of these three cases, they cannot correspond to a positive matrix.

The necessary and sufficient conditions are:

$$\begin{cases} \alpha_1 = \varepsilon \\ \alpha_2 \leqslant \beta_1 \otimes \beta_1 \\ \alpha_3 \leqslant \beta_1 \otimes \beta_2 \text{ or } \alpha_3 < \beta_1 \otimes \alpha_2 \\ \text{for Case A: no extra conditions} \\ \text{for Case B: } \beta_1 \otimes \alpha_4 \leqslant \beta_2 \otimes \alpha_3 \text{ or } \beta_1 \otimes \alpha_4 < \beta_2 \otimes \beta_3 \\ \text{for Case C: } \beta_1 \otimes \alpha_3 = \beta_2 \otimes \alpha_2 \text{ and } \beta_1 \otimes \alpha_4 = \beta_2 \otimes \alpha_3 \ . \end{cases}$$

3.4.1 Extra conditions

First we derive some extra conditions that automatically follow from the necessary and sufficient conditions.

Property 3.1 *In Case B and Case C we have*

1) $\alpha_4 \neq \varepsilon$
2) $\beta_2 \neq \varepsilon$
3) $\alpha_2 \leqslant \beta_2$
4) $\beta_1 \otimes \beta_1 \leqslant \beta_2$.

Proof: The condition $\alpha_4 > \beta_1 \otimes \beta_3$ can only be fulfilled if $\alpha_4 \neq \varepsilon$.

Since $\alpha_4 \leqslant \beta_2 \otimes \beta_2$ we then have that $\beta_2 \neq \varepsilon$ or equivalently $\alpha_2 \leqslant \beta_2$.

Assume that $\beta_2 < \beta_1 \otimes \beta_1$. This means that $\beta_1 \neq \varepsilon$.
If we use this in the first necessary and sufficient condition for Case B we get $\beta_1 \otimes \alpha_4 \leqslant \beta_2 \otimes \alpha_3 < \beta_1 \otimes \beta_1 \otimes \alpha_3$ or $\alpha_4 < \beta_1 \otimes \alpha_3$. But this is in contradiction with the fact that $\alpha_4 \geqslant \beta_1 \otimes \alpha_3$ in Case B. The second necessary and sufficient condition would lead to $\alpha_4 < \beta_1 \otimes \beta_3$ whereas $\alpha_4 > \beta_1 \otimes \beta_3$ in Case B.
The necessary and sufficient conditions for Case C would also lead to $\alpha_4 < \beta_1 \otimes \beta_3$, which is impossible since $\alpha_4 \geqslant \beta_1 \otimes \alpha_3$ in Case C.
So clearly our initial assumption was false and therefore we conclude that $\beta_1 \otimes \beta_1 \leqslant \beta_2$.

Property 3.2 *In Case C we have*

1) $\beta_1 \neq \varepsilon$
2) $\alpha_2 = \beta_2 = \beta_1 \otimes \beta_1$
3) $\alpha_3 = \beta_1 \otimes \beta_2 = (\beta_1)^{\otimes^3}$
4) $\beta_3 = \varepsilon$
5) $\alpha_4 = \beta_2 \otimes \beta_2 = \beta_1 \otimes \alpha_3 = (\beta_1)^{\otimes^4}$.

Proof: We know that $\alpha_2 \leqslant \beta_1 \otimes \beta_1$. Since $\alpha_2 \neq \varepsilon$ we should have that $\beta_1 \neq \varepsilon$.
From property 3.1 we know that $\beta_1 \otimes \beta_1 \leqslant \beta_2$. So $\alpha_2 \leqslant \beta_1 \otimes \beta_1 \leqslant \beta_2$. Since $\alpha_2 = \beta_2$ this leads to $\alpha_2 = \beta_2 = \beta_1 \otimes \beta_1$.

The condition $\beta_1 \otimes \alpha_3 = \beta_2 \otimes \alpha_2$ then results in $\alpha_3 = \beta_1 \otimes \alpha_2 = \beta_1 \otimes \beta_2 = \beta_1 \otimes \beta_1 \otimes \beta_1 = (\beta_1)^{\otimes^3}$.

The condition $\beta_1 \otimes \alpha_4 = \beta_2 \otimes \alpha_3$ leads to $\alpha_4 = \beta_1 \otimes \alpha_3 = \beta_1 \otimes (\beta_1)^{\otimes^3} = (\beta_1)^{\otimes^4} = \beta_2 \otimes \beta_2$.

Since $\alpha_4 = \alpha_3 \otimes \beta_1$ and $\alpha_4 > \beta_3 \otimes \beta_1$ we have $\alpha_3 > \beta_3$ or equivalently $\beta_3 = \varepsilon$.

3.4.2 Necessary conditions

Now we prove that the conditions for Case B and Case C are necessary.

Remark: The following properties are also valid for the coefficients of an arbitrary $n \times n$ matrix with $n \geqslant 4$.

We shall need some expressions that can be derived from formulas (1)−(4) :

$$a_1^- = \bigoplus_{\rho \in \mathcal{C}_n^1} a_{i_1 i_1} \tag{9}$$

$$a_2^+ = \bigoplus_{\varphi \in \mathcal{C}_n^2} a_{j_1 j_1} \otimes a_{j_2 j_2} \tag{10}$$

$$a_2^- = \bigoplus_{\varphi \in \mathcal{C}_n^2} a_{j_1 j_2} \otimes a_{j_2 j_1} \tag{11}$$

$$a_3^+ = \bigoplus_{\chi \in \mathcal{C}_n^3} a_{k_1 k_1} \otimes a_{k_2 k_3} \otimes a_{k_3 k_2} \tag{12}$$

$$a_3^- = \bigoplus_{\chi \in \mathcal{C}_n^3} a_{k_1 k_1} \otimes a_{k_2 k_2} \otimes a_{k_3 k_3} \oplus \bigoplus_{\chi \in \mathcal{C}_n^3} a_{k_1 k_2} \otimes a_{k_2 k_3} \otimes a_{k_3 k_1} \tag{13}$$

$$\begin{aligned} a_4^+ = & \bigoplus_{\psi \in \mathcal{C}_n^4} a_{l_1 l_1} \otimes a_{l_2 l_2} \otimes a_{l_3 l_3} \otimes a_{l_4 l_4} \oplus \bigoplus_{\psi \in \mathcal{C}_n^4} a_{l_1 l_1} \otimes a_{l_2 l_3} \otimes a_{l_3 l_4} \otimes a_{l_4 l_2} \\ & \oplus \bigoplus_{\psi \in \mathcal{C}_n^4} a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3} \end{aligned} \tag{14}$$

$$a_4^- = \bigoplus_{\psi \in \mathcal{C}_n^4} a_{l_1 l_1} \otimes a_{l_2 l_2} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3} \oplus \bigoplus_{\psi \in \mathcal{C}_n^4} a_{l_1 l_2} \otimes a_{l_2 l_3} \otimes a_{l_3 l_4} \otimes a_{l_4 l_1} \tag{15}$$

with
$$\begin{aligned} \rho &= \{i_1\} \\ \varphi &= \{j_1, j_2\} \\ \chi &= \{k_1, k_2, k_3\} \\ \psi &= \{l_1, l_2, l_3, l_4\} \ . \end{aligned}$$

Property 3.3 *If* $\alpha_4 > \beta_1 \otimes \beta_3$ *and* $\alpha_4 \geqslant \beta_1 \otimes \alpha_3$ *then*

1) $\alpha_4 \neq \varepsilon$

2) $\alpha_4 = \bigoplus_{\psi \in \mathcal{C}_n^4} a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3}$ *with* $\psi = \{l_1, l_2, l_3, l_4\}$

3) $\beta_1 \otimes \beta_1 \leqslant \beta_2$

4) $\alpha_2 \leqslant \beta_2$

5) $\alpha_4 \leqslant \beta_2 \otimes \beta_2$.

Proof: $\alpha_4 > \beta_1 \otimes \beta_3$ is only possible if $\alpha_4 \neq \varepsilon$ and thus $\alpha_4 = a_4^+$. From formula (14) we know that

$$\alpha_4 = t_1 \oplus t_2 \oplus t_3$$

$$\begin{aligned} \text{with} \quad t_1 &= \bigoplus_{\psi \in \mathcal{C}_n^4} a_{l_1 l_1} \otimes a_{l_2 l_2} \otimes a_{l_3 l_3} \otimes a_{l_4 l_4} \\ t_2 &= \bigoplus_{\psi \in \mathcal{C}_n^4} a_{l_1 l_1} \otimes a_{l_2 l_3} \otimes a_{l_3 l_4} \otimes a_{l_4 l_2} \\ t_3 &= \bigoplus_{\psi \in \mathcal{C}_n^4} a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3} \quad . \end{aligned}$$

We know that $\beta_1 = a_1^-$. If $a_3^+ > a_3^-$ then we have $\alpha_3 > a_3^-$ and if $a_3^+ \leqslant a_3^-$ then we have $\beta_3 = a_3^-$. So $\alpha_4 > \beta_1 \otimes \beta_3$ and $\alpha_4 \geqslant \beta_1 \otimes \alpha_3$ means that $\alpha_4 > a_1^- \otimes a_3^-$.
Using formulas (9) and (13) we find that

$$a_1^- \otimes a_3^- = t_4 \oplus t_5$$

$$\begin{aligned} \text{with} \quad t_4 &= \bigoplus_{\rho \in \mathcal{C}_n^1, \chi \in \mathcal{C}_n^3} a_{i_1 i_1} \otimes a_{k_1 k_1} \otimes a_{k_2 k_2} \otimes a_{k_3 k_3} \\ t_5 &= \bigoplus_{\rho \in \mathcal{C}_n^1, \chi \in \mathcal{C}_n^3} a_{i_1 i_1} \otimes a_{k_1 k_2} \otimes a_{k_2 k_3} \otimes a_{k_3 k_1} \quad . \end{aligned}$$

If we compare t_1 and t_4 we see that t_4 contains more terms than t_1 so we have that $t_1 \leqslant t_4$. Analogously we find $t_2 \leqslant t_5$. Combining these inequalities leads to

$$t_1 \oplus t_2 \leqslant t_4 \oplus t_5 \quad . \tag{16}$$

But we know that $\alpha_4 > a_1^- \otimes a_3^-$ or equivalently $t_1 \oplus t_2 \oplus t_3 > t_4 \oplus t_5$. Because of equation (16) we necessarily have that $t_1 < t_3$ and $t_2 < t_3$ and thus

$$\begin{aligned} \alpha_4 &= t_3 \\ &= \bigoplus_{\psi \in \mathcal{C}_n^4} a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3} \quad . \end{aligned} \tag{17}$$

Suppose that the maximum of t_3 is reached for $\psi = \{\delta_1, \delta_2, \delta_3, \delta_4\}$ and that $a_{\delta_1 \delta_2} \otimes a_{\delta_2 \delta_1} \geqslant a_{\delta_3 \delta_4} \otimes a_{\delta_4 \delta_3}$. We know that $t_4 < \alpha_4$. First take $k_2 = \delta_3$ and $k_3 = \delta_4$. Then we know that $k_1 \neq \delta_3$ and $k_1 \neq \delta_4$. The inequality $t_4 < \alpha_4$ then reduces to

$$\bigoplus_{i_1; k_1 \neq \delta_3, k_1 \neq \delta_4} a_{i_1 i_1} \otimes a_{k_1 k_1} \otimes a_{\delta_3 \delta_4} \otimes a_{\delta_4 \delta_3} \leqslant a_{\delta_1 \delta_2} \otimes a_{\delta_2 \delta_1} \otimes a_{\delta_3 \delta_4} \otimes a_{\delta_4 \delta_3}$$

or

$$\bigoplus_{i_1; k_1 \neq \delta_3, k_1 \neq \delta_4} a_{i_1 i_1} \otimes a_{k_1 k_1} \leqslant a_{\delta_1 \delta_2} \otimes a_{\delta_2 \delta_1} \leqslant a_2^-$$

because $a_2^- = \bigoplus_{j_1 \neq j_2} a_{j_1 j_2} \otimes a_{j_2 j_1}$.
Analogously we find that

$$\bigoplus_{i_1; k_1 \neq \delta_1, k_1 \neq \delta_2} a_{i_1 i_1} \otimes a_{k_1 k_1} \leqslant a_2^-$$

if we take $k_2 = \delta_1$ and $k_3 = \delta_2$.

Combining the last two inequalities yields

$$\bigoplus_{i_1, k_1} a_{i_1 i_1} \otimes a_{k_1 k_1} \leqslant a_2^- \quad .$$

This leads to

$$\beta_1 \otimes \beta_1 = a_1^- \otimes a_1^- = \bigoplus_{p_1, q_1} a_{p_1 p_1} \otimes a_{q_1 q_1} \leqslant a_2^- \quad .$$

From property 2.10 we know that $a_2^+ \leqslant a_1^- \otimes a_1^-$, so we have $a_2^+ \leqslant a_2^-$ or $a_2^- = \beta_2$ and thus $\beta_1 \otimes \beta_1 \leqslant \beta_2$.

$a_2^+ \leqslant a_2^-$ also leads to $\alpha_2 \leqslant \beta_2$.

We already know that $\alpha_4 > a_1^- \otimes a_3^-$ so according to property 2.10 we should have that $\alpha_4 = a_4^- \leqslant a_2^- \otimes a_2^- = \beta_2 \otimes \beta_2$.

Now we prove that the conditions for Case B are necessary:

Property 3.4 *If $\alpha_4 > \beta_1 \otimes \beta_3$ and $\alpha_4 \geqslant \beta_1 \otimes \alpha_3$ then at least one of the following statements is true:*

1) $\beta_1 \otimes \alpha_4 \leqslant \beta_2 \otimes \alpha_3$
2) $\beta_1 \otimes \alpha_4 < \beta_2 \otimes \beta_3$.

Proof: Using equation (17) and the fact that $\alpha_4 = a_4^+$ if $\alpha_4 > \beta_1 \otimes \beta_3$, we find

$$a_1^- \otimes a_4^+ = \bigoplus_{\rho \in \mathcal{C}_n^1, \psi \in \mathcal{C}_n^4} a_{i_1 i_1} \otimes a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3} \quad .$$

Combining formulas (11) and (12) leads to

$$a_2^- \otimes a_3^+ = \bigoplus_{\varphi \in \mathcal{C}_n^2, \chi \in \mathcal{C}_n^3} a_{j_1 j_2} \otimes a_{j_2 j_1} \otimes a_{k_1 k_1} \otimes a_{k_2 k_3} \otimes a_{k_3 k_2} \quad .$$

Take an arbitrary term $a_{i_1 i_1} \otimes a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3}$ of $a_1^- \otimes a_4^+$.
If $i_1 = l_1$ or if $i_1 = l_2$ we know that $i_1 \neq l_3$ and $i_1 \neq l_4$ and then we see that the term $a_{i_1 i_1} \otimes a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3}$ corresponds to the term of $a_2^- \otimes a_3^+$ with $j_1 = l_1, j_2 = l_2, k_1 = i_1, k_2 = l_3$ and $k_3 = l_4$.
Otherwise we have that $i_1 \neq l_1$ and that $i_1 \neq l_2$ and then we can take the term of $a_2^- \otimes a_3^+$ with $j_1 = l_3, j_2 = l_4, k_1 = i_1, k_2 = l_1$ and $k_3 = l_2$.
So we have demonstrated that each term of $a_1^- \otimes a_4^+$ also appears in $a_2^- \otimes a_3^+$ and thus that $a_1^- \otimes a_4^+ \leqslant a_2^- \otimes a_3^+$.
We have that $\beta_1 = a_1^-$. From property 3.3 we know that $a_4^+ = \alpha_4$ and that $a_2^- = \beta_2$.
If $a_3^+ \geqslant a_3^-$ we have $\alpha_3 = a_3^+$ and thus $\beta_1 \otimes \alpha_4 \leqslant \beta_2 \otimes \alpha_3$.
On the other hand if $a_3^+ < a_3^-$ we have $\beta_3 > a_3^+$ and this would lead to $\beta_1 \otimes \alpha_4 < \beta_2 \otimes \beta_3$.

The conditions for Case C are also necessary:

Property 3.5 *If* $\alpha_4 > \beta_1 \otimes \beta_3$ *and* $\alpha_4 \geqslant \beta_1 \otimes \alpha_3$ *and* $\alpha_2 = \beta_2 \neq \varepsilon$ *and* $\beta_4 = \varepsilon$ *then*

1) $\beta_1 \otimes \alpha_3 = \beta_2 \otimes \alpha_2$
2) $\beta_1 \otimes \alpha_4 = \beta_2 \otimes \alpha_3$.

Proof: First we prove that under the conditions $\alpha_4 > \beta_1 \otimes \beta_3$ and $\alpha_4 \geqslant \beta_1 \otimes \alpha_3$ and $\alpha_2 = \beta_2$ we should have that $a_4^+ \leqslant a_3^+ \otimes a_1^-$.
We know that

$$a_2^+ = \bigoplus_{\varphi \in \mathcal{C}_n^2} a_{j_1 j_1} \otimes a_{j_2 j_2} \ .$$

Suppose that the maximum of a_2^+ is reached for $\varphi = \{\gamma_1, \gamma_2\}$. Because $\alpha_2 = \beta_2$ we have that

$$a_2^+ = a_2^- = \bigoplus_{\varphi \in \mathcal{C}_n^2} a_{j_1 j_2} \otimes a_{j_2 j_1}$$

and thus

$$a_{j_1 j_2} \otimes a_{j_2 j_1} \leqslant a_{\gamma_1 \gamma_1} \otimes a_{\gamma_2 \gamma_2} \ , \quad \forall j_1, j_2 \in \{1, \ldots, n\} \ .$$

We already know that

$$a_4^+ = \bigoplus_{\psi \in \mathcal{C}_n^4} a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3} \ .$$

Formulas (9) and (12) result in

$$a_1^- \otimes a_3^+ = \bigoplus_{\rho \in \mathcal{C}_n^1, \chi \in \mathcal{C}_n^3} a_{i_1 i_1} \otimes a_{k_1 k_1} \otimes a_{k_2 k_3} \otimes a_{k_3 k_2} \ .$$

Take an arbitrary term $a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3}$ of a_4^+.
If $\gamma_2 = l_1$ or $\gamma_2 = l_2$ we know that $\gamma_2 \neq l_3$ and $\gamma_2 \neq l_3$ and then we see that $a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3}$ is less than or equal to the term of $a_1^- \otimes a_3^+$ with $i_1 = \gamma_1 \, , k_1 = \gamma_2 \, , k_2 = l_3$ and $k_3 = l_4$.
Otherwise we have that $\gamma_2 \neq l_1$ and $\gamma_2 \neq l_2$ and then we take the term with $i_1 = \gamma_1 \, , k_1 = \gamma_2 \, , k_2 = l_1$ and $k_3 = l_2$.
So we have demonstrated that each term of a_4^+ also appears in $a_1^- \otimes a_3^+$ and thus that

$$a_4^+ \leqslant a_1^- \otimes a_3^+ \ . \tag{18}$$

From property 3.3 we know that $a_4^+ = \alpha_4$.
If $a_3^+ < a_3^-$ we have $\beta_3 > a_3^+$. Then we get $a_4^+ = \alpha_4 > \beta_1 \otimes \beta_3 > a_1^- \otimes a_3^+$ but this is impossible because of equation (18). So we have to conclude that we always have that $a_3^+ \geqslant a_3^-$. So

$$\alpha_3 = a_3^+$$

and thus $\alpha_4 \leqslant \beta_1 \otimes \alpha_3$. If we combine this with the condition $\alpha_4 \geqslant \beta_1 \otimes \alpha_3$ we find

$$\alpha_4 = \beta_1 \otimes \alpha_3 \ .$$

Because $\alpha_2 = \beta_2 \neq \varepsilon$ we have $\alpha_2 = a_2^+ = a_2^- = \beta_2$. In general we have that $\alpha_2 \leqslant \beta_1 \otimes \beta_1$ but from property 3.3 we know that $\beta_1 \otimes \beta_1 \leqslant \beta_2 = \alpha_2$. This leads to

$$\beta_1 \otimes \beta_1 = \alpha_2 = \beta_2 \quad .$$

So $\beta_1 \otimes \alpha_4 = \beta_1 \otimes \beta_1 \otimes \alpha_3 = \beta_2 \otimes \alpha_3$.
If $\beta_4 = \varepsilon$ then $a_4^- < a_4^+$ because we already know that $\alpha_4 \neq \varepsilon$. Assume that the maximum in equation (17) is reached for $\psi = \{\delta_1, \delta_2, \delta_3, \delta_4\}$. Since $\alpha_4 \neq \varepsilon$ we have that $a_{\delta_1\delta_2} \otimes a_{\delta_2\delta_1} \neq \varepsilon$ and that $a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3} \neq \varepsilon$.
Formula (15) then leads to

$$a_4^- = \bigoplus_{\psi \in \mathcal{C}_n^4} a_{l_1l_1} \otimes a_{l_2l_2} \otimes a_{l_3l_4} \otimes a_{l_4l_3} < a_{\delta_1\delta_2} \otimes a_{\delta_2\delta_1} \otimes a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3} \quad .$$

Considering the terms with $l_3 = \delta_3$ and $l_4 = \delta_4$ leads to

$$\bigoplus_{l_1 \neq l_2, l_1, l_2 \notin \{\delta_1, \delta_2\}} a_{l_1l_1} \otimes a_{l_2l_2} \otimes a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3} < a_{\delta_1\delta_2} \otimes a_{\delta_2\delta_1} \otimes a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3}$$

and since $a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3} \neq \varepsilon$ we then have that

$$\bigoplus_{l_1 \neq l_2, l_1, l_2 \notin \{\delta_1, \delta_2\}} a_{l_1l_1} \otimes a_{l_2l_2} < a_{\delta_1\delta_2} \otimes a_{\delta_2\delta_1} \leqslant a_2^- = a_2^+ \quad .$$

Using an analogous reasoning with $l_3 = \delta_1$ and $l_4 = \delta_2$ we find

$$\bigoplus_{l_1 \neq l_2, l_1, l_2 \notin \{\delta_3, \delta_4\}} a_{l_1l_1} \otimes a_{l_2l_2} < a_2^+$$

and combining this with the previous result and formula (10) we get

$$a_2^+ = a_{\delta_1\delta_1} \otimes a_{\delta_3\delta_3} \oplus a_{\delta_1\delta_1} \otimes a_{\delta_4\delta_4} \oplus a_{\delta_2\delta_2} \otimes a_{\delta_3\delta_3} \oplus a_{\delta_2\delta_2} \otimes a_{\delta_4\delta_4}$$

because all other terms of the form $a_{j_1j_1} \otimes a_{j_2j_2}$ are less than a_2^+. Since we haven't put any restrictions or conditions on the indices, we may assume without loss of generality that $\alpha_2 = a_2^+ = a_{\delta_1\delta_1} \otimes a_{\delta_3\delta_3}$. We already know that $\alpha_2 = \beta_1 \otimes \beta_1$. So we conclude that

$$\beta_1 = a_{\delta_1\delta_1} = a_{\delta_3\delta_3} \quad .$$

Because $a_{\delta_1\delta_2} \otimes a_{\delta_2\delta_1} \leqslant a_2^+ = a_{\delta_1\delta_1} \otimes a_{\delta_3\delta_3}$ we have

$$\alpha_4 = a_{\delta_1\delta_2} \otimes a_{\delta_2\delta_1} \otimes a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3} \leqslant a_{\delta_3\delta_3} \otimes a_{\delta_1\delta_1} \otimes a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3} \leqslant a_1^- \otimes a_3^+ = \beta_1 \otimes \alpha_3 \quad .$$

But since we know that $\alpha_4 = \beta_1 \otimes \alpha_3$ the inequalities in the previous expression should be equalities. This leads to

$$a_{\delta_1\delta_2} \otimes a_{\delta_2\delta_1} = a_{\delta_3\delta_3} \otimes a_{\delta_1\delta_1} = \beta_1 \otimes \beta_1 = \beta_2 \quad .$$

Analogously we also find that

$$\beta_2 = a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3}$$

and thus

$$\alpha_4 = \beta_2 \otimes \beta_2 \quad .$$

Then we get

$$\beta_1 \otimes \alpha_3 = \alpha_4 = \beta_2 \otimes \beta_2 = \alpha_2 \otimes \beta_2 \quad .$$

Finally we have to demonstrate that we have considered all possible cases:

Property 3.6 *The coefficients of the characteristic equation of a positive n by n matrix with $n \geqslant 4$ always fall into exactly one of the following cases:*

Case A: $\alpha_4 \leqslant \beta_1 \otimes \beta_3$ *or* $\alpha_4 < \beta_1 \otimes \alpha_3$

Case B: $\alpha_4 > \beta_1 \otimes \beta_3$ *and* $\alpha_4 \geqslant \beta_1 \otimes \alpha_3$ *and* $\alpha_4 \leqslant \beta_2 \otimes \beta_2$ *and* $(\ \beta_1 = \varepsilon$ *or* $\alpha_2 = \varepsilon$ *or* $\beta_4 = \alpha_4\ )$

Case C: $\alpha_4 > \beta_1 \otimes \beta_3$ *and* $\alpha_4 \geqslant \beta_1 \otimes \alpha_3$ *and* $\alpha_4 \leqslant \beta_2 \otimes \beta_2$ *and* $\alpha_2 = \beta_2 \neq \varepsilon$ *and* $\beta_4 = \varepsilon$.

Proof: According to property 2.12 we have that

Case 1: $\alpha_4 \leqslant \beta_1 \otimes \beta_3$ or
Case 2: $\alpha_4 < \beta_1 \otimes \alpha_3$ or
Case 3: $\alpha_4 \leqslant \beta_2 \otimes \beta_2$ or
Case 4: $\alpha_4 < \alpha_2 \otimes \alpha_2$.

From property 3.3 we know that if we are neither in Case 1 nor in Case 2 then we are in Case 3. So it is not necessary to consider Case 4.
Case 1 and Case 2 correspond to Case A.
From now on we assume that Case 1 and Case 2 are not true if we are in Case 3. Then we know from property 3.1 that $\alpha_4 \neq \varepsilon$ so we have that either $\beta_4 = \alpha_4$ or $\beta_4 = \varepsilon$.
If we have $\beta_1 = \varepsilon$ or $\alpha_2 = \varepsilon$ or $\beta_4 = \alpha_4$, then we are in Case B.
Otherwise we know that $\beta_1 \neq \varepsilon$, $\beta_4 = \varepsilon$ and $\alpha_2 \neq \varepsilon$. Because of property 3.1 we have that $\beta_2 \neq \varepsilon$. This means that $\alpha_2 = \beta_2$ and thus we have Case C.
So we have indeed considered all possible cases and since the three cases are mutually exclusive for the coefficients of a positive matrix, we always have exactly one of the three cases.

Property 3.6 can be considered as an extra general necessary condition for an n-th degree polynomial ($n \geqslant 4$) to be the characteristic polynomial of a positive matrix.

3.4.3 Sufficient conditions

Now we demonstrate that the conditions are also sufficient.

If we search a matrix such that its characteristic equation is $\lambda^{\otimes^3} \oplus \alpha_2 \otimes \lambda \oplus \alpha_3 \nabla \beta_1 \otimes \lambda^{\otimes^2} \oplus \beta_2 \otimes \lambda \oplus \beta_3$ we find

for Case A: $\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 0 & \kappa_{1,2} & \kappa_{1,3} & \kappa_{1,4} \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon \end{bmatrix}$, for Case B: $\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 0 & \kappa_{1,2} & \kappa_{1,3} & \varepsilon \\ \varepsilon & 0 & \varepsilon & \kappa_{2,4} \\ \varepsilon & \varepsilon & 0 & \varepsilon \end{bmatrix}$ and for Case C: $\begin{bmatrix} \beta_1 & \beta_2 & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \kappa_{2,3} & \kappa_{2,4} \\ \varepsilon & \varepsilon & 0 & \varepsilon \end{bmatrix}$.

Proof for Case A :
From the 2×2 and the 3×3 case we already know that

$\kappa_{1,2} \leqslant \beta_1$
$\kappa_{1,3} \leqslant \beta_2$ or $\kappa_{1,3} < \alpha_2$
$\beta_1 \otimes \kappa_{1,2} = \alpha_2$
$\beta_1 \otimes \kappa_{1,3} = \alpha_3$.

Using the same reasoning we find for Case A:

$\kappa_{1,4} \leqslant \beta_3$ or $\kappa_{1,4} < \alpha_3$
$\beta_1 \otimes \kappa_{1,4} = \alpha_4$.

This leads to

$$\begin{aligned} a_1 &= \ominus\beta_1 \ominus \kappa_{1,2} \\ &= \ominus\beta_1 \\ a_2 &= \beta_1 \otimes \kappa_{1,2} \ominus \beta_2 \ominus \kappa_{1,3} \\ &= \alpha_2 \ominus \beta_2 \\ a_3 &= \beta_1 \otimes \kappa_{1,3} \ominus \beta_3 \ominus \kappa_{1,4} \\ &= \alpha_3 \ominus \beta_3 \\ a_4 &= \beta_1 \otimes \kappa_{1,4} \ominus \beta_4 \\ &= \alpha_4 \ominus \beta_4 \ . \end{aligned}$$

Proof for Case B :
Because $\alpha_4 \leqslant \beta_2 \otimes \beta_2$ we have that $\kappa_{2,4} \leqslant \beta_2$.

Now we use the necessary and sufficient conditions for Case B to prove that $\beta_1 \otimes \kappa_{2,4} \leqslant \alpha_3$ if $\alpha_3 \geqslant \beta_3$ and that $\beta_1 \otimes \kappa_{2,4} < \beta_3$ if $\alpha_3 < \beta_3$:

If $\alpha_3 \geqslant \beta_3$ and one of the necessary conditions is fulfilled we have that $\beta_2 \otimes \alpha_3 \geqslant \beta_1 \otimes \alpha_4$.

From property 3.1 we know that $\beta_2 \neq \varepsilon$. This leads to $\alpha_3 \geqslant \beta_1 \otimes \frac{\alpha_4}{\beta_2} = \beta_1 \otimes \kappa_{2,4}$.

If $\alpha_3 < \beta_3$ the necessary conditions result in $\beta_2 \otimes \beta_3 > \beta_1 \otimes \alpha_4$ and this leads to $\beta_3 > \beta_1 \otimes \kappa_{4,2}$.

So we always have that $\alpha_3 \oplus \beta_1 \otimes \kappa_{2,4} \ominus \beta_3 = \alpha_3 \ominus \beta_3$.

Now we prove that under the conditions of Case B we have that $\beta_1 \otimes \kappa_{1,2} \otimes \kappa_{2,4} \leqslant \beta_4$:

If $\beta_1 = \varepsilon$ or $\alpha_2 = \varepsilon$ then $\beta_1 \otimes \kappa_{1,2} \otimes \kappa_{2,4} = \varepsilon \leqslant \beta_4$.

Otherwise we have $\alpha_4 = \beta_4$ and $\alpha_2 = \beta_2$. So $\beta_1 \otimes \kappa_{1,2} \otimes \kappa_{2,4} = \beta_1 \otimes \frac{\alpha_2}{\beta_1} \otimes \frac{\alpha_4}{\beta_2} = \alpha_4 = \beta_4$.

We find

$$
\begin{aligned}
a_1 &= \ominus\beta_1 \ominus \kappa_{1,2} \\
&= \ominus\beta_1 \\
a_2 &= \beta_1 \otimes \kappa_{1,2} \ominus \beta_2 \ominus \kappa_{1,3} \ominus \kappa_{2,4} \\
&= \alpha_2 \ominus \beta_2 \\
a_3 &= \beta_1 \otimes \kappa_{1,3} \oplus \beta_1 \otimes \kappa_{2,4} \oplus \kappa_{1,2} \otimes \kappa_{2,4} \ominus \beta_3 \\
&= \alpha_3 \oplus \beta_1 \otimes \kappa_{2,4} \ominus \beta_3 \qquad\qquad \text{since } \kappa_{1,2} \leqslant \beta_1 \\
&= \alpha_3 \ominus \beta_3 \\
a_4 &= \beta_2 \otimes \kappa_{2,4} \ominus \beta_4 \ominus \beta_1 \otimes \kappa_{1,2} \otimes \kappa_{2,4} \\
&= \alpha_4 \ominus \beta_4 \ .
\end{aligned}
$$

Proof for Case C :

We know that $\beta_2 \neq \varepsilon$ and from property 3.2 we also know that $\beta_1 \neq \varepsilon$. If $\beta_1 \otimes \alpha_3 = \beta_2 \otimes \alpha_2$ then $\kappa_{2,3} = \frac{\alpha_3}{\beta_2} = \frac{\alpha_2}{\beta_1} \leqslant \beta_1$ and $\beta_1 \otimes \kappa_{2,3} = \beta_1 \otimes \frac{\alpha_3}{\beta_2} = \alpha_2$. Because $\alpha_4 \leqslant \beta_2 \otimes \beta_2$ we know that $\kappa_{2,4} \leqslant \beta_2$. $\beta_3 = \varepsilon$ according to property 3.2. We also have that $\beta_2 \otimes \kappa_{2,3} = \alpha_3$ and that $\beta_2 \otimes \kappa_{2,4} = \alpha_4$.

Finally $\beta_1 \otimes \alpha_4 = \beta_2 \otimes \alpha_3$ leads to $\beta_1 \otimes \kappa_{2,4} = \beta_1 \otimes \frac{\alpha_4}{\beta_2} = \alpha_3$.

We find

$$
\begin{aligned}
a_1 &= \ominus\beta_1 \ominus \kappa_{2,3} \\
&= \ominus\beta_1 \\
a_2 &= \beta_1 \otimes \kappa_{2,3} \ominus \beta_2 \ominus \kappa_{2,4} \\
&= \alpha_2 \ominus \beta_2 \\
a_3 &= \beta_1 \otimes \kappa_{2,4} \oplus \beta_2 \otimes \kappa_{2,3} \\
&= \alpha_3 \\
a_4 &= \beta_2 \otimes \kappa_{2,4} \\
&= \alpha_4 \ .
\end{aligned}
$$

Example 3.7 Consider the monic polynomial $\lambda^{\otimes^4} \ominus 2 \otimes \lambda^{\otimes^3} \ominus 7 \otimes \lambda^{\otimes^2} \oplus 9 \otimes \lambda \oplus 15$.

We have that
$$
\begin{aligned}
\alpha_4 = 15 &> \varepsilon = 2 \otimes \varepsilon = \beta_1 \otimes \beta_3 \\
\alpha_4 = 15 &\geqslant 11 = 2 \otimes 9 = \beta_1 \otimes \alpha_3 \\
\alpha_4 = 15 &> 14 = 7 \otimes 7 = \beta_2 \otimes \beta_2 \ .
\end{aligned}
$$

Since the coefficients don't belong to one of the three possible cases, the given polynomial cannot be the characteristic polynomial of a positive matrix.

Example 3.8 Consider $\lambda^{\otimes^4} \ominus 3 \otimes \lambda^{\otimes^3} \oplus 6^{\bullet} \otimes \lambda^{\otimes^2} \oplus 6 \otimes \lambda \oplus 9^{\bullet} \ \nabla \ \varepsilon$.

We have that
$$
\begin{aligned}
\alpha_4 = 9 &> \varepsilon = 3 \otimes \varepsilon = \beta_1 \otimes \beta_3 \\
\alpha_4 = 9 &\geqslant 9 = 3 \otimes 6 = \beta_1 \otimes \alpha_3 \\
\alpha_4 = 9 &\leqslant 12 = 6 \otimes 6 = \beta_2 \otimes \beta_2 \\
\alpha_4 = 9 &= 9 = \beta_4
\end{aligned}
$$

so we are in Case B. The necessary and sufficient conditions are fulfilled:

$\alpha_1 = \varepsilon$

$\alpha_2 = 6 \leqslant 6 = 3 \otimes 3 = \beta_1 \otimes \beta_1$

$\alpha_3 = 6 \leqslant 9 = 3 \otimes 6 = \beta_1 \otimes \beta_2$

$\beta_1 \otimes \alpha_4 = 3 \otimes 9 = 12 \leqslant 12 = 6 \otimes 6 = \beta_2 \otimes \alpha_3$.

The matrix $A = \begin{bmatrix} 3 & 6 & \varepsilon & 9 \\ 0 & 3 & 3 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 3 \\ \varepsilon & \varepsilon & 0 & \varepsilon \end{bmatrix}$ has the given monic balance as its characteristic equation.

3.5 The general case

Here we have not yet found sufficient conditions, but we shall outline a heuristic algorithm that will (in most cases) result in a positive matrix for which the characteristic polynomial will be equal to a given polynomial.
Extrapolating the results of the previous subsections and supported by many examples we state the following conjecture:

Conjecture 3.9 *If* $\lambda^{\otimes^n} \oplus \bigoplus_{i=2}^{n} \alpha_i \otimes \lambda^{\otimes^{n-i}} \ \nabla \ \beta_1 \otimes \lambda^{\otimes^{n-1}} \oplus \bigoplus_{j=2}^{n} \beta_j \otimes \lambda^{\otimes^{n-j}}$ *is the characteristic equation of a matrix* $A \in \mathbb{R}_{\max}^{n\times n}$ *then it is also the characteristic equation of an upper Hessenberg matrix of the form*

$$K = \begin{bmatrix} k_{0,1} & k_{0,2} & k_{0,3} & \dots & k_{0,n-1} & k_{0,n} \\ 0 & k_{1,2} & k_{1,3} & \dots & k_{1,n-1} & k_{1,n} \\ \varepsilon & 0 & k_{2,3} & \dots & k_{2,n-1} & k_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \varepsilon & \varepsilon & \varepsilon & \dots & 0 & k_{n-1,n} \end{bmatrix} .$$

We shall use this conjecture in our heuristic algorithm to construct a matrix for which the characteristic polynomial will be equal to a given polynomial. However in [4] we have presented a method to construct such a matrix that works even if Conjecture 3.9 would not be true. The major disadvantage of this method is its computational complexity. Therefore we now present a heuristic algorithm that will on the average be much faster. If a result is returned, it is right. But it could be possible that sometimes no result is returned although there is a solution (in which case we have to fall back on the method of [4]).

A heuristic algorithm:
First we check whether the coefficients of the given polynomial satisfy the conditions of Property 2.12. Then we reconstruct the a_p^-'s by setting $a_1^- = \beta_1$ and $a_p^- = \max(\alpha_p - \delta, \beta_p)$ for $p = 2, 3, \dots, n$ with δ a small strictly positive real number.

Consider $K_1 = \begin{bmatrix} a_1^- & a_2^- & a_3^- & \dots & a_n^- \\ 0 & a_1^- & a_2^- & \dots & a_{n-1}^- \\ \varepsilon & 0 & a_1^- & \dots & a_{n-2}^- \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \varepsilon & \dots & a_1^- \end{bmatrix}$ and $K_2 = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \dots & \varepsilon \\ \varepsilon & \kappa_{1,2} & \kappa_{1,3} & \dots & \kappa_{1,n} \\ \varepsilon & \varepsilon & \kappa_{2,3} & \dots & \kappa_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \varepsilon & \dots & \kappa_{n-1,n} \end{bmatrix}$

where $\kappa_{i,j} = \dfrac{\alpha_j}{a_i^-}$ if $a_i^- \neq \varepsilon$,

$= \varepsilon$ if $a_i^- = \varepsilon$.

We shall make a judicious choice out of the elements of K_1 and K_2 to compose a matrix for which the characteristic equation will coincide with (8):

We start with $A = \begin{bmatrix} a_1^- & a_2^- & a_3^- & \dots & a_n^- \\ 0 & \varepsilon & \varepsilon & \dots & \varepsilon \\ \varepsilon & 0 & \varepsilon & \dots & \varepsilon \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \varepsilon & \dots & \varepsilon \end{bmatrix}$. Now we shall column by column transfer non-ε elements of K_2 to A (one element per column) such that the coefficients of the characteristic equation of A are less than or equal to those of (8). If this doesn't lead to a valid result we shift a_1^- along its diagonal and repeat the procedure. We keep shifting a_1^- until it reaches the n-th column. If this still doesn't yield a result we put a_1^- back in the first column and repeat the procedure but now with a_2^-, and so on. Finally, if we have found A we remove redundant entries: these are elements that can be removed without altering the characteristic equation.

The results of this section will now be used to determine a minimal state space realization of a SISO discrete event system.

4 Minimal state space realization

4.1 Realization and minimal realization

Suppose that we have a single input single output (SISO) discrete event system that can be described by an n-th order state space model

$$x[k+1] = A \otimes x[k] \oplus B \otimes u[k] \tag{19}$$

$$y[k] = C \otimes x[k] \tag{20}$$

with $A \in \mathbb{R}_{\max}^{n \times n}$, $B \in \mathbb{R}_{\max}^{n \times 1}$ and $C \in \mathbb{R}_{\max}^{1 \times n}$. u is the input, y is the output and x is the state vector.

We define the unit impulse e as: $e[k] = 0$ if $k = 0$,

$= \varepsilon$ otherwise .

If we apply a unit impulse to the system and if we assume that the initial state $x[0]$ satisfies $x[0] = \varepsilon$ or $A \otimes x[0] \leqslant B$, we get the impulse response as the output of the system:

$$\begin{aligned} x[1] &= B \\ x[2] &= A \otimes B \\ &\vdots \\ x[k] &= A^{\otimes^{k-1}} \otimes B \\ \Rightarrow\ y[k] &= C \otimes A^{\otimes^{k-1}} \otimes B \ . \end{aligned}$$

Let $g_k = C \otimes A^{\otimes^k} \otimes B$. The g_k's are called the *Markov parameters*.

Let us now reverse the process: suppose that A, B and C are unknown, and that we only know the Markov parameters (e.g. from experiments – where we assume that the system is max-linear and time-invariant and that there is no noise present). How can we construct

A, B and C from the g_k's? This process is called realization. If we make the dimension of A minimal, we have a minimal realization. Although there have been some attempts to solve this problem [2, 7, 9], this problem has at present – to the authors' knowledge – not been solved entirely.

4.2 A lower bound for the minimal system order

In this section we shall use the following property:

Property 4.1 *Consider $A \in \mathbb{S}_{\max}^{n\times n}, B \in \mathbb{S}_{\max}^{n\times 1}$ and $C \in \mathbb{S}_{\max}^{1\times n}$. If A satisfies an equation of the form*

$$\bigoplus_{p=0}^{n} a_p \otimes A^{\otimes^{n-p}} \,\nabla\, \varepsilon$$

(e.g. its characteristic equation) then the Markov parameters satisfy

$$\bigoplus_{p=0}^{n} a_p \otimes g_{k+n-p} \,\nabla\, \varepsilon \quad \textit{for } k = 0\,, 1,\, 2,\, \ldots\ .$$

Proof: We know that $\bigoplus_{p=0}^{n} a_p \otimes A^{\otimes^{n-p}} \,\nabla\, \varepsilon$. After left multiplication by $C \otimes A^{\otimes^k}$ and right multiplication by B we get $\bigoplus_{p=0}^{n} a_p \otimes C \otimes A^{\otimes^{k+n-p}} \otimes B \,\nabla\, \varepsilon$ and since $g_k = C \otimes A^{\otimes^k} \otimes B$ we finally get $\bigoplus_{p=0}^{n} a_p \otimes g_{k+n-p} \,\nabla\, \varepsilon$.

Suppose that we have a system that can be described by equations (19) and (20), with unknown system matrices. If we want to find a minimal realization of this system the first question that has to be answered is that of the minimal system order.

Consider the semi-infinite Hankel matrix $H = \begin{bmatrix} g_0 & g_1 & g_2 & \cdots \\ g_1 & g_2 & g_3 & \cdots \\ g_2 & g_3 & g_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$.

As a direct consequence of theorem 1.5 and property 4.1 we have that the columns of H satisfy

$$\bigoplus_{p=0}^{n} a_p \otimes H(:, k+n-p) \,\nabla\, \varepsilon \qquad \text{for } k = 1,\, 2,\, \ldots \tag{21}$$

where the coefficients a_p are the coefficients of the characteristic equation of the system matrix A.

Now we shall reverse this reasoning: first we construct a p by q Hankel matrix

$$H_{p,q} = \begin{bmatrix} g_0 & g_1 & g_2 & \cdots & g_{q-1} \\ g_1 & g_2 & g_3 & \cdots & g_q \\ g_2 & g_3 & g_4 & \cdots & g_{q+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{p-1} & g_p & g_{p+1} & \cdots & g_{p+q-2} \end{bmatrix}$$

with p and q large enough: $p, q \gg n$, where n is the real (but unknown) system order. Then we try to find n and $a_0, a_1, \dots, a_n$ such that the columns of $H_{p,q}$ satisfy an equation of the form (21), which will lead to the characteristic equation of the unknown system matrix A. We propose the following procedure:

First we look for the largest square submatrix of $H_{p.q}$ with consecutive column indices:

$$H_{\mathrm{sub},r} = H_{p,q}([i_1, i_2, \dots, i_r], [j+1, j+2, \dots, j+r]) enspace,$$

the determinant of which is not balanced: $\det H_{\mathrm{sub},r} \not\nabla \varepsilon$. If we add one row and the $j+r+1$-st column we get an $r+1$ by $r+1$ matrix $H_{\mathrm{sub},r+1}$ that has a balanced determinant. So according to theorem 1.3 the set of linear balances

$$H_{\mathrm{sub},r+1} \otimes a \nabla \varepsilon$$

has a signed solution $a = \begin{bmatrix} a_r & a_{r-1} & \dots & a_0 \end{bmatrix}^t$. We now search a solution a that corresponds to the characteristic equation of a matrix with elements in $\mathbb{R}_{\max}$ (this should not necessarily be a signed solution – a signed solution would correspond to the a_p^{Olsder}'s). First of all we normalize a_0 to 0 and then we check if the necessary (and sufficient) conditions of section 2 are satisfied. If they are not satisfied we augment r and repeat the procedure.
We continue until we get the following stable relation among the columns of $H_{p,q}$:

$$H_{p,q}(:, k+r) \ \oplus \ a_1 \otimes H_{p,q}(:, k+r-1) \ \oplus \ \dots \ \oplus \ a_r \otimes H_{p,q}(:, k) \nabla \varepsilon \tag{22}$$

for $k \in \{1, \dots, q-r\}$. Since we assume that the system can be described by equations (19) and (20) and that p, $q \gg n$, we can always find such a stable relationship, by gradually augmenting r. The r that results from this procedure is a lower bound for the minimal system order.

4.3 Determination of the system matrices

In [4] we have described a method to find all solutions of a set of multivariate polynomial (in)equalities in the max algebra. Now we can use this method to find the A, B and C matrices of an r-th order SISO system with Markov parameters $g_0, g_1, g_2, \dots$. If the algorithm doesn't find any solutions, this means that the output behavior can't be described by an r-th order SISO system. In that case we have to augment our estimate of the system order and repeat the procedure. Since we assume that the system can be described by the state space model (19) – (20) we shall always get a minimal realization.
However in many cases we can use the results of section 2 to find a minimal realization. Starting from the coefficients $a_1, a_2, \dots, a_r$ of equation (22) we search a matrix A with elements in $\mathbb{R}_{\max}$ such that its characteristic equation is

$$\lambda^{\otimes^r} \ \oplus \ \bigoplus_{p=1}^{r} a_p \otimes \lambda^{\otimes^{r-p}} \nabla \varepsilon \ . \tag{23}$$

Once we have found the A matrix, we have to find a B and a C with elements in $\mathbb{R}_{\max}$ such that

$$C \otimes A^{\otimes^k} \otimes B = g_k \quad \text{for } k = 0, 1, 2, \dots \ .$$

In practice it seems that we only have to take the transient behavior and the first cycles of the steady-state behavior into account. So we may limit ourselves to the first, say, N Markov parameters.
Let's take a closer look at equations of the form $C \otimes R \otimes B = s$ with $C \in \mathbb{R}_{\max}^{1\times n}, R \in \mathbb{R}_{\max}^{n\times n}, B \in \mathbb{R}_{\max}^{n\times 1}$ and $s \in \mathbb{R}_{\max}$. This equation can be rewritten as

$$\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} c_i \otimes r_{ij} \otimes b_i = s \ .$$

So if we take the first N Markov parameters into account, we get a set of N multivariate polynomial equations in the max algebra, with the elements of B and C as unknowns and $R = A^{\otimes^{k-1}}$ and $s = g_{k-1}$ in the k-th equation. This problem can also be solved using the algorithm described in [4].
However one has to be careful since it is not always possible to find a B and a C for every matrix that has equation (23) as its characteristic equation as will be shown in example 5.2. In that case we have to search another A matrix or we could fall back on the method described in [3, 4], which finds all possible minimal realizations.
The reason that it is not always possible to find a B and C for A is that in $\mathbb{S}_{\max}$ all triples (A, B, C) that result in the same output behavior are connected by a kind of similarity transformation. We have to pick a triple $(\tilde{A}, \tilde{B}, \tilde{C})$ that is completely in $\mathbb{R}_{\max}$.

5 Examples

We now illustrate the procedure of the preceding section with a few examples.

Example 5.1

Here we reconsider the example of [2, 9]. We start from a system with system matrices

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 0 \\ -3 & 1 & 2 \end{bmatrix} \ , \quad B = \begin{bmatrix} 0 \\ \varepsilon \\ \varepsilon \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & \varepsilon & \varepsilon \end{bmatrix} \ .$$

Now we are going to construct the system matrices from the impulse response of the system. This impulse response is given by

$$\{g_k\} = 0\,, 1\,, 2\,, 3\,, 4\,, 5\,, 6\,, 8\,, 10\,, 12\,, 14\,, \ldots \ .$$

First we construct the Hankel matrix

$$H_{8,8} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 8 & 10 \\ 2 & 3 & 4 & 5 & 6 & 8 & 10 & 12 \\ 3 & 4 & 5 & 6 & 8 & 10 & 12 & 14 \\ 4 & 5 & 6 & 8 & 10 & 12 & 14 & 16 \\ 5 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\ 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 \\ 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 \end{bmatrix} \ .$$

The determinant of $H_{\mathrm{sub},2} = H_{8,8}([1,7],[1,2]) = \begin{bmatrix} 0 & 1 \\ 6 & 8 \end{bmatrix}$ is not balanced. We add one row and the third column and then we search a solution of the set of linear balances

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 6 & 8 & 10 \end{bmatrix} \otimes \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} \nabla \varepsilon \ .$$

The solution $a_0 = 0$, $a_1 = \ominus 2$, $a_2 = 3$ satisfies the necessary and sufficient conditions for the 2 by 2 case since $\alpha_1 = \varepsilon$ and $\alpha_2 = 3 \leqslant 4 = 2 \otimes 2 = \beta_1 \otimes \beta_1$. This solution also corresponds to a stable relation among the columns of $H_{8,8}$:

$$H_{8,8}(:,k+2) \ \oplus \ 3 \otimes H_{8,8}(:,k) = 2 \otimes H_{8,8}(:,k+1) \qquad \text{for } k \in \{1,2,\ldots,6\} \ ,$$

or to the following characteristic equation:

$$\lambda^{\otimes^2} \ominus 2 \otimes \lambda \ \oplus \ 3 \nabla \varepsilon \ .$$

This leads to a second order system with $A = \begin{bmatrix} 2 & \varepsilon \\ 0 & 1 \end{bmatrix}$. Using the technique of [4] we get a whole set of solutions for B and C. One of the solutions is $B = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$ and $C = \begin{bmatrix} \varepsilon & 0 \end{bmatrix}$. Apart from a permutation of the two state variables this result is the same as that of [9].

We now give another example that doesn't satisfy the assumptions of [9], where only impulse responses with a uniformly up-terrace behavior are considered.

Example 5.2

We start from the system (A,B,C) with

$$A = \begin{bmatrix} 3 & 1 & 0 \\ \varepsilon & 3 & 2 \\ 0 & 5 & \varepsilon \end{bmatrix} \ , \quad B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & \varepsilon & \varepsilon \end{bmatrix} \ .$$

The impulse response of this system is: $0\,,3\,,6\,,9\,,13\,,16\,,20\,,23\,,27\,,\ldots$. Since there are two different alternating increments in steady state (3 and 4), we can't use the technique of [9]. First we construct the Hankel matrix

$$H_{8,8} = \begin{bmatrix} 0 & 3 & 6 & 9 & 13 & 16 & 20 & 23 \\ 3 & 6 & 9 & 13 & 16 & 20 & 23 & 27 \\ 6 & 9 & 13 & 16 & 20 & 23 & 27 & 30 \\ 9 & 13 & 16 & 20 & 23 & 27 & 30 & 34 \\ 13 & 16 & 20 & 23 & 27 & 30 & 34 & 37 \\ 16 & 20 & 23 & 27 & 30 & 34 & 37 & 41 \\ 20 & 23 & 27 & 30 & 34 & 37 & 41 & 44 \\ 23 & 27 & 30 & 34 & 37 & 41 & 44 & 48 \end{bmatrix} .$$

The determinant $H_{\mathrm{sub},3} = H_{8,8}([1,3,4],[1,2,3]) = \begin{bmatrix} 0 & 3 & 6 \\ 6 & 9 & 13 \\ 9 & 13 & 16 \end{bmatrix}$ is not balanced.

The set of linear balances

$$\begin{bmatrix} 0 & 3 & 6 & 9 \\ 3 & 6 & 9 & 13 \\ 6 & 9 & 13 & 16 \\ 9 & 13 & 16 & 20 \end{bmatrix} \otimes \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} \nabla \varepsilon$$

has a solution $a_0 = 0$, $a_1 = \ominus 3$, $a_2 = \ominus 7$, $a_3 = 10$ that satisfies the necessary and sufficient conditions of subsection 3.3: $\begin{cases} \alpha_1 = \varepsilon \\ \alpha_2 = \varepsilon \leqslant 6 = 3 \otimes 3 = \beta_1 \otimes \beta_1 \\ \alpha_3 = 10 \leqslant 10 = 3 \otimes 7 = \beta_1 \otimes \beta_2 \ . \end{cases}$

This solution also corresponds to a stable relation among the columns of $H_{8,8}$:

$$H_{8,8}(:,k+3) \ \oplus \ 10 \otimes H_{8,8}(:,k) = 3 \otimes H_{8,8}(:,k+2) \ \oplus \ 7 \otimes H_{8,8}(:,k+1)$$

for $k \in \{1,2,\ldots,5\}$, or to the following characteristic equation:

$$\lambda^{\otimes^3} \ominus 3 \otimes \lambda^{\otimes^2} \ominus 7 \otimes \lambda \oplus 10 \ \nabla \ \varepsilon \ .$$

This would lead to $A_1 = \begin{bmatrix} 3 & 7 & \varepsilon \\ 0 & \varepsilon & 7 \\ \varepsilon & 0 & \varepsilon \end{bmatrix}$. But it is impossible to find positive vectors B_1 and C_1 such that (A_1, B_1, C_1) is a realization of the given impulse response.

However the 7 on the first row and in the second column of A_1 is redundant, because if we remove it, we get $A_2 = \begin{bmatrix} 3 & \varepsilon & \varepsilon \\ 0 & \varepsilon & 7 \\ \varepsilon & 0 & \varepsilon \end{bmatrix}$ which has the same characteristic equation as A_1, but for A_2 it is possible to find a corresponding B_2 and C_2: $B_2 = \begin{bmatrix} 0 \\ -3 \\ \varepsilon \end{bmatrix}$ and $C_2 = \begin{bmatrix} 0 & 2 & \varepsilon \end{bmatrix}$.

6 Conclusions and future research

We have derived necessary and for some cases also sufficient conditions for an $\mathbb{S}_{\max}$ polynomial to be the characteristic polynomial of an $\mathbb{R}_{\max}$ matrix. Then we have indicated how such a matrix can be constructed. The results were then applied to develop a procedure to find a minimal state space realization of a SISO system, given its Markov parameters. This procedure is an alternative to the method of [3], which finds all possible minimal realizations but which has one disadvantage: its computational complexity. Since we allow an upper Hessenberg form for the matrix A, our method incorporates both the companion form of [2] and the bidiagonal form of [9].

In the future we shall expand our theory and fill the gaps that are still left: how many Markov parameters are necessary to find the system matrices, how do we select a new matrix A if the matrix that resulted from the heuristic algorithm didn't lead to a realization of the given impulse response, etc.. We shall also try to improve the performance of our heuristic algorithm. Then we shall turn our attention to multiple input multiple output (MIMO) discrete event systems.

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