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The Characteristic Equation and Minimal State Space Realization of SISO Systems in the Max Algebra

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1 Introduction

1.1 Overview

There exist many modeling and analysis frameworks for discrete event systems: Petri nets, formal languages, generalized semi-Markov processes, perturbation analysis and so on. In this paper we consider systems that can be modeled using max algebra.

In the first part we study the characteristic equation of a matrix in the max algebra ($\mathbb{R}_{\max}$). We determine necessary and for some cases also sufficient conditions for a polynomial to be the characteristic polynomial of a matrix with elements in $\mathbb{R}_{\max}$. Then we indicate how to construct a matrix such that its characteristic polynomial is equal to a given monic polynomial in $\mathbb{S}_{\max}$, the extension of $\mathbb{R}_{\max}$.

In the second part of this paper we address the minimal state space realization problem. Based on the results of the first part we propose a procedure to find a minimal state space realization of a single input single output (SISO) discrete event system in the max algebra, given its Markov parameters. Finally we illustrate this procedure with an example.

1.2 Notations

One of the mathematical tools used in this paper is the max algebra. In this introduction we only explain the notations we use to represent the max-algebraic operations. A complete introduction to the max algebra can be found in [1].

In this paper we use the following notations: $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$. $\epsilon = -\infty$ is the neutral element for $\oplus$ in $\mathbb{R}_{\max} = \mathbb{R} \cup \{\epsilon\}, \oplus, \otimes$. The inverse element of $a \neq \epsilon$ for $\otimes$ in $\mathbb{R}_{\max}$ is denoted by $a^{\otimes^{-1}}$. The division is defined as follows: $\dfrac{a}{b} = a \otimes b^{\otimes^{-1}}$ if $b \neq \epsilon$. E_n is the n by n identity matrix in $\mathbb{R}_{\max}$.

We also use the extension of the max algebra $\mathbb{S}_{\max}$ that was introduced in [1, 6]. $\mathbb{S}_{\max}$ is a kind of symmetrization of $\mathbb{R}_{\max}$. We shall restrict ourselves to the most important features of $\mathbb{S}_{\max}$. For a more formal derivation the interested reader is referred to [6].

There are three kinds of elements in $\mathbb{S}_{\max}$: the positive elements ($\mathbb{S}^{\oplus}_{\max} = \mathbb{R}_{\max}$), the negative elements ($\mathbb{S}^{\ominus}_{\max}$) and the balanced elements ($\mathbb{S}^{\bullet}_{\max}$). The positive and the negative elements are called signed ($\mathbb{S}^{\vee}_{\max} = \mathbb{S}^{\oplus}_{\max} \cup \mathbb{S}^{\ominus}_{\max}$). The $\ominus$ operation in $\mathbb{S}_{\max}$ is defined as follows:

$$\begin{aligned} a \ominus b &= a \quad \text{if } a > b\,, \\ a \ominus b &= \ominus b \quad \text{if } a < b\,, \\ a \ominus a &= a^{\bullet}\;. \end{aligned}$$

If $a \in \mathbb{S}_{\max}$ then it can be written as $a = a^{+} \ominus a^{-}$ where a^{+} is the positive part of a and a^{-} is the negative part of a. If $a \in \mathbb{S}^{\oplus}_{\max}$ then $a^{+} = a$ and $a^{-} = \epsilon$, if $a \in \mathbb{S}^{\ominus}_{\max}$ then $a^{+} = \epsilon$ and $a^{-} = \ominus a$ and if $a \in \mathbb{S}^{\bullet}_{\max}$ then $a^{+} = a^{-}$.

In $\mathbb{S}_{\max}$ we have to use balances (∇) instead of equalities. Loosely speaking an $\ominus$ sign in a balance indicates that the element should be at the other side. If both sides of a balance are signed we can replace the balance by an equality. So $x \ominus 4 \;\nabla\; 2$ means $x \;\nabla\; 2 \oplus 4$ and if x is signed we get $x = 2 \oplus 4 = 4$ as a solution.

To select submatrices of a matrix we use the following notation: $A([i_1, \ldots, i_k], [j_1, \ldots, j_l])$ is the matrix resulting from A by eliminating all rows except for rows $i_1, \ldots, i_k$ and all columns except for columns $j_1, \ldots, j_l$.

1.3 Some Definitions and Properties

Definition 1. (Determinant) Consider a matrix $A \in \mathbb{S}^{n\times n}_{\max}$. The determinant of A is defined as

$$\det A = \bigoplus_{\sigma \in \mathcal{P}_n} \operatorname{sgn}(\sigma) \otimes \bigotimes_{i=1}^{n} a_{i\sigma(i)}$$

where $\mathcal{P}_n$ is the set of all permutations of $\{1, \ldots, n\}$, and $\operatorname{sgn}(\sigma) = 0$ if the permutation σ is even and $\operatorname{sgn}(\sigma) = \ominus 0$ if the permutation is odd.

Theorem 2. *Let $A \in \mathbb{S}^{n\times n}_{\max}$. The homogeneous linear balance $A \otimes x \;\nabla\; \epsilon$ has a non-trivial signed solution if and only if* $\det A \;\nabla\; \epsilon$.

Proof. See [6]. The proof given there is constructive so it can be used to find a solution. □

Definition 3. (Characteristic equation) The characteristic equation of a matrix $A \in \mathbb{S}^{n\times n}_{\max}$ is defined as $\det(A \ominus \lambda \otimes E_n) \;\nabla\; \epsilon$.

This leads to

$$\lambda^{\otimes^{n}} \;\oplus\; \bigoplus_{p=1}^{n} a_p \otimes \lambda^{\otimes^{n-p}} \;\nabla\; \epsilon \tag{1}$$

with

$$a_p = (\ominus 0)^{\otimes^{p}} \otimes \bigoplus_{\varphi \in \mathcal{C}^n_p} \det A([i_1, i_2, \ldots, i_p], [i_1, i_2, \ldots, i_p]) \tag{2}$$

where $\mathcal{C}_p^n$ is the set of all combinations of p numbers out of $\{1, \ldots, n\}$ and $\varphi = \{i_1, i_2, \ldots, i_p\}$. Equation (1) will be called a *monic* balance, since the coefficient of $\lambda^{\otimes^n}$ equals 0 (i.e. the identity element for $\otimes$).

In $\mathbb{S}_{\max}$ every monic n-th order linear balance is the characteristic equation of an $n \times n$ matrix. However, this is not the case in $\mathbb{R}_{\max}$ as will be shown in the next section.

2 Necessary Conditions for a Polynomial to Be the Characteristic Polynomial of a Positive Matrix

A positive matrix is a matrix the elements of which lie in $\mathbb{R}_{\max}$. In this section we state necessary conditions for the coefficients of the characteristic polynomial of a positive matrix. These conditions will play an important role when one wants to determine the minimal order of a SISO system in the max algebra.

From now on we assume that $A \in \mathbb{R}_{\max}^{n \times n}$. If we define $\alpha_p = a_p^+$ and $\beta_p = a_p^-$ ($\alpha_p, \beta_p \in \mathbb{R}_{\max}$) and if we move all terms with negative coefficients to the right hand side (1) becomes

$$\lambda^{\otimes^n} \oplus \bigoplus_{i=2}^{n} \alpha_i \otimes \lambda^{\otimes^{n-i}} \; \nabla \; \beta_1 \otimes \lambda^{\otimes^{n-1}} \oplus \bigoplus_{j=2}^{n} \beta_j \otimes \lambda^{\otimes^{n-j}} \quad . \tag{3}$$

There are three possible cases: $\alpha_p = \epsilon$, $\beta_p = \epsilon$ or $\alpha_p = \beta_p$. We already have omitted α_1, since we always have that $a_1 \in \mathbb{S}_{\max}^{\ominus}$ and thus $\alpha_1 = a_1^+ = \epsilon$.

The most stringent property for α_p and β_p that was proven in [3] is:

Property 4. $\forall i \in \{2, \ldots, n\}$ *at least one of the following statements is true:*

$$\alpha_i \leq \bigoplus_{r=1}^{\lfloor \frac{i}{2} \rfloor} \beta_r \otimes \beta_{i-r} \quad \text{or} \quad \alpha_i < \bigoplus_{r=2}^{\lfloor \frac{i}{2} \rfloor} \alpha_r \otimes \alpha_{i-r} \quad \text{or} \quad \alpha_i < \bigoplus_{r=2}^{i-1} \alpha_r \otimes \beta_{i-r} \;,$$

where $\lfloor x \rfloor$ *stands for the largest integer number less than or equal to* x.

This property gives necessary conditions for the coefficients of an $\mathbb{S}_{\max}$ polynomial to be the characteristic polynomial of a positive matrix. For more properties and extensive proofs the reader is referred to [3].

3 Necessary and Sufficient Conditions for a Polynomial to Be the Characteristic Polynomial of a Positive Matrix

In the next subsections we determine case by case necessary and sufficient conditions for (3) to be the characteristic equation of a positive matrix and indicate how such a matrix can be found (see [3] for proofs). For the lower dimensional cases we can give an analytic description of the matrix we are looking for. For higher dimensional cases we shall first state a conjecture and then sketch a heuristic algorithm that will (in most cases) find a solution.

In all cases we have $\alpha_1 = \epsilon$ as a necessary condition. We also define $\kappa_{i,j} = \dfrac{\alpha_j}{\beta_i}$ if $\beta_i \neq \epsilon$ and $\kappa_{i,j} = \epsilon$ if $\beta_i = \epsilon$.

3.1 The 1 × 1 Case

There is no extra condition. The matrix $[\beta_1]$ has $\lambda \nabla \beta_1$ as its characteristic equation.

3.2 The 2 × 2 Case

The necessary and sufficient condition is: $\alpha_2 \leq \beta_1 \otimes \beta_1$. The characteristic equation of the matrix $\begin{bmatrix} \beta_1 & \beta_2 \\ 0 & \kappa_{1,2} \end{bmatrix}$ is $\lambda^{\otimes^2} \oplus \alpha_2 \nabla \beta_1 \otimes \lambda \oplus \beta_2$.

3.3 The 3 × 3 Case

The necessary and sufficient conditions are $\begin{cases} \alpha_2 \leq \beta_1 \otimes \beta_1 \\ \alpha_3 \leq \beta_1 \otimes \beta_2 \text{ or } \alpha_3 < \beta_1 \otimes \alpha_2 \end{cases}$.

The corresponding matrix is $\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \\ 0 & \kappa_{1,2} & \kappa_{1,3} \\ \epsilon & 0 & \epsilon \end{bmatrix}$.

3.4 The 4 × 4 Case

First we distinguish three possible cases:

Case A: $\alpha_4 \leq \beta_1 \otimes \beta_3$ or $\alpha_4 < \beta_1 \otimes \alpha_3$
Case B: $\alpha_4 > \beta_1 \otimes \beta_3$ and $\alpha_4 \geq \beta_1 \otimes \alpha_3$ and $\alpha_4 \leq \beta_2 \otimes \beta_2$ and ($\beta_1 = \epsilon$ or $\alpha_2 = \epsilon$ or $\beta_4 = \alpha_4$)
Case C: $\alpha_4 > \beta_1 \otimes \beta_3$ and $\alpha_4 \geq \beta_1 \otimes \alpha_3$ and $\alpha_4 \leq \beta_2 \otimes \beta_2$ and $\alpha_2 = \beta_2 \neq \epsilon$ and $\beta_4 = \epsilon$.

If the coefficients don't fall into exactly one of these three cases, they cannot correspond to a positive matrix.
The necessary and sufficient conditions are:

$$\begin{cases} \alpha_2 \leq \beta_1 \otimes \beta_1 \\ \alpha_3 \leq \beta_1 \otimes \beta_2 \text{ or } \alpha_3 < \beta_1 \otimes \alpha_2 \\ \text{for Case A: no extra conditions} \\ \text{for Case B: } \beta_1 \otimes \alpha_4 \leq \beta_2 \otimes \alpha_3 \text{ or } \beta_1 \otimes \alpha_4 < \beta_2 \otimes \beta_3 \\ \text{for Case C: } \beta_1 \otimes \alpha_3 = \beta_2 \otimes \alpha_2 \text{ and } \beta_1 \otimes \alpha_4 = \beta_2 \otimes \alpha_3 \end{cases}.$$

We find for Case A: $\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 0 & \kappa_{1,2} & \kappa_{1,3} & \kappa_{1,4} \\ \epsilon & 0 & \epsilon & \epsilon \\ \epsilon & \epsilon & 0 & \epsilon \end{bmatrix}$, for Case B: $\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 0 & \kappa_{1,2} & \kappa_{1,3} & \epsilon \\ \epsilon & 0 & \epsilon & \kappa_{2,4} \\ \epsilon & \epsilon & 0 & \epsilon \end{bmatrix}$ and

for Case C: $\begin{bmatrix} \beta_1 & \beta_2 & \epsilon & \epsilon \\ 0 & \epsilon & \epsilon & \epsilon \\ \epsilon & 0 & \kappa_{2,3} & \kappa_{2,4} \\ \epsilon & \epsilon & 0 & \epsilon \end{bmatrix}$.

3.5 The General Case

Here we have not yet found sufficient conditions, but we shall outline a heuristic algorithm that will in most cases result in a positive matrix for which the characteristic polynomial will be equal to the given polynomial.

Extrapolating the results of the previous subsections and supported by many examples we state the following conjecture:

Conjecture 5. *If* $\lambda^{\otimes^n} \oplus \bigoplus_{i=2}^{n} \alpha_i \otimes \lambda^{\otimes^{n-i}} \ \nabla\ \beta_1 \otimes \lambda^{\otimes^{n-1}} \oplus \bigoplus_{j=2}^{n} \beta_j \otimes \lambda^{\otimes^{n-j}}$ *is the characteristic equation of a matrix* $A \in \mathbb{R}_{\max}^{n \times n}$ *then it is also the characteristic equation of an upper Hessenberg matrix of the form*

$$K = \begin{bmatrix} k_{0,1} & k_{0,2} & k_{0,3} & \dots & k_{0,n-1} & k_{0,n} \\ 0 & k_{1,2} & k_{1,3} & \dots & k_{1,n-1} & k_{1,n} \\ \epsilon & 0 & k_{2,3} & \dots & k_{2,n-1} & k_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \epsilon & \epsilon & \epsilon & \dots & 0 & k_{n-1,n} \end{bmatrix} .$$

We shall use this conjecture in our heuristic algorithm to construct a matrix for which the characteristic polynomial will be equal to a given polynomial. However, in [5] we have presented a method to construct such a matrix that works even if Conjecture 5 would not be true. The major disadvantage of this method is its computational complexity. Therefore we now present a heuristic algorithm that will be much faster on the average. If a result is returned, it is right. But it could be possible that sometimes no result is returned although there is a solution (in which case we have to fall back on the method of [5]).

A heuristic algorithm:

First we check whether the coefficients of the given polynomial satisfy the conditions of Property 4. Then we reconstruct the a_p^-'s by setting $a_1^- = \beta_1$ and $a_p^- = \max(\alpha_p - \delta, \beta_p)$ for $p = 2, 3, \dots, n$ where δ is a small strictly positive real number.

Consider $K_1 = \begin{bmatrix} a_1^- & a_2^- & a_3^- & \dots & a_n^- \\ 0 & a_1^- & a_2^- & \dots & a_{n-1}^- \\ \epsilon & 0 & a_1^- & \dots & a_{n-2}^- \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \epsilon & \epsilon & \epsilon & \dots & a_1^- \end{bmatrix}$ and $K_2 = \begin{bmatrix} \epsilon & \epsilon & \epsilon & \dots & \epsilon \\ \epsilon & \kappa_{1,2} & \kappa_{1,3} & \dots & \kappa_{1,n} \\ \epsilon & \epsilon & \kappa_{2,3} & \dots & \kappa_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \epsilon & \epsilon & \epsilon & \dots & \kappa_{n-1,n} \end{bmatrix}$

where $\kappa_{i,j} = \dfrac{\alpha_j}{a_i^-}$ if $a_i^- \neq \epsilon$ and $\kappa_{i,j} = \epsilon$ if $a_i^- = \epsilon$.

We shall make a judicious choice out of the elements of K_1 and K_2 to compose a matrix for which the characteristic equation will coincide with (3).

We start with $A = \begin{bmatrix} a_1^- & a_2^- & \dots & a_n^- \\ 0 & \epsilon & \dots & \epsilon \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon & \epsilon & \dots & \epsilon \end{bmatrix}$. Now we shall column by column transfer non-ϵ elements of K_2 to A (one element per column) such that the coefficients of the characteristic equation of A are less than or equal to those of (3). If this doesn't lead to a valid result we shift a_1^- along its diagonal and repeat the procedure. We keep shifting a_1^- until it reaches the n-th column. If this still doesn't yield a result we put a_1^- back in the first column and repeat the procedure but now with a_2^-, and so on. Finally, if we have found A we remove redundant entries: these are elements that can be removed without altering the characteristic equation.

4 Minimal State Space Realization

4.1 Realization and Minimal Realization

Suppose that we have a single input single output (SISO) discrete event system that can be described by an n-th order state space model

$$x[k+1] = A \otimes x[k] \ \oplus \ b \otimes u[k] \tag{4}$$

$$y[k] = c \otimes x[k] \tag{5}$$

with $A \in \mathbb{R}_{\max}^{n \times n}$, $b \in \mathbb{R}_{\max}^{n \times 1}$ and $c \in \mathbb{R}_{\max}^{1 \times n}$. u is the input, y is the output and x is the state vector.

We define the unit impulse e as: $e[k] = 0$ if $k = 0$ and $e[k] = \epsilon$ otherwise. If we apply a unit impulse to the system and if we assume that the initial state $x[0]$ satisfies $x[0] = \epsilon$ or $A \otimes x[0] \leq b$, we get the impulse response as the output of the system:

$$x[1] = b\,,\ x[2] = A \otimes b\,,\ \dots\,,\ x[k] = A^{\otimes^{k-1}} \otimes b \ \Rightarrow\ y[k] = c \otimes A^{\otimes^{k-1}} \otimes b\ . \tag{6}$$

Let $g_k = c \otimes A^{\otimes^k} \otimes b$. The g_k's are called the Markov parameters.

Let us now reverse the process: suppose that A, b and c are unknown, and that we only know the Markov parameters (e.g. from experiments – where we assume that the system is max-linear and time-invariant and that there is no noise present). How can we construct A, b and c from the g_k's? This process is called realization. If we make the dimension of A minimal, we have a minimal realization. Although there have been some attempts to solve this problem [2, 7, 8], this problem has at present – to the authors' knowledge – not been solved entirely.

4.2 A Lower Bound for the Minimal System Order

Property 6. *The Markov parameters of the system with system matrix* $A \in \mathbb{S}_{\max}^{n \times n}$ *satisfy the characteristic equation of* A*:*

$$\bigoplus_{p=0}^{n} a_p \otimes g_{k+n-p} \; \nabla \; \epsilon \quad \textit{for } k = 0\,, 1,\, 2,\, \ldots \quad ,$$

where $a_0 = 0$*.*

Suppose that we have a system that can be described by (4) − (5), with unknown system matrices. If we want to find a minimal realization of this system the first question that has to be answered is that of the minimal system order. Consider the semi-infinite Hankel matrix $H = \begin{bmatrix} g_0 & g_1 & g_2 & \cdots \\ g_1 & g_2 & g_3 & \cdots \\ g_2 & g_3 & g_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$. Let $H(:, i)$ be the i-th column of H. As a direct consequence of Property 6 we have that

$$\bigoplus_{p=0}^{n} a_p \otimes H(:, k+n-p) \; \nabla \; \epsilon \qquad \text{for } k = 1,\, 2,\, \ldots \quad . \tag{7}$$

Now we reverse this reasoning: first we construct a p by q Hankel matrix

$$H_{p,q} = \begin{bmatrix} g_0 & g_1 & \cdots & g_{q-1} \\ g_1 & g_2 & \cdots & g_q \\ \vdots & \vdots & \ddots & \vdots \\ g_{p-1} & g_p & \cdots & g_{p+q-2} \end{bmatrix}$$

with p and q large enough: $p, q \gg n$, where n is the real (but unknown) system order. Then we try to find n and $a_0, a_1, \ldots, a_n$ such that the columns of $H_{p,q}$ satisfy an equation of the form (7), which will lead to the characteristic equation of the unknown system matrix A.

We propose the following procedure:
First we look for the largest square submatrix of $H_{p,q}$ with consecutive column indices,

$$H_{\text{sub},r} = H_{p,q}([i_1, i_2, \ldots, i_r], [j+1, j+2, \ldots, j+r]) \ ,$$

the determinant of which is not balanced: $\det H_{\text{sub},r} \not\nabla\, \epsilon$. If we add one arbitrary row and the $j+r+1$-st column to $H_{\text{sub},r}$ we get an $r+1$ by $r+1$ matrix $H_{\text{sub},r+1}$ that has a balanced determinant. So according to Theorem 2 the set of linear balances $H_{\text{sub},r+1} \otimes a \; \nabla \; \epsilon$ has a signed solution $a = \begin{bmatrix} a_r \; a_{r-1} \; \ldots \; a_0 \end{bmatrix}^t$. We now search a solution a that corresponds to the characteristic equation of a matrix with elements in $\mathbb{R}_{\max}$ (this should not necessarily be a signed solution). First of all we normalize a_0 to 0 and then we check if the necessary (and sufficient)

conditions of section 3 for α_p and β_p are satisfied, where $\alpha_p = a_p^+$ and $\beta_p = a_p^-$. If they are not satisfied we augment r and repeat the procedure.
We continue until we get the following stable relation among the columns of $H_{p,q}$:

$$H_{p,q}(:,k+r) \oplus a_1 \otimes H_{p,q}(:,k+r-1) \oplus \ldots \oplus a_r \otimes H_{p,q}(:,k) \nabla \epsilon \qquad (8)$$

for $k \in \{1,\ldots,q-r\}$. Since we assumed that the system can be described by (4) – (5) and that p, $q \gg n$, we can always find such a stable relationship by gradually augmenting r. The r that results from this procedure is a lower bound for the minimal system order.

4.3 Determination of the System Matrices

In [5] we have described a method to find all solutions of a set of multivariate polynomial equalities in the max algebra. Now we can use this method to find the A, b and c matrices of an r-th order SISO system with Markov parameters $g_0, g_1, g_2, \ldots$. If the algorithm doesn't find any solutions, this means that the output behavior can't be described by an r-th order SISO system. In that case we have to augment our estimate of the system order and repeat the procedure. Since we assume that the system can be described by the state space model (4) – (5) we shall always get a minimal realization.

However, in many cases we can use the results of the previous section to find a minimal realization. Starting from the coefficients $a_1, a_2, \ldots, a_r$ of (8) we search a matrix A with elements in $\mathbb{R}_{\max}$ such that its characteristic equation is

$$\lambda^{\otimes^n} \oplus \bigoplus_{p=1}^{r} a_p \otimes \lambda^{\otimes^{r-p}} \nabla \epsilon \ . \qquad (9)$$

Once we have found the A matrix, we have to find b and c with elements in $\mathbb{R}_{\max}$ such that

$$c \otimes A^{\otimes^k} \otimes b = g_k \quad \text{for } k = 0,1,2,\ldots \ . \qquad (10)$$

In practice it seems that we only have to take the transient behavior and the first cycles of the steady-state behavior into account. So we may limit ourselves to the first, say, N Markov parameters.

Let's take a closer look at equations of the form $c \otimes R \otimes b = s$ with $c \in \mathbb{R}_{\max}^{1\times n}, R \in \mathbb{R}_{\max}^{n\times n}, b \in \mathbb{R}_{\max}^{n\times 1}$ and $s \in \mathbb{R}_{\max}$. This equation can be rewritten as

$$\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} c_i \otimes r_{ij} \otimes b_i = s \ . \qquad (11)$$

So if we take the first N Markov parameters into account, we get a set of N multivariate polynomial equations in the max algebra, with the elements of b and c as unknowns and $R = A^{\otimes^{k-1}}$ and $s = g_{k-1}$ in the k-th equation. This problem can also be solved using the algorithm described in [5].

However, one has to be careful since it is not always possible to find a b and a c for every matrix that has (9) as its characteristic equation (see [3] for an example). In that case we have to search another A matrix or we could fall back on the method described in [4, 5], which finds all possible minimal realizations.

5 Example

We now illustrate the procedure of the preceding section with an example.

Example 1. Here we reconsider the example of [2, 8]. We start from a system with system matrices

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 0 \\ -3 & 1 & 2 \end{bmatrix} \ , \ b = \begin{bmatrix} 0 \\ \epsilon \\ \epsilon \end{bmatrix} \ \text{ and } \ c = \begin{bmatrix} 0 \ \epsilon \ \epsilon \end{bmatrix} \ .$$

Now we are going to construct the system matrices from the impulse response of the system. This impulse response is given by $\{g_k\} = 0\,,1\,,2\,,3\,,4\,,5\,,6\,,8\,,10\,,12\,,14\,,\ldots$. First we construct the Hankel matrix

$$H_{8,8} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 8 & 10 \\ 2 & 3 & 4 & 5 & 6 & 8 & 10 & 12 \\ 3 & 4 & 5 & 6 & 8 & 10 & 12 & 14 \\ 4 & 5 & 6 & 8 & 10 & 12 & 14 & 16 \\ 5 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\ 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 \\ 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 \end{bmatrix} \ .$$

The determinant of $H_{\text{sub},2} = H_{8,8}([1,7],[1,2]) = \begin{bmatrix} 0 & 1 \\ 6 & 8 \end{bmatrix}$ is not balanced. We add the second row and the third column and then we search a solution of the set of linear balances

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 6 & 8 & 10 \end{bmatrix} \otimes \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} \ \nabla \ \epsilon \ .$$

The solution $a_0 = 0$, $a_1 = \ominus 2$, $a_2 = 3$ satisfies the necessary and sufficient conditions for the 2 by 2 case since $\alpha_1 = \epsilon$ and $\alpha_2 = 3 \leq 4 = 2 \otimes 2 = \beta_1 \otimes \beta_1$. This solution also corresponds to a stable relation among the columns of $H_{8,8}$:

$$H_{8,8}(:,k+2) \ \oplus \ 3 \otimes H_{8,8}(:,k) = 2 \otimes H_{8,8}(:,k+1) \ ,$$

for $k \in \{1, 2, \ldots, 6\}$, or to the following characteristic equation:

$$\lambda^{\otimes^2} \ominus 2 \otimes \lambda \ \oplus \ 3 \ \nabla \ \epsilon \ .$$

This leads to a second order system with $A = \begin{bmatrix} 2 & \epsilon \\ 0 & 1 \end{bmatrix}$. Using the technique of [5] we get a whole set of solutions for b and c. One of the solutions is $b = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$ and $c = \begin{bmatrix} \epsilon \ 0 \end{bmatrix}$.
Apart from a permutation of the two state variables this result is the same as that of [8].

Another example, that doesn't satisfy the assumptions of [8] – where only impulse responses that exhibit a uniformly up-terrace behavior are considered –, can be found in [3].

6 Conclusions and Future Research

We have derived necessary and for some cases also sufficient conditions for an $\mathbb{S}_{\max}$ polynomial to be the characteristic polynomial of an $\mathbb{R}_{\max}$ matrix. So if we have a monic polynomial in $\mathbb{S}_{\max}$ these results allow us

1. to check whether the given polynomial can be the characteristic polynomial of a positive matrix and
2. to construct a matrix such that its characteristic polynomial is equal to the given polynomial.

Based on these results we have proposed a procedure to find a minimal state space realization of a SISO system, given its Markov parameters. This procedure is an alternative to the method of [4], which finds all possible minimal realizations but which has one disadvantage: its computational complexity. Since we allow a Hessenberg form for the system matrix A, our method incorporates both the companion form of [2] and the bidiagonal form of [8].

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