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The Extended Linear Complementarity Problem[1]

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Abstract. In this paper we define the Extended Linear Complementarity Problem (ELCP), an extension of the well-known Linear Complementarity Problem (LCP). We show that the ELCP can be viewed as a kind of unifying framework for the LCP and its various generalizations. We study the general solution set of an ELCP and we develop an algorithm to find all its solutions. We also show that the general ELCP is an NP-hard problem.

Keywords: linear complementarity problem, generalized linear complementarity problem, double description method.

1 Introduction

1.1 Overview

In this paper we propose the Extended Linear Complementarity Problem (ELCP), an extension of the well-known Linear Complementarity Problem (LCP), which is one of the fundamental problems of mathematical programming. We show that the ELCP can be viewed as a unifying framework for the LCP and its various extensions, such as the Vertical LCP of Cottle and Dantzig [5], the Generalized LCP of De Moor et al. [8, 9], the Extended Generalized Order LCP of Gowda and Sznajder [17], the Extended LCP of Mangasarian and Pang [20] and so on.

The formulation of the ELCP arose from our work in the study of discrete event systems, examples of which are flexible manufacturing systems, subway traffic networks, parallel processing systems and telecommunication networks. Some of these systems can be described using the so called max algebra [2, 7]. In [11, 12] we have demonstrated that many important problems in the max algebra such as solving a set of multivariate polynomial equalities and inequalities, matrix decompositions, state space transformations, minimal state space realization of max-linear discrete event systems and so on, can be reformulated as an ELCP. We shall illustrate this with an example. Although these problems do not always ask for the generation of the entire solution set of the corresponding ELCP, in some cases such as e.g. the (minimal) realization problem it can be interesting to obtain the entire solution set.

Therefore we also derive an algorithm to find all solutions of an ELCP. The core of this algorithm is formed by an adaptation and extension of Motzkin's double description method [25] for solving sets of linear inequalities. Our algorithm yields a description of the complete solution set of an ELCP by extreme rays and a basis for the linear subspace

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associated to the largest affine subspace of the solution set. In that way it provides a geometric insight in the solution set of the kind of problems mentioned above. The algorithm we propose is in fact a generalization and extension of the algorithm of [8, 9] for solving Generalized LCPs. We also use some concepts of graph theory such as *clique*. This paper thus borrows from a broad range of domains such as max algebra, linear algebra, mathematical programming and graph theory.

In Section 1 we introduce the notations and some of the concepts and definitions that will be used in this paper. We also give a concise introduction to the Linear Complementarity Problem. In Section 2 we propose the Extended Linear Complementarity Problem (ELCP) and show how it is linked to other Linear Complementarity Problems. Next we make a thorough study of the general solution set of an ELCP and we develop an algorithm to find all solutions of this problem. We also discuss the computational complexity of the ELCP. We conclude with an example of the application of the ELCP in the max algebra.

1.2 Notations and definitions

All the vectors that appear in this paper are assumed to be column vectors. If a is a vector then a_i or $(a)_i$ represents the ith component of a. If A is a matrix then the entry on the ith row and the jth column is denoted by a_{ij} or $(A)_{ij}$. We use $A_{i.}$ to denote the ith row of A and $A_{.j}$ to denote the jth column of A. The submatrix of A obtained by extracting the first k rows of A is represented by $A_{1:k,.}$. The n by n identity matrix is represented by I_n and the m by n zero matrix by $O_{m \times n}$. The transpose of A is denoted by A^T. If a is a vector with n components then $a \geqslant 0$ means that $a_i \geqslant 0$ for $i = 1, 2, \ldots, n$. Likewise $a = 0$ means $a_i = 0$ for $i = 1, 2, \ldots, n$.

If $\mathcal{A}$ is a set then $\#\mathcal{A}$ is the cardinality of $\mathcal{A}$. Consider a set of vectors $\mathcal{A} = \{a_1, a_2, \ldots, a_l\}$ with $a_i \in \mathbb{R}^n$ and define $a = \sum_{i=1}^{l} \alpha_i a_i$. If $\alpha_i \in \mathbb{R}$ then a is a linear combination of the vectors of $\mathcal{A}$. If $\alpha_i \geqslant 0$ we have a nonnegative combination. A nonnegative combination that also satisfies $\sum_{i=1}^{l} \alpha_i = 1$ is a convex combination.

Definition 1.1 (Polyhedron) *A polyhedron is the solution set of a finite system of linear inequalities.*

Definition 1.2 (Polyhedral cone) *A polyhedral cone is the set of solutions of a finite system of homogeneous linear inequalities.*

The definitions of the remainder of this subsection are based on [28].

Definition 1.3 (Face) *A subset F of a polyhedron $\mathcal{P}$ is called a face of $\mathcal{P}$ if $F = \mathcal{P}$ or if F is the intersection of $\mathcal{P}$ with a supporting hyperplane of $\mathcal{P}$.*

Note that each face of a polyhedron is also a (nonempty) polyhedron and that a k-dimensional face of a polyhedron $\mathcal{P}$ in $\mathbb{R}^n$ is the intersection of $\mathcal{P}$ and $n-k$ linearly independent hyperplanes from the constraints defining the polyhedron.

Definition 1.4 (Minimal face) *A minimal face of a polyhedron is a face not containing any other face.*

Definition 1.5 (Lineality space) *Let $\mathcal{P}$ be a polyhedron defined by $\mathcal{P} = \{x \mid Ax \geqslant b\}$. The lineality space of $\mathcal{P}$, denoted by $\mathcal{L}(\mathcal{P})$, is the linear subspace associated to the largest affine subspace of $\mathcal{P}$: $\mathcal{L}(\mathcal{P}) = \{x \mid Ax = 0\}$.*

A set $\mathcal{C}$ of basis vectors for $\mathcal{L}(\mathcal{P})$ is called a set of *central rays*. The dimension of $\mathcal{L}(\mathcal{P})$ is equal to $t = n - \text{rank}\,(A)$. If t is equal to 0, then $\mathcal{P}$ is called a *pointed* polyhedron. The minimal faces of $\mathcal{P}$ are translations of $\mathcal{L}(\mathcal{P})$. Hence the dimension of a minimal face of $\mathcal{P}$ is equal to t. Now consider a *polyhedral cone* $\mathcal{K}$ defined by $\mathcal{K} = \{x \mid Ax \geqslant 0\}$. Clearly, the only minimal face of $\mathcal{K}$ is its lineality space. Let t be the dimension of $\mathcal{L}(\mathcal{K})$. A face of $\mathcal{K}$ of dimension $t+1$ is called a *minimal proper face*. If G is a minimal proper face of the polyhedral cone $\mathcal{K}$ and if $e \in G$ with $e \neq 0$, then any arbitrary point u of G can be represented as

$$u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \kappa e \qquad \text{with } \lambda_k \in \mathbb{R} \text{ and } \kappa \geqslant 0$$

where $\mathcal{C}$ is a set of central rays of $\mathcal{K}$. We call e an *extreme ray* corresponding to G. If $\mathcal{K}_{\text{red}}$ is the pointed polyhedral cone obtained by subtracting the lineality space from $\mathcal{K}$, then extreme rays of $\mathcal{K}$ correspond to edges of $\mathcal{K}_{\text{red}}$. If $\mathcal{C}$ is a set of central rays of $\mathcal{K}$ and if $\mathcal{E}$ is a set of extreme rays of $\mathcal{K}$, obtained by selecting exactly one point of each minimal proper face of $\mathcal{K}$, then any arbitrary point u of $\mathcal{K}$ can be uniquely represented as

$$u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}} \kappa_k e_k \qquad \text{with } \lambda_k \in \mathbb{R} \text{ and } \kappa_k \geqslant 0 \ .$$

Definition 1.6 (Adjacency) *Two minimal faces of a polyhedron $\mathcal{P}$ are called adjacent if they are contained in one face of dimension $t+1$, where $t = \dim \mathcal{L}(\mathcal{P})$.*
Two minimal proper faces of a polyhedral cone $\mathcal{K}$ are called adjacent if they are contained in one face of dimension $t+2$, where $t = \dim \mathcal{L}(\mathcal{K})$. Extreme rays corresponding to these faces are then also called adjacent.

1.3 The Linear Complementarity Problem

One of the possible formulations of the LCP is the following [6]:

Given a matrix $M \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, find two vectors $w, z \in \mathbb{R}^n$ such that

$$w, z \geqslant 0$$
$$w = q + Mz$$
$$z^T w = 0 \ ,$$

or show that no such vectors w and z exist.

The LCP has numerous applications such as linear and quadratic programming problems, the bimatrix game problem, the market equilibrium problem, the optimal invariant capital stock problem, the optimal stopping problem, etc. [6]. This makes the LCP one of the fundamental problems of mathematical programming.

2 The Extended Linear Complementarity Problem

In this section we introduce the Extended Linear Complementarity Problem (ELCP) and we establish a link between the ELCP and the Linear Complementarity Problem (LCP). We also show that many generalizations of the LCP can be considered as special cases of the ELCP.

2.1 Problem formulation

Consider the following problem:

> Given two matrices $A \in \mathbb{R}^{p \times n}, B \in \mathbb{R}^{q \times n}$, two vectors $c \in \mathbb{R}^p, d \in \mathbb{R}^q$ and m subsets ϕ_j, $j = 1, 2, \ldots, m$, of $\{1, 2, \ldots, p\}$, find a vector $x \in \mathbb{R}^n$ such that
>
> $$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i = 0 \tag{1}$$
>
> $$\text{subject to} \quad \begin{array}{rcl} Ax & \geqslant & c \\ Bx & = & d \end{array} ,$$
>
> or show that no such vector exists.

In Section 2.3 we demonstrate that this problem is an extension of the Linear Complementarity Problem (LCP). Therefore we call it the Extended Linear Complementarity Problem (ELCP). Equation (1) represents the *complementarity condition.* One possible interpretation of this condition is the following: since $Ax \geqslant c$, condition (1) is equivalent to

$$\prod_{i \in \phi_j} (Ax - c)_i = 0 \qquad \text{for } j = 1, 2, \ldots, m \ .$$

So we could say that each set ϕ_j corresponds to a subgroup of inequalities of $Ax \geqslant c$ and that in each group at least one inequality should hold with equality:

$$\forall j \in \{1, 2, \ldots, m\} : \exists i \in \phi_j \text{ such that } (Ax - c)_i = 0 \ .$$

2.2 The homogeneous ELCP

Now we homogenize the ELCP: we introduce a scalar $\alpha \geqslant 0$ and define $u = \begin{bmatrix} x \\ \alpha \end{bmatrix}$, $P = \begin{bmatrix} A & -c \\ O_{1 \times n} & 1 \end{bmatrix}$ and $Q = [\, B \ \ -d\,]$. Then we get an *homogeneous* ELCP:

> Given two matrices $P \in \mathbb{R}^{p \times n}, Q \in \mathbb{R}^{q \times n}$ and m subsets ϕ_j of $\{1, 2, \ldots, p\}$, find a (non-trivial) vector $u \in \mathbb{R}^n$ such that
>
> $$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Pu)_i = 0 \tag{2}$$
>
> $$\text{subject to} \quad \begin{array}{rcl} Pu & \geqslant & 0 \\ Qu & = & 0 \end{array} ,$$
>
> or show that no such vector u exists.

In Section 4 we shall develop an algorithm to solve this homogeneous ELCP. Afterwards we shall extract the solutions of the original inhomogeneous ELCP.
It is sometimes advantageous to use an alternative form of the complementarity condition: since $Pu \geqslant 0$, condition (2) is equivalent to

$$\prod_{i \in \phi_j} (Pu)_i = 0 \qquad \text{for } j = 1, 2, \ldots, m \ . \tag{3}$$

2.3 Link with the LCP

The LCP can be considered as a particular case of the ELCP: if we set $x = \begin{bmatrix} w \\ z \end{bmatrix}$, $A = I_{2n}$, $B = [\, I_n \ \ -M \,]$, $c = O_{2n \times 1}$, $d = q$ and $\phi_j = \{j, j+n\}$ for $j = 1, 2, \ldots, n$ in the formulation of the ELCP we get an LCP.

2.4 Link with the Horizontal LCP

A problem that is slightly more general than the LCP is the so called Horizontal Linear Complementarity Problem (HLCP), which can be formulated as follows [6]:

> Given 2 matrices $M, N \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, find two non-trivial vectors $w, z \in \mathbb{R}^n$ such that
>
> $$w, z \geqslant 0$$
> $$Mz + Nw = q$$
> $$z^T w = 0 \ .$$

The term *horizontal* is used to characterize the geometric shape of the matrix $[\, M \ N \,]$ since the number of rows of this matrix is less than the number of columns. It is obvious that the HLCP is also a particular case of the ELCP.

2.5 Link with the Vertical LCP

In [5] Cottle and Dantzig introduced a generalization of the LCP which is now called the Vertical Linear Complementarity Problem (VLCP) and is defined as follows [6]:

> Let M be a matrix of order $m \times n$ with $m \geqslant n$, and let q be a vector with m components. Suppose that M and q are partitioned in the following form:
>
> $$M = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$
>
> where each $M_i \in \mathbb{R}^{m_i \times n}$ and $q_i \in \mathbb{R}^{m_i}$ with $\sum_{i=1}^{n} m_i = m$. Now find a vector $z \in \mathbb{R}^n$ such that
>
> $$q + Mz \geqslant 0$$

$$z \geqslant 0$$

$$z_i \prod_{j=1}^{m_i} (q_i + M_i z)_j \;=\; 0 \qquad \text{for } i = 1, 2, \ldots, n \;\;.$$

Since the number of rows of M is greater than or equal to the number of columns this problem is a *vertical* generalization of the LCP.

The VLCP is also a particular case of the inhomogeneous ELCP: let $x = z$, $A = \begin{bmatrix} M \\ I_n \end{bmatrix}$, $B = [\,]$, $c = \begin{bmatrix} -q \\ O_{n\times 1} \end{bmatrix}$, $d = [\,]$ and $\phi_j = \{s_j + 1, s_j + 2, \ldots, s_j + m_j, m + j\}$ for $j = 1, 2, \ldots, n$ with $s_1 = 0$ and $s_{j+1} = s_j + m_j$.

2.6 Link with the GLCP

In [8, 9] De Moor introduced the following Generalized Linear Complementarity Problem (GLCP):

> Given a matrix $Z \in \mathbb{R}^{p\times n}$ and m subsets ϕ_j of $\{1, 2, \ldots, p\}$, find a non-trivial vector $u \in \mathbb{R}^n$ such that
>
> $$\sum_{j=1}^{m} \prod_{i\in\phi_j} u_i \;=\; 0$$
>
> $$\text{subject to} \quad \begin{aligned} u &\geqslant 0 \\ Zu &= 0 \;. \end{aligned}$$

Now we show that the homogeneous ELCP and the GLCP are equivalent: that is, if we can solve the homogeneous ELCP we can also solve the GLCP and vice versa.

Theorem 2.1 *The homogeneous ELCP and the GLCP are equivalent.*

Proof:
The GLCP is a special case of the homogeneous ELCP since setting $P = I_n$ and $Q = Z$ in the definition of the homogeneous ELCP yields a GLCP.

Now we prove that an homogeneous ELCP can be transformed into a GLCP.
First we define the sign decomposition of u: $u = u^+ - u^-$ with $u^+, u^- \geqslant 0$ and $(u^+)^T u^- = 0$. Next we introduce a vector of nonnegative slack variables $s \in \mathbb{R}^p$ such that $Pu - I_p s = 0$. Since $Pu = I_p s$, the complementarity condition $\sum_{j=1}^{m} \prod_{i\in\phi_j} (Pu)_i = 0$ is equivalent to $\sum_{j=1}^{m} \prod_{i\in\phi_j} s_i = 0$. Because the components of u^+, u^- and s are nonnegative we can combine the latter condition with the condition $(u^+)^T u^- = 0$, which yields the new complementarity condition $\sum_{i=1}^{n} u_i^+ u_i^- + \sum_{j=1}^{m} \prod_{i\in\phi_j} s_i = 0$. Finally we define $n + m$ subsets ϕ'_j such that

$$\begin{aligned} \phi'_j \;&=\; \{j, j+n\} && \text{for } j = 1, 2, \ldots, n\,, \\ &=\; \{i + 2n \,|\, i \in \phi_{j-n}\} && \text{for } j = n+1, n+2, \ldots, n+m \;\;. \end{aligned}$$

This leads to the following GLCP:

$$\text{Find } v = \begin{bmatrix} u^+ \\ u^- \\ s \end{bmatrix} \text{ such that } \sum_{j=1}^{n+m} \prod_{i \in \phi'_j} v_i = 0$$

$$\text{subject to } v \geqslant 0 \text{ and } \begin{bmatrix} P & -P & -I_p \\ Q & -Q & O_{q\times p} \end{bmatrix} v = 0 \ .$$

Hence we have proved that the ELCP and the GLCP are equivalent. □

In [8] an algorithm was derived to find all solutions of a GLCP. Since the GLCP and the ELCP are equivalent we could use that algorithm to solve the ELCP. However, there are a few drawbacks:

- To convert the ELCP into a GLCP we have introduced extra variables: u^- and the slack variables (one for each inequality). This increases the complexity of the problem. Because the execution time of the algorithm of [8] grows rapidly as the number of unknowns grows, it is not advantageous to have a large number of variables. Since the number of intermediate solutions and thus the required storage space also grows with the number of variables, the problem can even become intractable in practice if the number of variables is too large. Moreover, we do not need the extra slack variables, since they will be dropped at the end anyway.

- The solutions set of a GLCP is characterized by a set of extreme rays $\mathcal{E}$ and a set Γ of so called cross-complementary subsets of $\mathcal{E}$ such that any arbitrary solution of the GLCP can be written as $u = \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k$ with $\kappa_k \geqslant 0$ for some subset $\mathcal{E}_s \in \Gamma$[4]. Even if there is no redundancy in the description of the solution set of the GLCP after dropping the slack variables, it is possible that the transition from u^+ and u^- to u results in redundant rays. It is also possible that some of the cross-complementary sets can be taken together. This means that in general we do not get a minimal description of the solution set of the ELCP.

We certainly do much unnecessary work if we use the detour via the GLCP. Therefore we shall develop a separate algorithm to solve the ELCP, in which we do not have to introduce extra variables and that will yield a concise description of the solution set. This algorithm will also be much faster than an algorithm that uses the transformation into a GLCP.

2.7 Link with other generalizations of the LCP

In [17] Gowda and Sznajder have introduced the Generalized Order Linear Complementarity Problem (GOLCP) and the Extended Generalized Order Linear Complementarity Problem (EGOLCP). The EGOLCP is defined as follows:

Given $k+1$ matrices $B_0, B_1, \ldots, B_k \in \mathbb{R}^{n\times n}$ and $k+1$ vectors $b_0, b_1, \ldots, b_k \in \mathbb{R}^n$, find a vector $x \in \mathbb{R}^n$ such that

$$(B_0 x + b_0) \wedge (B_1 x + b_1) \wedge \ldots \wedge (B_k x + b_k) \ = \ 0$$

where $\wedge$ is the entrywise minimum: if $x, y \in \mathbb{R}^n$ then $x \wedge y \in \mathbb{R}^n$ and $(x \wedge y)_i = \min\{x_i, y_i\}$.

[4] In Section 4 we shall show how the sets $\mathcal{E}$ and Γ can be calculated and what they represent.

If we take $B_0 = I_n$ and $b_0 = O_{n\times 1}$ then we have a GOLCP.
The EGOLCP is a special case of the ELCP: since the entrywise minimum of the vectors $B_i x + b_i$ is equal to 0 we should have that $B_i x \geqslant -b_i$ and for every $j \in \{1, 2, \ldots, n\}$ there should exist at least one i such that $(B_i x + b_i)_j = 0$. So if we put all matrices B_i in one large matrix $A = \begin{bmatrix} B_0 \\ \vdots \\ B_k \end{bmatrix}$ and all vectors b_i in one large vector $c = \begin{bmatrix} -b_0 \\ \vdots \\ -b_k \end{bmatrix}$ and if we define n sets ϕ_j such that $\phi_j = \{j, j+n, \ldots, j+kn\}$ for $j = 1, 2, \ldots, n$, then we get an ELCP:

$$\text{Find } x \in \mathbb{R}^n \text{ such that } \sum_{j=1}^{n} \prod_{i \in \phi_j} (Ax - c)_i = 0 \text{ subject to } Ax \geqslant c\,,$$

that is equivalent to the original EGOLCP.

The Extended Linear Complementarity Problem of Mangasarian and Pang [16, 20]:

Given two matrices $M, N \in \mathbb{R}^{m\times n}$ and a polyhedral set $\mathcal{P}$ in $\mathbb{R}^m$, find two vectors $x, y \in \mathbb{R}^n$ such that

$$x, y \geqslant 0$$

$$Mx - Ny \in \mathcal{P}$$

$$x^T y = 0 \ ,$$

is also a special case of our ELCP:
We may assume without loss of generality that $\mathcal{P}$ can be represented as $\mathcal{P} = \{u \in \mathbb{R}^m \mid Au \geqslant b\}$ for some matrix $A \in \mathbb{R}^{l\times m}$ and some vector $b \in \mathbb{R}^l$. Hence the condition $Mx - Ny \in \mathcal{P}$ is equivalent to $AMx - ANy \geqslant b$. If we define $v = \begin{bmatrix} x \\ y \end{bmatrix}$ then we get the following ELCP:

$$\text{Find } v \in \mathbb{R}^{2n} \text{ such that } \sum_{i=1}^{n} v_i v_{i+n} = 0 \text{ subject to } v \geqslant 0 \text{ and } [\, AM \quad -AN \,]\, v \geqslant b\,.$$

Furthermore, it is easy to show that the Generalized LCP of Ye [33]:

Given $A, B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{m\times k}$ and $q \in \mathbb{R}^m$, find $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}^k$ such that

$$x, y, z \geqslant 0$$

$$Ax + By + Cz = q$$

$$x^T y = 0 \ ,$$

the mixed LCP [6]:

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, $C \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{m\times n}$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, find $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ such that

$$a + Au + Cv = 0$$

$$b + Du + Bv \geqslant 0$$

$$v \geqslant 0$$

$$v^T (b + Du + Bv) = 0 \ ,$$

and the Extended HLCP of Sznajder and Gowda [30]:

Given $k+1$ matrices $C_0, C_1, \ldots, C_k \in \mathbb{R}^{n \times n}$, $q \in \mathbb{R}^n$ and $k-1$ vectors $d_1, d_2, \ldots, d_{k-1} \in \mathbb{R}^n$ with positive components, find $x_0, x_1, \ldots, x_k \in \mathbb{R}^n$ such that

$$C_0 x_0 = q + \sum_{j=1}^{k} C_j x_j$$

$$x_0, x_1, \ldots, x_k \geqslant 0$$

$$d_j - x_j \geqslant 0 \qquad \text{for } j = 1, 2, \ldots, k-1$$

$$x_0^T x_1 = 0$$

$$(d_j - x_j)^T x_{j+1} = 0 \qquad \text{for } j = 1, 2, \ldots, k-1 \ ,$$

are also special cases of the ELCP.

Conclusion: As can be seen from this and the previous subsections, the ELCP can indeed be considered as a unifying framework for the LCP and its various generalizations.

The underlying geometric explanation for the fact that all these generalizations of the LCP can be considered as particular cases of the ELCP is that they all have a solution set that either is empty or consists of the union of faces of a polyhedron, and that the union of any arbitrary set of faces of an arbitrary polyhedron can be described by an ELCP (see Theorem 4.15). For more information on the LCP and the various generalizations discussed above and for applications, properties and methods to solve these problems the interested reader is referred to [5, 6, 8, 9, 13, 16, 17, 18, 20, 29, 30, 31, 32, 33, 34] and the references therein.

3 The solution set of the homogeneous ELCP

In this section we discuss some properties of the solution set of the homogeneous ELCP. Note that the homogeneous ELCP can be considered as a system of homogeneous linear equalities and inequalities subject to a complementarity condition. The solution set of the system of homogeneous linear inequalities and equalities

$$\begin{aligned} Pu &\geqslant 0 \\ Qu &= 0 \ , \end{aligned}$$

is a polyhedral cone $\mathcal{K}$. We already know that an arbitrary point of $\mathcal{K}$ can be uniquely represented as

$$u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}} \kappa_k e_k \qquad \text{with } \lambda_k \in \mathbb{R} \text{ and } \kappa_k \geqslant 0$$

where $\mathcal{C}$ is a set of central rays of $\mathcal{K}$ and $\mathcal{E}$ is a set of extreme rays of $\mathcal{K}$. If c is a central ray then we have that $Pc = 0$. By analogy we call all points $u \in \mathcal{K}$ that satisfy $Pu = 0$ *central solutions* of $\mathcal{K}$ and all points $u \in \mathcal{K}$ that satisfy $Pu \neq 0$ *non-central solutions.* Note that if e

is an extreme ray then we have that $Pe \neq 0$.
Later we shall show that every solution of the homogeneous ELCP can be written as

$$u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \qquad \text{with } \lambda_k \in \mathbb{R} \text{ and } \kappa_k \geqslant 0$$

for some subset $\mathcal{E}_s$ of $\mathcal{E}$ (see Theorem 4.11). Note that we always have a trivial solution for the homogeneous ELCP: $u = [\,0\;\;0\;\;\ldots\;\;0\,]^T$.
In the next section we shall present an algorithm to calculate $\mathcal{C}$ and $\mathcal{E}$. But first we give some properties of the solution set of the homogeneous ELCP.

Property 3.1 *If c is a central solution of the polyhedral cone defined by $Pu \geqslant 0$ and $Qu = 0$ then we have that $\forall \lambda \in \mathbb{R} : \lambda c$ is a solution of the homogeneous ELCP.*

Proof: Since c is a central solution, we have that

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} \left(P(\lambda c)\right)_i = \sum_{j=1}^{m} \prod_{i \in \phi_j} \lambda (Pc)_i = 0 \ .$$

Furthermore, we have that $P(\lambda c) = \lambda(Pc) = 0 \geqslant 0$ and $Q(\lambda c) = \lambda(Qc) = 0$. So λc is indeed a solution of the ELCP. □

Note that every central solution of the polyhedral cone defined by $Pu \geqslant 0$ and $Qu = 0$ automatically satisfies the complementarity condition.

Property 3.2 *If u is a solution of the homogeneous ELCP then $\forall \kappa \geqslant 0 : \kappa u$ is also a solution of the homogeneous ELCP.*

Proof:

$$\begin{aligned}
\sum_{j=1}^{m} \prod_{i \in \phi_j} \left(P(\kappa u)\right)_i &= \sum_{j=1}^{m} \prod_{i \in \phi_j} \kappa (Pu)_i \\
&= \sum_{j=1}^{m} \kappa^{\#\phi_j} \prod_{i \in \phi_j} (Pu)_i \\
&= 0 \qquad\qquad \text{because of complementarity condition (3)} \ .
\end{aligned}$$

We have that $P(\kappa u) = \kappa(Pu) \geqslant 0$ because $Pu \geqslant 0$ and $\kappa \geqslant 0$. We also have that $Q(\kappa u) = \kappa(Qu) = 0$. So κu is a solution of the ELCP. □

Now we prove that extreme rays that do not satisfy the complementarity condition cannot yield a solution of the ELCP. In our algorithm such rays will therefore immediately be removed from $\mathcal{E}$.

Property 3.3 *If $e_l \in \mathcal{E}$ does not satisfy the complementarity condition then we have that $\forall \mathcal{E}_s \subset \mathcal{E}$ with $e_l \in \mathcal{E}_s$, $\forall \lambda_k \in \mathbb{R}$, $\forall \kappa_k \geqslant 0$ with $\kappa_l > 0$: $u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k$ does not satisfy the complementarity condition.*

Proof: If e_l does not satisfy the complementarity condition then

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Pe_l)_i \neq 0 \ .$$

Since $Pe_l \geqslant 0$ this is only possible if

$$\exists j \in \{1, 2, \ldots, m\} \text{ such that } \forall i \in \phi_j : (Pe_l)_i \neq 0 \ . \tag{4}$$

Now assume that $u = \sum\limits_{c_k \in \mathcal{C}} \lambda_k c_k + \sum\limits_{e_k \in \mathcal{E}_s} \kappa_k e_k$ satisfies the complementarity condition. Then we have that

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} \left(P \left(\sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \right) \right)_i = 0$$

or

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} \left(\sum_{c_k \in \mathcal{C}} \lambda_k (Pc_k)_i + \sum_{e_k \in \mathcal{E}_s} \kappa_k (Pe_k)_i \right) = 0$$

and since $\forall c \in \mathcal{C} : Pc = 0$, we get

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} \left(\kappa_l (Pe_l)_i + \sum_{e_k \in \mathcal{E}_s \setminus \{e_l\}} \kappa_k (Pe_k)_i \right) = 0 \ .$$

Because $Pe_k \geqslant 0, Pe_l \geqslant 0, \kappa_k \geqslant 0$ and $\kappa_l > 0$ this is only possible if

$$\forall j \in \{1, 2, \ldots, m\}, \exists i \in \phi_j \text{ such that } \kappa_l (Pe_l)_i + \sum_{e_k \in \mathcal{E}_s \setminus \{e_l\}} \kappa_k (Pe_k)_i = 0$$

and also

$$\forall j \in \{1, 2, \ldots, m\}, \exists i \in \phi_j \text{ such that } (Pe_l)_i = 0$$

since $\kappa_l > 0$. But this is in contradiction with (4). Hence our initial assumption was false, which means that u does not satisfy the complementarity condition. □

4 An algorithm to find all solutions of an ELCP

In this section we shall derive an algorithm to find all solutions of a general ELCP. As was already indicated in Section 2.2 we shall first solve the corresponding homogeneous ELCP and afterwards we shall extract the solutions of the inhomogeneous ELCP.
So now we consider an homogeneous ELCP. To enhance the efficiency of the algorithm we first extract the inequalities of $Pu \geqslant 0$ that appear in the complementarity condition and put them in $P_1 u \geqslant 0$. The remaining inequalities are put in $P_2 u \geqslant 0$. If we also adapt the sets ϕ_j accordingly we get an ELCP of the following form:

Given three matrices $P_1 \in \mathbb{R}^{p_1 \times n}, Q \in \mathbb{R}^{q \times n}, P_2 \in \mathbb{R}^{p_2 \times n}$ and m subsets ϕ_j of $\{1, 2, \ldots, p_1\}$, find a vector $u \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (P_1 u)_i = 0$$

$$\text{subject to} \quad \begin{aligned} P_1 u &\geqslant 0 \\ Q u &= 0 \\ P_2 u &\geqslant 0 \ . \end{aligned}$$

Note that we now have that $\bigcup_{j=1}^{m} \phi_j = \{1, 2, \ldots, p_1\}$.

The ELCP algorithm consists of 3 parts:

Part 1: Find all solutions of $P_1 u \geqslant 0$ that satisfy the complementarity condition. We describe the solution set of this problem with central and extreme rays.

Part 2: Take the conditions $Qu = 0$ and $P_2 u \geqslant 0$ into account.

Part 3: Determine which combinations of the central and extreme rays are solutions of the ELCP: i.e. determine the so called cross-complementary sets.

Now we go through the algorithm part by part. We represent the different parts of the algorithm in their most rudimentary form. In the remarks after each algorithm we indicate how one can improve the numerical stability and the performance of the algorithm.

4.1 Find all solutions of a system of linear inequalities that satisfy the complementarity condition

The algorithm of this subsection is an extension and adaptation of the double description method of [25] to find all solutions of a system of linear inequalities. We have adapted it to get a more concise description of the solution set and we have added tests to reject solutions that do not satisfy the complementarity condition. In this iterative algorithm we take a new inequality into account in each step and we determine the intersection of the current polyhedral cone – described by a set of central rays $\mathcal{C}$ and a set of extreme rays $\mathcal{E}$ – with the half-space determined by this inequality. We also immediately remove the rays that do not satisfy the complementarity condition.
We give the algorithm to calculate $\mathcal{C}$ and $\mathcal{E}$ in a pseudo programming language. $\leftarrow$ indicates an assignment. Italic text inside curly brackets $\{\}$ is meant to be a comment.

Algorithm 1: solve a system of linear inequalities subject to the complementarity condition

Input: $p_1, n, P_1 \in \mathbb{R}^{p_1 \times n}, \{\phi_j\}_{j=1}^{m}$

Initialization:

$\mathcal{C} \leftarrow \{c_i \,|\, c_i = (I_n)_{.i} \text{ for } i = 1, 2, \ldots, n\}$

$\mathcal{E} \leftarrow \emptyset$

$P_{\text{nec}} \leftarrow [\,]$

Main loop:

for $k = 1, 2, \dots, p_1$ **do** { *The rows of P_1 are taken one by one.* }

  $\forall s \in \mathcal{C} \cup \mathcal{E} : \mathrm{res}\,(s) \leftarrow (P_1)_{k.}\, s$ { *Calculate the residues.* }

  $\mathcal{C}^+ \leftarrow \{c \in \mathcal{C} \mid \mathrm{res}\,(c) > 0\}$

  $\mathcal{C}^- \leftarrow \{c \in \mathcal{C} \mid \mathrm{res}\,(c) < 0\}$

  $\mathcal{C}^0 \leftarrow \{c \in \mathcal{C} \mid \mathrm{res}\,(c) = 0\}$

  $\mathcal{E}^+ \leftarrow \{e \in \mathcal{E} \mid \mathrm{res}\,(e) > 0\}$

  $\mathcal{E}^- \leftarrow \{e \in \mathcal{E} \mid \mathrm{res}\,(e) < 0\}$

  $\mathcal{E}^0 \leftarrow \{e \in \mathcal{E} \mid \mathrm{res}\,(e) = 0\}$

  if $\mathcal{C}^+ = \emptyset$ **and** $\mathcal{C}^- = \emptyset$ **and** $\mathcal{E}^- = \emptyset$ **then** { *Case 1* }

    { *The kth inequality is redundant.* }

    $\mathcal{E} \leftarrow \mathcal{E}^0 \cup \{ e \in \mathcal{E}^+ \mid e$ satisfies the partial complementarity condition $\}$

  else

    if $\mathcal{C}^+ = \emptyset$ **and** $\mathcal{C}^- = \emptyset$ **then** { *Case 2* }

      $\mathcal{E} \leftarrow \mathcal{E}^0 \cup \{ e \in \mathcal{E}^+ \mid e$ satisfies the partial complementarity condition $\}$

      for all pairs $(e^+, e^-) \in \mathcal{E}^+ \times \mathcal{E}^-$ **do**

        if e^+ and e^- are adjacent **then**

          $e^{\mathrm{new}} \leftarrow \mathrm{res}\,(e^+)\, e^- - \mathrm{res}\,(e^-)\, e^+$

          if e^{new} satisfies the partial complementarity condition **then**

            $\mathcal{E} \leftarrow \mathcal{E} \cup \{ e^{\mathrm{new}} \}$

          endif

        endif

      endfor

    else { *Case 3* }

      $\mathcal{C} \leftarrow \mathcal{C}^0$

      $\mathcal{E} \leftarrow \mathcal{E}^0$

      $\mathcal{C}^+ \leftarrow \mathcal{C}^+ \cup \{-s \mid s \in \mathcal{C}^-\}$

      $\forall s \in \mathcal{C}^- : \mathrm{res}\,(-s) \leftarrow -\mathrm{res}\,(s)$ { *Adapt the residues.* }

      Take one ray $c \in \mathcal{C}^+$.

      if c satisfies the partial complementarity condition **then**

        $\mathcal{E} \leftarrow \mathcal{E} \cup \{c\}$

      endif

      $\forall c^+ \in \mathcal{C}^+ \setminus \{c\} : \mathcal{C} \leftarrow \mathcal{C} \cup \{ \mathrm{res}\,(c^+)\, c - \mathrm{res}\,(c)\, c^+ \}$

      for all $e \in \mathcal{E}^+ \cup \mathcal{E}^-$ **do**

        $e^{\mathrm{new}} \leftarrow \mathrm{res}\,(c)\, e - \mathrm{res}\,(e)\, c$ **do**

        if e^{new} satisfies the partial complementarity condition **then**

          $\mathcal{E} \leftarrow \mathcal{E} \cup \{ e^{\mathrm{new}} \}$

        endif

      endfor

    endif

Add the kth row of P_1 to P_{nec}.

endif

endfor

Output: $\mathcal{C}, \mathcal{E}, P_{\text{nec}}$

Remarks:

1. If s_1 and s_2 are two rays in the kth step and if $\text{res}\,(s_1)\,\text{res}\,(s_2) < 0$ then the new ray

$$s = |\text{res}\,(s_1)\,|\,s_2 + |\text{res}\,(s_2)\,|\,s_1 \tag{5}$$

will satisfy $(P_1)_{k.}s = 0$, in other words, ray s will lie in the hyperplane defined by the kth row of P_1.

Proof: Without loss of generality we may assume that $\text{res}\,(s_1) > 0$ and $\text{res}\,(s_2) < 0$. Then we have that

$$\begin{aligned}
(P_1)_{k.}s &= (P_1)_{k.}\,(|\text{res}\,(s_1)\,|s_2 + |\text{res}\,(s_2)\,|s_1) \\
&= (P_1)_{k.}\,(\text{res}\,(s_1)\,s_2 - \text{res}\,(s_2)\,s_1) \\
&= \text{res}\,(s_1)\,(P_1)_{k.}s_2 - \text{res}\,(s_2)\,(P_1)_{k.}s_1 \\
&= \text{res}\,(s_1)\,\text{res}\,(s_2) - \text{res}\,(s_2)\,\text{res}\,(s_1) \\
&= 0\ . \qquad \Box
\end{aligned}$$

In our algorithm we have worked out the absolute values in (5), which leads to the different expressions for constructing new rays.

2. Because in each main loop we have to combine intermediate rays it is advantageous to have as few intermediate rays as possible. The complementarity test is one way to reject rays. We cannot use the complete complementarity condition (2) when we are processing the kth inequality since this complementarity condition takes all inequalities into account. However, if we consider the equivalent complementarity condition (3) then it is obvious that we can apply the condition for ϕ_j to eliminate extreme rays as soon as we have considered all inequalities that correspond to that particular ϕ_j. That is why we use a partial complementarity test. In the kth step the *partial complementarity condition* is:

$$\prod_{i\in\phi_j} (P_1u)_i = 0 \qquad \forall j \in \{1,2,\ldots,m\} \text{ such that } \phi_j \subset \{1,2,\ldots,k\}\ . \tag{6}$$

If there are no sets ϕ_j such that $\phi_j \subset \{1,2,\ldots,k\}$ then the partial complementarity condition is satisfied by definition.
We know that extra rays can only be constructed by taking positive combinations of other rays as indicated by (5). Because of Property 3.3, which is also valid for the partial complementarity condition, any ray that does not satisfy the (partial) complementarity condition cannot yield a ray that satisfies the complementarity condition. Therefore we

can reject such rays immediately.
Since central rays automatically satisfy the complementarity condition, we only have to check the extreme rays. We can even be more specific. According to the following property we only have to test new extreme rays and extreme rays that have a non-zero residue.

Property 4.1 *If $e \in \mathcal{E}^0$ in step k and if e satisfied the partial complementarity condition of step $k-1$, then e will also satisfy the partial complementarity condition for step k.*

Proof: If $e \in \mathcal{E}^0$ then we have that $(P_1)_{k.}e = 0$ or equivalently $(P_1 e)_k = 0$ and thus

$$\prod_{i \in \phi_j} (P_1 e)_i \;=\; 0 \qquad \forall j \in \{1, 2, \ldots, m\} \text{ such that } k \in \phi_j \;\; . \tag{7}$$

Since e satisfies the partial complementarity condition of step $k-1$ we know that

$$\prod_{i \in \phi_j} (P_1 e)_i \;=\; 0 \qquad \forall j \in \{1, 2, \ldots, m\} \text{ such that } \phi_j \subset \{1, 2, \ldots, k-1\} \;\; . \tag{8}$$

Combining (7) and (8) leads to

$$\prod_{i \in \phi_j} (P_1 u)_i \;=\; 0 \qquad \forall j \in \{1, 2, \ldots, m\} \text{ such that } \phi_j \subset \{1, 2, \ldots, k\} \;\; .$$

So e satisfies the partial complementarity condition of step k. □

3. The matrix P_{nec} is used to determine whether two extreme rays are adjacent. The reason that we only combine adjacent extreme rays is that we do not want any redundancy in the description of the solution set. Note that at the beginning of the kth step P_{nec} contains all the inequalities that define the current polyhedral cone. Furthermore, it is obvious that we do not have to include redundant inequalities in P_{nec}.
Let $\mathcal{K}$ be the polyhedral cone defined by $\mathcal{K} = \{u \,|\, P_{\text{nec}} u \geqslant 0\}$ at the beginning of step k. Let $\mathcal{E}_{\mathcal{K}}$ be a set of extreme rays of $\mathcal{K}$ and let t be the dimension of the lineality space of $\mathcal{K}$. So t is equal to the number of central rays of $\mathcal{K}$. The zero index set $\mathcal{I}_0(e)$ of an extreme ray $e \in \mathcal{E}_{\mathcal{K}}$ is defined as follows:

$$\mathcal{I}_0(e) \;=\; \{\, i \,|\, (P_{\text{nec}}\, e)_i = 0 \,\} \;\; .$$

Now we shall determine some necessary and sufficient conditions for two extreme rays e_1 and e_2 of the polyhedral cone $\mathcal{K}$ to be adjacent.
If e_1 and e_2 are adjacent then by Definition 1.6 there exist two minimal proper faces G_1 and G_2 of $\mathcal{K}$ with $e_1 \in G_1$ and $e_2 \in G_2$ and a $(t+2)$-dimensional face F of $\mathcal{K}$ such that $G_1 \subset F$ and $G_2 \subset F$. This means that both e_1 and e_2 have to belong to the same $(t+2)$-dimensional face F of $\mathcal{K}$. Since each element of such a face satisfies at least $n-t-2$ linearly independent equality constraints taken from $P_{\text{nec}}\, u = 0$, this leads to the following property:

Property 4.2 (Necessary condition for adjacency) *A necessary condition for two extreme rays* e_1 *and* e_2 *to be adjacent is that the zero index sets of* e_1 *and* e_2 *should contain at least* $n-t-2$ *common indices.*

If we consider a pointed polyhedral cone then $t = 0$ and then this condition reduces to the necessary condition for adjacency described in [22].
Since F is a $(t+2)$-dimensional face of $\mathcal{K}$ and since $e_1 \in G_1 \subset F$ and $e_2 \in G_2 \subset F$, we have that

$$F = \left\{ x \,\middle|\, x = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \kappa_1 e_1 + \kappa_2 e_2 \text{ with } \lambda_k \in \mathbb{R} \text{ and } \kappa_1, \kappa_2 \geqslant 0 \right\} .$$

Since e_1 and e_2 belong to F and since there is exactly one extreme ray in $\mathcal{E}_\mathcal{K}$ for each minimal proper face of $\mathcal{K}$, there are no other extreme rays in $\mathcal{E}_\mathcal{K}$ that also belong to F, which leads to:

Property 4.3 (Necessary and sufficient condition for adjacency)
Two extreme rays $e_1, e_2 \in \mathcal{E}_\mathcal{K}$ *are adjacent if and only if there is no other extreme ray* $e \in \mathcal{E}_\mathcal{K}$ *such that* $\mathcal{I}_0(e_1) \cap \mathcal{I}_0(e2) \subset \mathcal{I}_0(e)$ *.*

The conditions of Properties 4.2 and 4.3 can be considered as an extension and a generalization of the necessary and/or sufficient conditions for the adjacency of two extreme rays of a *pointed* polyhedral cone of [8, 9].
It is possible that some of the extreme rays of $\mathcal{K}$ have already been eliminated in a previous step of the ELCP algorithm because they did not satisfy the complementarity condition. In that case the condition of Property 4.3 is not sufficient any more since we do not consider *all* extreme rays.
Therefore we apply the following procedure in the ELCP algorithm to determine whether 2 extreme rays $e_1, e_2 \in \mathcal{E}$ are adjacent:

Adjacency Test 1: First we determine the common zero indices. If there are less than $n-t-2$ common zero indices then e_1 and e_2 are not adjacent.

Adjacency Test 2: Next we test whether there are other extreme rays $e \in \mathcal{E}$ such that $\mathcal{I}_0(e_1) \cap \mathcal{I}_0(e2) \subset \mathcal{I}_0(e)$. If such rays exist then e_1 and e_2 are not adjacent.

Note that the first test takes far less time to perform than the second especially if the number of extreme rays is large. That is why we use it first.
It is possible that two non-adjacent extreme rays pass Adjacency Test 2 if some other extreme rays of $\mathcal{K}$ have already been eliminated. However, in that case the following proposition provides a sufficient condition for adjacency:

Proposition 4.4 *If two non-adjacent extreme rays* $e_1, e_2 \in \mathcal{E}$ *pass Adjacency Test 2, then every positive combination of these rays will not satisfy the (partial) complementarity condition.*

Proof: Since adjacency only depends on the extreme rays, we can assume without loss of generality that $t = 0$.

If all sets ϕ_j are empty then the complementarity condition is always satisfied by definition and no extreme ray of the polyhedral cone will be eliminated, which means that the condition of Property 4.3 is still sufficient. Therefore non-adjacent extreme rays cannot pass Adjacency Test 2.
If at least one set ϕ_j is nonempty, then we have a complementarity condition. Since the complementarity condition requires that some of the inequalities of $P_{1:k,.}u \geqslant 0$ should hold with equality, there are only two possible cases:

- Every point of the polyhedral cone $\mathcal{K}$ satisfies the (partial) complementarity condition.
- The (partial) complementarity condition only selects points that lie on the border of the polyhedral cone $\mathcal{K}$. So interior points of the polyhedral cone will not satisfy the (partial) complementarity condition.

In the first case no extreme rays will be rejected because of the (partial) complementarity condition, which means that the condition of Adjacency Test is still sufficient. Therefore we can limit ourselves to the second case.
Now consider a face F of the polyhedral cone $\mathcal{K}$ that contains both e_1 and e_2. Note that F is in itself also a polyhedral cone. If the non-adjacent rays e_1 and e_2 pass Adjacency Test 2 then this is only possible if another extreme ray of F has already been eliminated because it did not satisfy the (partial) complementarity condition. Since either all points of F satisfy the (partial) complementarity condition or only points on the border of F satisfy the (partial) complementarity condition, this means that every positive combination of the non-adjacent extreme rays e_1 and e_2 – which always lies in the interior of F – will not satisfy the (partial) complementarity condition. □

So if two non-adjacent rays e_1 and e_2 pass both Adjacency Tests then the new ray e^{new} will not satisfy the (partial) complementarity condition and hence will be automatically rejected. Therefore no redundant rays will be created.
Together these three tests provide necessary and sufficient conditions for adjacency.
Note that the final P_{nec} is also considered as an output of this algorithm because we need it in the second part of the ELCP algorithm, when we process $P_2 u \geqslant 0$.

4. If $c \in \mathcal{C}$ at the beginning of step k then both c and $-c$ are solutions of $(P_1)_{1:k-1,.}\, u \geqslant 0$. We have that $\text{res}\,(-c) = (P_1)_{k,.}\,(-c) = -(P_1)_{k,.}\, c = -\text{res}\,(c)$. So if $c \in \mathcal{C}^+$ then $-c \in \mathcal{C}^-$ and vice versa. This explains why we may set $\mathcal{C}^+ \leftarrow \mathcal{C}^+ \cup \{-s \,|\, s \in \mathcal{C}^-\}$ and why we have adapted the residues in the next step. After this step all central rays have a nonnegative residue.

5. If we multiply a central or extreme ray by a positive real number it will stay a central or extreme ray because of Properties 3.1 and 3.2. This means that we can normalize all new rays after each pass through the main loop in order to avoid problems such as overflow.
To avoid problems arising from round-off errors it is better to test the residues against a threshold $\tau > 0$ instead of against 0 when determining the subsets $\mathcal{C}^+$, $\mathcal{C}^-$, $\mathcal{C}^0$, $\mathcal{E}^+$, $\mathcal{E}^-$ and $\mathcal{E}^0$.

6. If both $\mathcal{C}$ and $\mathcal{E}$ are empty after a pass through the main loop, we can stop the algorithm. In that case the homogeneous ELCP will not have any solutions except for the trivial solution $u = [\,0\ \ 0\ \ldots\ 0\,]^T$.

For more information about the method used to find all solutions of a system of linear inequalities the interested reader is referred to [25]. One of the main differences between our algorithm and that of [25] is that we only store one version of each central ray c, whereas in the double description method both c and $-c$ are stored. We have also added the test on the (partial) complementarity condition to eliminate as many rays as soon as possible.

4.2 Take the remaining equality and inequality constraints into account

The next algorithm is an adaptation of Algorithm 1. Since we have already processed all rows of P_1 in Algorithm 1, we can now test for the complete complementarity condition.

Algorithm 2: add the equality constraints

```
Input:   m, p1, q, n, P1 ∈ R^{p1×n}, Q ∈ R^{q×n}, {φj}_{j=1}^m, C, E, P_nec
Main loop:
  for k = 1, 2, ..., q do                        { The rows of Q are taken one by one. }
      ∀s ∈ C ∪ E : res(s) ← Q_{k.} s                         { Calculate the residues. }
      C+ ← {c ∈ C | res(c) > 0}
      C- ← {c ∈ C | res(c) < 0}
      C0 ← {c ∈ C | res(c) = 0}
      E+ ← {e ∈ E | res(e) > 0}
      E- ← {e ∈ E | res(e) < 0}
      E0 ← {e ∈ E | res(e) = 0}
      if C+ = ∅ and C- = ∅ and E+ = ∅ and E- = ∅ then                      { Case 1 }
          { The kth equation is redundant. }
      else
          if C+ = ∅ and C- = ∅ then                                          { Case 2 }
              E ← E0
              for all pairs (e+, e-) ∈ E+ × E- do
                  if e+ and e- are adjacent then
                      e_new ← res(e+) e- − res(e-) e+
                      if e_new satisfies the complementarity condition then
                          E ← E ∪ { e_new }
                      endif
                  endif
              endfor
          else                                                               { Case 3 }
              C ← C0
              E ← E0
              C+ ← C+ ∪ {−s | s ∈ C-}
```

$\forall s \in \mathcal{C}^- : \text{res}(-s) \leftarrow -\text{res}(s)$ { *Adapt the residues.* }
Take one ray $c \in \mathcal{C}^+$.
$\forall c^+ \in \mathcal{C}^+ \setminus \{c\} : \mathcal{C} \leftarrow \mathcal{C} \cup \{\text{res}(c^+)\, c - \text{res}(c)\, c^+\}$
for all $e \in \mathcal{E}^+ \cup \mathcal{E}^-$ **do**
$e^{\text{new}} \leftarrow \text{res}(c)\, e - \text{res}(e)\, c$
if e^{new} satisfies the complementarity condition **then**
$\mathcal{E} \leftarrow \mathcal{E} \cup \{ e^{\text{new}} \}$
endif
endfor
endif
endif
endfor

Output: $\mathcal{C}, \mathcal{E}$

Remarks:

1. We do not have to add any rows to P_{nec} since after the kth step every ray s will satisfy $Q_{1:k,.}s = 0$. So adding the kth row of Q to P_{nec} would yield the same extra element in all zero index sets. As a consequence Adjacency Test 1 for the kth step of Algorithm 2 becomes: if there are less than $n - (k-1) - t - 2$ common indices in the zero index sets of e_1 and e_2 then e_1 and e_2 are not adjacent.

2. The main difference with Algorithm 1 is that now we have to satisfy equality constraints. That is why we only keep those rays that have a zero residue, whereas in Algorithm 1 we kept all rays with a positive or zero residue.

3. If we construct extra rays we immediately test whether the complementarity condition is satisfied. We do not have to test the rays that are copied from the previous loop since they already satisfy the complete complementarity condition. Since each new central ray c will still satisfy $P_1 c = 0$ and thus also $\sum_{j=1}^{m} \prod_{i \in \phi_j} (P_1 c)_i = 0$, we only have to test new extreme rays.

4. If one is only interested in one solution, one could use the equality constraints to eliminate some of the variables. However, since we want a minimal description of the entire solution set with central and extreme rays, we do not eliminate any variables. Furthermore, the matrix Q is not necessarily invertible.

To take the remaining inequalities into account we again apply Algorithm 1 but we skip the initialization step and continue with the sets $\mathcal{C}$ and $\mathcal{E}$ that resulted from Algorithm 2 and the matrix P_{nec} from Algorithm 1. Adjacency Test 1 now becomes: if there are less than $n - q - t - 2$ common indices in $\mathcal{I}_0(e_1)$ and $\mathcal{I}_0(e_2)$ then e_1 and e_2 are not adjacent. In the main loop we only have to test whether newly constructed extreme rays satisfy the complete complementarity condition.

To avoid unnecessary calculations and to limit the required amount of storage space, it is advantageous to have as few intermediate rays as possible. That is why we split the inequalities of $Pu \geqslant 0$ and why we process $Qu = 0$ before $P_2 u \geqslant 0$:

- The complementarity condition is one way to reject rays. Therefore we already use a partial complementarity condition in Algorithm 1. Since we want to apply this test as soon as possible we removed the inequalities that did not appear in the complementarity condition and put them in $P_2 u \geqslant 0$.
- In the next steps we then further reduce the solution set by taking the extra equality and inequality constraints into account. Unless we have a priori knowledge about the coefficients of the equalities and the inequalities, it is logical to assume that an equality will yield less intermediate rays than an inequality, since we only retain existing rays with a zero residue for an equality and rays with a positive or zero residue for an inequality. That is why we first take $Qu = 0$ into account and only then $P_2 u \geqslant 0$.

4.3 Determine the cross-complementary sets

Let $\mathcal{K}$ be the polyhedral cone defined by $P_1 u \geqslant 0$, $Qu = 0$ and $P_2 u \geqslant 0$. As a direct consequence of the way in which $\mathcal{C}$ and $\mathcal{E}$ are constructed, we have that every u that is defined as

$$u = \sum_{c_k \in \mathcal{C}_s} \lambda_k c_k + \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \qquad \text{with } \lambda_k \in \mathbb{R} \text{ and } \kappa_k \geqslant 0 \tag{9}$$

for an arbitrary subset $\mathcal{C}_s \subset \mathcal{C}$ and an arbitrary subset $\mathcal{E}_s \subset \mathcal{E}$, belongs to $\mathcal{K}$. Since the complementarity condition requires that in each subgroup of inequalities of $P_1 u \geqslant 0$ that corresponds to some ϕ_j at least one inequality should be satisfied by equality, the complementarity condition is satisfied either by all points of $\mathcal{K}$ or only by (some) points on the border of $\mathcal{K}$. Since we have only rejected rays that did not satisfy the complementarity condition and hence certainly would not yield solutions of the ELCP, any arbitrary solution of the homogeneous ELCP can be represented by (9).

However, if we take arbitrary subsets of $\mathcal{C}$ and $\mathcal{E}$ then normally not every combination of the form (9) will be a solution of the ELCP. The complementarity condition determines for which subsets of $\mathcal{C}$ and $\mathcal{E}$ (9) will yield a solution of the homogeneous ELCP. This is where the concept "cross-complementarity" appears.
In [8] two solutions of a GLCP are called cross-complementary if every nonnegative combination of the two solutions satisfies the complementarity condition. This definition can be extended to an arbitrary number of solutions. However, for the ELCP we have to adapt this definition:

Definition 4.5 (Cross-complementarity) *A set of solutions $\mathcal{S}$ of an ELCP is called cross-complementary if every sum of an arbitrary linear combination of the central solutions in $\mathcal{S}$ and an arbitrary nonnegative combination of the non-central solutions in $\mathcal{S}$:*

$$u = \sum_{s_k \in \mathcal{S}^{\mathrm{cen}}} \lambda_k s_k + \sum_{s_k \in \mathcal{S}^{\mathrm{nc}}} \kappa_k s_k \qquad \textit{with } \lambda_k \in \mathbb{R} \textit{ and } \kappa_k \geqslant 0 \tag{10}$$

where $\mathcal{S}^{\mathrm{cen}} = \{ s \in \mathcal{S} \,|\, Ps = 0 \}$ and $\mathcal{S}^{\mathrm{nc}} = \{ s \in \mathcal{S} \,|\, Ps \neq 0 \}$, satisfies the complementarity condition.

Note that every combination of the form (10) always belongs to $\mathcal{K}$. So if $\mathcal{S}$ is a cross-complementary set then every combination of the form

$$u = \sum_{s_k \in \mathcal{S}^{\text{cen}}} \lambda_k s_k + \sum_{s_k \in \mathcal{S}^{\text{nc}}} \kappa_k s_k \qquad \text{with } \lambda_k \in \mathbb{R} \text{ and } \kappa_k \geqslant 0$$

where $\mathcal{S}^{\text{cen}} = \{s \in \mathcal{S} \,|\, Ps = 0\}$ and $\mathcal{S}^{\text{nc}} = \{s \in \mathcal{S} \,|\, Ps \neq 0\}$ is a solution of the ELCP.
Now we shall determine the maximal sets of cross-complementary solutions. The following property tells us that we can always set $\mathcal{C}_s = \mathcal{C}$ in (9):

Property 4.6 *If u_1 is a solution of the ELCP then the set $\mathcal{C} \cup \{u_1\}$ is cross-complementary.*

Proof: First we define a set A such that $A = \mathbb{R}$ if u_1 is a central solution and $A = \{r \in \mathbb{R} \,|\, r \geqslant 0\}$ if u_1 is a non-central solution. So now we have to prove that $\forall \lambda_k \in \mathbb{R}$, $\forall \mu \in A$: $u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \mu u_1$ satisfies the complementarity condition. We have that

$$\begin{aligned}
\sum_{j=1}^{m} \prod_{i \in \phi_j} \left(P_1 \left(\sum_{c_k \in \mathcal{C}} \lambda_k c_k + \mu u_1 \right) \right)_i &= \sum_{j=1}^{m} \prod_{i \in \phi_j} \left(\sum_{c_k \in \mathcal{C}} \lambda_k (P_1 c_k)_i + \mu (P_1 u_1)_i \right) \\
&= \sum_{j=1}^{m} \prod_{i \in \phi_j} \left(0 + \mu (P_1 u_1)_i \right) \qquad \text{since } P_1 c_k = 0 \\
&= \sum_{j=1}^{m} \mu^{\# \phi_j} \prod_{i \in \phi_j} (P_1 u_1)_i \\
&= 0
\end{aligned}$$

because of complementarity condition (3). So the set $\mathcal{C} \cup \{u_1\}$ is cross-complementary. □

So now we only have to consider the extreme rays. The following property tells us that we only have to test one positive combination to determine whether a set of extreme rays or non-central solutions is cross-complementary or not:

Property 4.7 *Let e_1, e_2, ..., e_k be arbitrary extreme rays (or non-central solutions) of the ELCP. Then we have that*

$$\forall \kappa_1, \kappa_2, \ldots, \kappa_k \geqslant 0 \, : \, \kappa_1 e_1 + \kappa_2 e_2 + \ldots + \kappa_k e_k \ \text{ satisfies the complementarity condition}$$

if and only if

$$\exists \mu_1, \mu_2, \ldots, \mu_k > 0 \text{ such that } \mu_1 e_1 + \mu_2 e_2 + \ldots + \mu_k e_k \ \text{ satisfies the complementarity condition.}$$

Proof: Since the proof of the only-if-part is trivial we only prove the if-part.
If there exist positive real numbers μ_1, μ_2, ..., μ_k such that $\mu_1 e_1 + \mu_2 e_2 + \ldots + \mu_k e_k$ satisfies the complementarity condition then we have that

$$\prod_{i \in \phi_j} \left(P_1 (\mu_1 e_1 + \mu_2 e_2 + \ldots + \mu_k e_k) \right)_i = 0 \qquad \text{for } j = 1, 2, \ldots, m$$

or

$$\prod_{i\in\phi_j} (\mu_1(P_1e_1)_i + \mu_2(P_1e_2)_i + \ldots + \mu_k(P_1e_k)_i) \;=\; 0 \qquad \text{for } j = 1,2,\ldots,m$$

and thus

$$\sum_{(\psi_1,\psi_2,\ldots,\psi_k)\in\Psi_j} \prod_{l=1}^{k} \prod_{i\in\psi_l} \mu_l(P_1e_l)_i \;=\; 0 \qquad \text{for } j = 1,2,\ldots,m$$

where Ψ_j is the set of all possible k-tuples of disjoint subsets of ϕ_j the union of which is equal to ϕ_j:

$$\Psi_j \;=\; \Bigg\{ (\psi_1,\psi_2,\ldots,\psi_k) \;\Bigg|\; \forall l \in \{1,2,\ldots,k\} \;:\; \psi_l \subset \phi_j \;;\; \bigcup_{l=1}^{k} \psi_l = \phi_j \;\text{ and } \\ \forall l_1,l_2 \in \{1,2,\ldots,k\} \;:\; \text{if } l_1 \neq l_2 \text{ then } \psi_{l_1} \cap \psi_{l_2} = \emptyset \Bigg\} \;.$$

Note that we also allow empty subsets ψ_l in the definition of Ψ_j.
So

$$\sum_{(\psi_1,\psi_2,\ldots,\psi_k)\in\Psi_j} \left(\prod_{l=1}^{k} \mu_l^{\#\psi_l} \right) \cdot \left(\prod_{l=1}^{k} \prod_{i\in\psi_l} (P_1e_l)_i \right) \;=\; 0 \qquad \text{for } j = 1,2,\ldots,m$$

and since $\mu_l > 0$ and $(P_1e_l)_i \geqslant 0$ for $l = 1,2,\ldots,k$, this is only possible if

$$\forall(\psi_1,\psi_2,\ldots,\psi_k) \in \Psi_j \;:\; \prod_{l=1}^{k} \prod_{i\in\psi_l} (P_1e_l)_i \;=\; 0 \qquad \text{for } j = 1,2,\ldots,m \;. \tag{11}$$

Now we show that $\forall \kappa_1,\kappa_2,\ldots,\kappa_k \geqslant 0$: $\kappa_1e_1 + \kappa_2e_2 + \ldots + \kappa_ke_k$ also satisfies the complementarity condition. Using the same reasoning as for $\mu_1e_1 + \mu_2e_2 + \ldots + \mu_ke_k$ we find that

$$\begin{aligned} &\sum_{j=1}^{m} \prod_{i\in\phi_j} (P_1(\kappa_1e_1 + \kappa_2e_2 + \ldots + \kappa_ke_k))_i \\ &\qquad = \sum_{j=1}^{m} \sum_{(\psi_1,\psi_2,\ldots,\psi_k)\in\Psi_j} \left(\prod_{l=1}^{k} \kappa_l^{\#\psi_l} \right) \cdot \left(\prod_{l=1}^{k} \prod_{i\in\psi_l} (P_1e_l)_i \right) \\ &\qquad = 0 \qquad \text{because of (11).} \end{aligned}$$

□

To determine whether a set of extreme rays of the ELCP is cross-complementary we take an arbitrary positive combination of these rays. If the combination satisfies the complementarity condition then the rays are cross-complementary. If it does not satisfy the condition then the rays cannot be cross-complementary.

Now we construct the cross-complementarity graph $\mathcal{G}$. This graph has $\#\mathcal{E}$ vertices – one for each extreme ray $e_k \in \mathcal{E}$ – and an edge between two different vertices k and l if the corresponding extreme rays e_k and e_l are cross-complementary. A subset $\mathcal{V}$ of vertices of a graph such that any two vertices of $\mathcal{V}$ are connected by an edge is called a *clique*. A *maximal clique* is a clique that is not a subset of any other clique of the graph. In contrast to what has been suggested in [9], finding all cross-complementary solutions does not amount to detecting all maximal cliques of the graph $\mathcal{G}$, as will be shown by the following trivial example.

Example 4.8 Consider the following GLCP:

$$\text{Find a vector } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \text{ such that } xyz = 0 \text{ subject to } x, y, z \geqslant 0 \text{ and } 0x + 0y + 0z = 0 \ .$$

The solution set of this GLCP has three extreme rays:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ .$$

$\{e_1, e_2\}$ is a set of cross-complementary solutions, and the same goes for $\{e_2, e_3\}$ and $\{e_3, e_1\}$. The graph $\mathcal{G}$ of cross-complementary rays is represented in Figure 1.

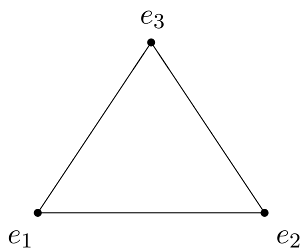

Figure 1: The cross-complementarity graph $\mathcal{G}$ of Example 4.8.

$\{e_1, e_2, e_3\}$ is clearly a clique of this graph, but it is not a cross-complementary set since $e_1 + e_2 + e_3 = [\,1 \ 1 \ 1\,]^T$ does not satisfy the complementarity condition. □

To find all cross-complementary solutions we have to construct all maximal subsets $\mathcal{E}_s$ of cross-complementary extreme rays. For Example 4.8 this would yield $\mathcal{E}_1 = \{e_1, e_2\}$, $\mathcal{E}_2 = \{e_2, e_3\}$ and $\mathcal{E}_3 = \{e_3, e_1\}$.
We can save much time if we make some extra provisions, as will be shown by the following properties.

Property 4.9 *If $e_1 \in \mathcal{E}$ satisfies $P_1 e_1 = 0$ then e_1 belongs to every maximal cross-complementary set.*

Proof: Assume that $\mathcal{E}_s \subset \mathcal{E} \setminus \{e_1\}$ is a cross-complementary set. Now we show that $\mathcal{E}_s \cup \{e_1\}$ is also a cross-complementary set. We have to prove that every nonnegative combination of the rays of $\mathcal{E}_s \cup \{e_1\}$:

$$u \ = \ \kappa e_1 + \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \qquad \text{with } \kappa \geqslant 0 \text{ and } \kappa_k \geqslant 0$$

satisfies the complementarity condition.
Since $\mathcal{E}_s$ is a cross-complementary set we know that

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} \left(P_1 \left(\sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \right) \right)_i \ = \ 0 \ . \tag{12}$$

Now we have that

$$\begin{aligned}
\sum_{j=1}^{m} \prod_{i \in \phi_j} (P_1 u)_i &= \sum_{j=1}^{m} \prod_{i \in \phi_j} \left(P_1 \left(\kappa e_1 + \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \right) \right)_i \\
&= \sum_{j=1}^{m} \prod_{i \in \phi_j} \left(\kappa (P_1 e_1)_i + \left(P_1 \left(\sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \right) \right)_i \right) \\
&= \sum_{j=1}^{m} \prod_{i \in \phi_j} \left(P_1 \left(\sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \right) \right)_i \qquad \text{since } P_1 e_1 = 0 \\
&= 0 \qquad \text{because of (12)}\,.
\end{aligned}$$

So $\mathcal{E}_s \cup \{e_1\}$ is indeed a cross-complementary set. □

Property 4.10 *If e_1 and e_2 are two extreme rays and if*

$$\forall i \in \{1, 2, \ldots, p_1\} \,:\, (P_1 e_1)_i = 0 \ \textit{ if and only if } \ (P_1 e_2)_i = 0 \ , \tag{13}$$

then e_1 will belong to a maximal cross-complementary set if and only if e_2 belongs to that set.

Proof: Consider an arbitrary subset $\mathcal{E}_s \subset \mathcal{E} \setminus \{e_1, e_2\}$. First we prove that if the set $\mathcal{E}_s \cup \{e_1\}$ is cross-complementary then $\mathcal{E}_s \cup \{e_1, e_2\}$ is also cross-complementary.
If $\mathcal{E}_s \cup \{e_1\}$ is cross-complementary then every nonnegative combination of its elements satisfies the complementarity condition. This means that

$$u \;=\; e_1 + \sum_{e_k \in \mathcal{E}_s} e_k = e_1 + v_s$$

with $v_s = \sum_{e_k \in \mathcal{E}_s} e_k$, satisfies the complementarity condition:

$$\prod_{i \in \phi_j} \left(P_1 (e_1 + v_s) \right)_i \;=\; 0 \qquad \text{for } j = 1, 2, \ldots, m$$

or

$$\prod_{i \in \phi_j} \left((P_1 e_1)_i + (P_1 v_s)_i \right) \;=\; 0 \qquad \text{for } j = 1, 2, \ldots, m$$

and thus

$$\sum_{\psi \in \mathcal{D}(\phi_j)} \prod_{i \in \psi} (P_1 e_1)_i \prod_{i \in \psi^{\mathrm{c}}} (P_1 v_s)_i \;=\; 0 \qquad \text{for } j = 1, 2, \ldots, m \tag{14}$$

where $\mathcal{D}(\phi_j)$ is the set of all subsets of ϕ_j and $\psi^{\mathrm{c}} = \phi_j \setminus \psi$ is the complement of ψ with respect to ϕ_j. Since $(P_1 e_1)_i \geqslant 0$ and $(P_1 v_s)_i \geqslant 0$, (14) can only hold if

$$\forall \psi \in \mathcal{D}(\phi_j) \,:\, \prod_{i \in \psi} (P_1 e_1)_i \prod_{i \in \psi^{\mathrm{c}}} (P_1 v_s)_i \;=\; 0 \qquad \text{for } j = 1, 2, \ldots, m$$

or equivalently

$$\forall \psi \in \mathcal{D}(\phi_j) \,:\, (\, \exists i \in \psi \text{ such that } (P_1 e_1)_i = 0\,) \text{ or } (\, \exists i \in \psi^{\mathrm{c}} \text{ such that } (P_1 v_s)_i = 0\,)$$

for $j = 1, 2, \ldots, m$. But if $(P_1 e_1)_i = 0$ then also $(P_1 e_2)_i = 0$ and thus $(P_1(e_1 + e_2))_i = 0$. This leads to

$$\begin{aligned} \forall \psi \in \mathcal{D}(\phi_j) \;:\; & (\, \exists i \in \psi \text{ such that } (P_1(e_1+e_2))_i = 0\,) \text{ or} \\ & (\, \exists i \in \psi^{\mathrm{c}} \text{ such that } (P_1 v_s)_i = 0\,) \end{aligned}$$

for $j = 1, 2, \ldots, m$, and consequently

$$\prod_{i \in \phi_j} (P_1(e_1 + e_2 + v_s))_i \;=\; 0 \qquad \text{for } j = 1, 2, \ldots, m \;.$$

Hence the nonnegative combination

$$v \;=\; e_1 + e_2 + v_s \;=\; e_1 + e_2 + \sum_{e_k \in \mathcal{E}_s} e_k$$

of the elements of $\mathcal{E}_s \cup \{e_1, e_2\}$ satisfies the complementarity condition. According to Property 4.7 this means that the set $\mathcal{E}_s \cup \{e_1, e_2\}$ is cross-complementary.
To prove the only-if-part we interchange e_1 and e_2 and repeat the above reasoning. $\Box$

This leads to the following procedure for determining the sets of cross-complementary extreme rays:
First we put all rays $e \in \mathcal{E}$ that satisfy $P_1 e = 0$ in $\mathcal{E}_0$. Because of Property 4.9 we know that these rays will belong to every maximal cross-complementary set.
Next we define an equivalence relation $\sim$ on $\mathcal{E} \setminus \mathcal{E}_0$:

$$e_1 \sim e_2 \quad \text{ if } \quad \forall i \in \{1, 2, \ldots, p_1\} \,:\, (\,(P_1 e_1)_i = 0\,) \Leftrightarrow (\,(P_1 e_2)_i = 0\,) \;\;,$$

and we construct the equivalence classes. Now we take one representative out of each equivalence class and put it in $\mathcal{E}_{\mathrm{red}}$. If we define $\mathcal{S} = \{P_1 e_k \,|\, e_k \in \mathcal{E}_{\mathrm{red}}\}$ then $\forall s_k \in \mathcal{S} : s_k \geqslant 0$. Since for any arbitrary pair of extreme rays $e_1 \,, e_2 \in \mathcal{E}_{\mathrm{red}}$ we have that

$$(P_1(\mu_1 e_1 + \mu_2 e_2))_i \;=\; \mu_1 (P_1 e_1)_i + \mu_2 (P_1 e_2)_i \;=\; \mu_1 (s_1)_i + \mu_2 (s_2)_i \;\;,$$

there is a one-to-one correspondence between the cross-complementary subsets of $\mathcal{E}_{\mathrm{red}}$ and the cross-complementary sets of solutions of the GLCP $\sum_{j=1}^{m} \prod_{i \in \phi_j} s_i = 0$ with $s \geqslant 0$. Therefore we now present an algorithm to determine the maximal cross-complementary sets of solutions of a GLCP:

Algorithm 3: Determine the maximal cross-complementary sets of solutions of a GLCP

Input: $m, \mathcal{S}, \{\phi_j\}_{j=1}^{m}$

Initialization:

$\Gamma \leftarrow \emptyset$

$\mathcal{B} \leftarrow \{\text{binary}(s_k) \mid s_k \in \mathcal{S}\}$ { *Construct the binary equivalents.* }

{ *Construct the cross-complementarity matrix:* }

for $k = 1, 2, \ldots, \#\mathcal{B} - 1$

  for $l = k+1, k+2, \ldots, \#\mathcal{B}$

    if $(b_k \vee b_l)$ satisfies the complementarity condition **then**

      $cross(k, l) \leftarrow 1$

    else

      $cross(k, l) \leftarrow 0$

    endif

  endfor

endfor

$depth \leftarrow 1$

$start(1) \leftarrow 0$

$last(1) \leftarrow \#\mathcal{B}$

$\forall k \in \{1, 2, \ldots, \#\mathcal{B}\}: \ vertices(1, k) \leftarrow k$

Main loop:

while $depth > 0$ **do**

  $start(depth) \leftarrow start(depth) + 1$

  $b \leftarrow \bigvee_{d=1}^{depth} b_{vertices(d, start(d))}$

  { *Determine the vertices for the next depth:* }

  $current\_vertex \leftarrow vertices(depth, start(depth))$

  $next\_depth \leftarrow depth + 1$

  $start(next\_depth) \leftarrow 0$

  $last(next\_depth) \leftarrow 0$

  for $k = start(depth) + 1, \ldots, last(depth)$ **do**

    $new\_vertex \leftarrow vertices(depth, k)$

    if $cross(current\_vertex, new\_vertex) = 1$ **then**

      if $(b \vee b_{new\_vertex})$ satisfies the complementarity condition **then**

        $last(next\_depth) \leftarrow last(next\_depth) + 1$

        $vertices(next\_depth, last(next\_depth)) \leftarrow new\_vertex$

      endif

    endif

  endfor

```
      { If the next depth does not contain any vertices, then a new maximal }
      { cross-complementary set has been found: }
      if last(next_depth) > 0 then
         depth ← next_depth
      else
         S^new ← ⋃_{d=1}^{depth} {s_{vertices(d,start(d))}}
         if ∀S_s ∈ Γ : S^new ⊄ S_s then
            Γ ← Γ ∪ {S^new}
         endif
         { Check whether the current subset contains all remaining vertices, }
         { otherwise return to the previous point where a choice was made: }
         if start(1) + depth − 1 = #B then
            depth ← 0
         else
            while start(depth) = last(depth) do
               depth ← depth − 1
            endwhile
         endif
      endif
   endwhile
Output: Γ = {S_1, S_2, ...}            { the set of all cross-complementary sets }
```

Remarks:

1. This algorithm is an adaptation of the algorithm of [3] to determine a maximum clique of a graph, i.e. a clique of maximum cardinality. We start with a set that contains one vertex and we keep adding extra vertices as long as the corresponding set of extreme rays stays cross-complementary. If no vertices can be added without violating the cross-complementarity, we have found a maximal cross-complementary set. Then we go back to the last point were a choice was made and repeat the procedure.
   For additional information about this algorithm the interested reader is referred to [3]. A recent survey of algorithms and applications of the maximum clique problem can be found in [27]. However, note that finding all maximal sets of cross-complementarity extreme rays does not amount to determining all maximal cliques of the cross-complementarity graph $\mathcal{G}$ (cf. Example 4.8).

2. If we encounter a set that is not cross-complementary, then according to Property 3.3 – which is also valid if e_l is a nonnegative combination of extreme rays – each superset of that set will also be not cross-complementary. So once we have found a set that is not cross-complementary, we do not have to add extra vertices.

3. If we have found a new maximal cross-complementary set, we should add it to the list Γ. But first we determine if there is any redundancy. Because of the order in which we

process the maximal cross-complementary sets it is impossible that a new subset is a superset of a set that is already in the list. So we only have to test whether the new set is a subset of one of the sets of Γ.

4. As was mentioned in [8] the cross-complementarity test can be done in binary arithmetic only. First we replace each solution by its binary equivalent:

$$\text{if } s \in \mathbb{R}^n \text{ then } b = \text{binary}(s) \in \mathbb{R}^n \text{ with } \begin{aligned} b_i &= 0 \quad \text{if } |s_i| < \tau\,, \\ &= 1 \quad \text{if } |s_i| \geqslant \tau\,, \end{aligned}$$

   where $\tau > 0$ is a threshold.
   In the complementarity condition we use logical **and** ($\wedge$) instead of multiplication and logical **or** ($\vee$) instead of addition. Example: the binary equivalent of the complementarity condition $s_1 s_2 + s_3 = 0$ is $[(b_1 = 0) \vee (b_2 = 0)] \wedge (b_3 = 0)$.
   To determine whether two (or more) solutions are cross-complementary we first construct a new vector by taking the entrywise **or** of the binary equivalents of the solutions and then we test whether this vector satisfies the (binary) complementarity condition. We have already included this technique in our algorithm since it will be much faster than doing everything in floating point arithmetic.
   Note that we can also use this technique in Algorithms 1 and 2 if we define $s = Pu$ because then complementarity condition (2) reduces to $\displaystyle\sum_{j=1}^{m} \prod_{i \in \phi_j} s_i = 0$.

5. We have only constructed the upper triangular part of the cross-complementarity matrix *cross* because in the test $\mathit{cross}(\mathit{current\_vertex}, \mathit{new\_vertex}) = 1$ we always have that $\mathit{new\_vertex} > \mathit{current\_vertex}$.

6. If we are only interested in obtaining one solution of the ELCP we can skip Algorithm 3. However, this is certainly not the most efficient way to get one solution of the ELCP (see also Section 4.5).

Once we have found the sets of cross-complementary solutions of the GLCP we reconstruct the corresponding subsets of $\mathcal{E}$ by replacing each s_k in each subset $\mathcal{S}_s$ by the corresponding e_k and all the other members of the equivalence class of e_k. If we also add all the elements of $\mathcal{E}_0$ to each subset we finally get Γ, the set of subsets $\mathcal{E}_s$ of cross-complementary extreme rays of the ELCP. Now we can characterize the solution set of the ELCP:

Theorem 4.11 *When $\mathcal{C}$, $\mathcal{E}$ and Γ are given, then u is a solution of the homogeneous ELCP if and only if there exists a subset of cross-complementary extreme rays $\mathcal{E}_s \in \Gamma$ such that*

$$u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \qquad \textit{with } \lambda_k \in \mathbb{R} \textit{ and } \kappa_k \geqslant 0 \ .$$

This leads to:

Theorem 4.12 *In general the solution set of an homogeneous ELCP consists of the union of faces of a polyhedral cone.*

Remark: The main difference between the ELCP and the GLCP is that the solution set of an homogeneous ELCP consists of the union of faces of a polyhedral cone – which means that

it can contain a linear subspace – whereas the solution set of a GLCP is the union of faces of a *pointed* polyhedral cone, which means that it cannot contain a linear subspace. Hence there are no central rays in the solution set of a GLCP.
The algorithm of [8] to calculate the solution set of a GLCP starts with $\mathcal{C} = \emptyset, \mathcal{E} = \{e_i \,|\, e_i = (I_n)_{.i}$ for $i = 1, 2, \ldots, n\}$ and $P_{\text{nec}} = I_n$ and directly goes to Algorithm 2 and skips all steps that deal with central rays.

4.4 Solutions of the inhomogeneous ELCP

Every solution of the homogeneous ELCP has the following form: $u = \begin{bmatrix} x \\ u_\alpha \end{bmatrix}$ with $u_\alpha \geqslant 0$.
First we normalize all nonzero α components:

- If c is a central ray then both c and $-c$ are solutions of the homogeneous ELCP. Since $c_\alpha \geqslant 0$ this is only possible if $c_\alpha = 0$.
- For an extreme ray there are two possibilities: $e_\alpha = 0$ or $e_\alpha > 0$. If $e_\alpha > 0$ then we divide each component of e by e_α such that the α component of e becomes 1. Because of Property 3.2 the new e will still be a solution of the homogeneous ELCP.

This results in two groups of extreme rays: $\mathcal{E}^{\text{inf}} = \{e_k \in \mathcal{E} \,|\, (e_k)_\alpha = 0\}$ and $\mathcal{E}^{\text{fin}} = \{e_k \in \mathcal{E} \,|\, (e_k)_\alpha = 1\}$. The rays in $\mathcal{C}$ and $\mathcal{E}^{\text{inf}}$ will correspond to solutions at infinity, whereas $\mathcal{E}^{\text{fin}}$ will yield finite solutions of the inhomogeneous ELCP. If we extract the x part out of each ray of $\mathcal{C}$, $\mathcal{E}^{\text{inf}}$ and $\mathcal{E}^{\text{fin}}$ we get $\mathcal{X}^{\text{cen}}$, the set of central rays; $\mathcal{X}^{\text{inf}}$, the set of *infinite rays* and $\mathcal{X}^{\text{fin}}$, the set of *finite rays* respectively.
Finally we construct for every subset $\mathcal{E}_s \in \Gamma$ the corresponding subsets $\mathcal{X}_s^{\text{inf}} \subset \mathcal{X}^{\text{inf}}$ and $\mathcal{X}_s^{\text{fin}} \subset \mathcal{X}^{\text{fin}}$. We only retain those pairs $\left\{\mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}}\right\}$ for which $\mathcal{X}_s^{\text{fin}}$ is not empty. This yields Λ, the set of pairs of cross-complementary infinite and finite rays.

Theorem 4.13 *When $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{inf}}$, $\mathcal{X}^{\text{fin}}$ and Λ are given, then x is a solution of the inhomogeneous ELCP if and only if there exists a pair $\left\{\mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}}\right\} \in \Lambda$ such that*

$$x = \sum_{x_k \in \mathcal{X}^{\text{cen}}} \lambda_k x_k + \sum_{x_k \in \mathcal{X}_s^{\text{inf}}} \kappa_k x_k + \sum_{x_k \in \mathcal{X}_s^{\text{fin}}} \mu_k x_k$$

with $\lambda_k \in \mathbb{R}$, $\kappa_k \geqslant 0$, $\mu_k \geqslant 0$ and $\sum_k \mu_k = 1$.

Proof:
First we prove the if-part:
Take an arbitrary pair $\left\{\mathcal{E}_s^{\text{inf}}, \mathcal{E}_s^{\text{fin}}\right\}$ with $\mathcal{E}_s^{\text{fin}} \neq \emptyset$. Consider

$$u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}_s^{\text{inf}}} \kappa_k e_k + \sum_{e_k \in \mathcal{E}_s^{\text{fin}}} \mu_k e_k$$

with arbitrary λ_k, arbitrary $\kappa_k \geqslant 0$ and arbitrary $\mu_k \geqslant 0$ such that $\sum_k \mu_k = 1$. Let $u = \begin{bmatrix} x \\ u_\alpha \end{bmatrix}$.
Then we have that

$$u_\alpha = \sum_k \lambda_k \cdot 0 + \sum_k \kappa_k \cdot 0 + \sum_k \mu_k \cdot 1 = \sum_k \mu_k = 1 \ .$$

Since u is a solution of the homogeneous ELCP and since $u_\alpha = 1$, we have that

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i \; = \; 0 \,, \; Ax - c \geqslant 0 \; \text{ and } \; Bx - d = 0 \;.$$

Hence

$$x \; = \sum_{x_k \in \mathcal{X}^{\text{cen}}} \lambda_k x_k + \sum_{x_k \in \mathcal{X}_s^{\text{inf}}} \kappa_k x_k + \sum_{x_k \in \mathcal{X}_s^{\text{fin}}} \mu_k x_k$$

is a solution of the inhomogeneous ELCP.

Now we prove the only-if-part:

Consider an arbitrary solution x of the inhomogeneous ELCP. Construct $u = \begin{bmatrix} x \\ 1 \end{bmatrix}$. Since x is a solution of the inhomogeneous ELCP, u is a solution of the homogeneous ELCP. So there exists a set $\mathcal{E}_s$ such that

$$u \; = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_l \in \mathcal{E}_s} \nu_l e_l$$

with $\lambda_k \in \mathbb{R}$ and $\nu_l \geqslant 0$ or if we extract $\mathcal{E}_s^{\text{inf}}$ and $\mathcal{E}_s^{\text{fin}}$:

$$u \; = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}_s^{\text{inf}}} \kappa_k e_k + \sum_{e_k \in \mathcal{E}_s^{\text{fin}}} \mu_k e_k$$

with $\lambda_k \in \mathbb{R}$, $\kappa_k \geqslant 0$, $\mu_k \geqslant 0$ and $u_\alpha = 1 = \sum_k \lambda_k \cdot 0 + \sum_k \kappa_k \cdot 0 + \sum_k \mu_k \cdot 1$. So $\sum_k \mu_k = 1$.

Note that the set $\mathcal{E}_s^{\text{fin}}$ cannot be empty since otherwise the α component of u would not be equal to 1. □

This leads to the following theorem:

Theorem 4.14 *The solution set of an ELCP either is empty or consists of the union of faces of a polyhedron.*

We can even reverse this theorem:

Theorem 4.15 *The union of any arbitrary set $\mathcal{S}$ of faces of an arbitrary polyhedron $\mathcal{P}$ can be described by an ELCP.*

Proof: Let the polyhedron $\mathcal{P}$ be defined by $\mathcal{P} = \{x \,|\, Ax \geqslant c\}$ for some matrix $A \in \mathbb{R}^{p \times n}$ and some vector $c \in \mathbb{R}^p$. Let $\mathcal{F}$ be the union of the faces in $\mathcal{S}$: $\mathcal{F} = \bigcup_{F_i \in \mathcal{S}} F_i$.

Now consider one face $F_i \in \mathcal{S}$ with dimension k_i. This face is the intersection of $\mathcal{P}$ and $l_i = n - k_i$ linearly independent hyperplanes from the constraints that define $\mathcal{P}$. Let ϕ_i be the set of indices that correspond to these hyperplanes. Then we have that $F_i = \{x \,|\, Ax \geqslant c \text{ and } \forall j \in \phi_i : (Ax - c)_j = 0\}$. Since $Ax - c \geqslant 0$, we also have that $F_i = \{x \,|\, Ax \geqslant c \text{ and } \sum_{j \in \phi_i} (Ax - c)_j = 0\}$. If we define a set ϕ_i of indices for each face $F_i \in \mathcal{S}$ then $\mathcal{F}$ coincides with the solution set of the following ELCP:

Find $x \in \mathbb{R}^n$ such that

$$\prod_{i=1}^{\#\mathcal{S}} \sum_{j \in \phi_i} (Ax - c)_j \; = \; 0 \tag{15}$$

subject to $Ax \geqslant c$.

Equation (15) really is a complementarity condition since we can always rewrite it as a sum of products. □

Note that the empty set can also be described by an ELCP (e.g. by taking an infeasible system of linear inequalities).

Remark: For the inhomogeneous ELCP we are only interested in cross-complementary subsets that contain at least one finite solution. Therefore it is advantageous to put the extreme rays that have $e_\alpha \neq 0$ at the top before determining the sets of cross-complementary rays. In that case we do not have to consider all sets of cross-complementary extreme rays but we can stop Algorithm 3 as soon as we have used all rays that have $e_\alpha \neq 0$.

4.5 The complexity of the ELCP and our ELCP algorithm

In each step of the algorithm to determine the central and extreme rays we have to make combinations of intermediate rays. This means that the execution time of this algorithm depends heavily on the number of extra rays that are generated in each step. In general one could say that the execution time and the required amount of storage space grow as the number of equations and variables grows. However, we have noticed that the execution time and the storage space requirements do not only depend on the number of variables and (in)equalities but also on the structure of the solution set.

In [22] Mattheiss and Rubin give a survey and comparison of methods for finding all vertices of polytopes or polyhedra with an empty lineality space. The worst case behavior of Algorithms 1 and 2 can be compared with these algorithms if we would take the trivial complementarity condition $\prod_{i=1}^{p} (Ax - c)_i = 0$, which means that at least one inequality should be satisfied by equality or that every border point of the polyhedron defined by $Ax \geqslant c$ is a solution of the ELCP. Note that we may assume that there are no central rays since we can first determine a set of central rays by solving the system of homogeneous linear equations $Ax = 0$ and $Bx = 0$, and then remove the central rays from the ELCP solution by imposing the condition that the other solutions should be orthogonal to the central rays. In [22] Mattheiss and Rubin report execution times of the order $O(v^\rho)$ with $\rho = 1.418$ and v the number of vertices of the polyhedron for the Chernikova algorithm, which is a special case of the double description method: the Chernikova algorithm requires the additional constraint that all variables should be nonnegative, so there are no central rays. However, note that according to the upper bound conjecture [22] the least upper bound of the number of vertices of a polytope defined by m (irredundant) inequality constraints in an n-dimensional space is given by

$$\binom{m - \left\lfloor \dfrac{n+1}{2} \right\rfloor}{m-n} + \binom{m - \left\lfloor \dfrac{n+2}{2} \right\rfloor}{m-n}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x and $\begin{pmatrix} m \\ n \end{pmatrix} = \dfrac{m!}{(m-n)!\,n!}$. This means that in the worst case the number of vertices v can be of the order $O\left(m^{\lfloor \frac{n}{2} \rfloor}\right)$ if $m \gg n \gg 1$. Fortunately, we can already use Property 3.3 to reject extreme rays that do not satisfy the complementarity condition during the iteration process. This means that on the average the execution times of our algorithm will be far less than the ones reported in [22]. Furthermore, we have noticed that the number of intermediate rays also depends on the order in which the inequalities and equalities are processed. It is still an open question how the optimal order can be determined.
The execution time of Algorithm 3 depends heavily on the structure of the solution set and on the number of finite rays (since we stop the algorithm as soon as all finite rays have been considered).
Since the execution time of our ELCP algorithm depends on so many factors it is difficult to give a neat characterization of the computational complexity as a function of the number of variables and inequalities.

From the above we can conclude that the ELCP algorithm as presented in this paper is not well suited for large problems with a large number of (in)equalities and variables and/or a complex solution set. For such kind of systems one could try to develop algorithms that only search one (non-trivial) solution, since in many cases we do not need all solutions. Possible approaches are:

- global minimization [21]: for the ELCP we have to minimize the left hand side of the complementarity condition over the equality and inequality constraints. The function value in the minimum will be equal to 0 and the minimum will be a solution of the ELCP. (See e.g. [15, 26] for methods and algorithms for constrained optimization);
- systems of polynomial equalities: by introducing slack variables the ELCP can be transformed into a system of multivariate polynomial equalities;
- adaptations and extensions of the existing methods for LCPs and GLCPs of e.g. [1, 6, 19, 23, 24, 29, 34].

However, the following theorem shows that the ELCP is intrinsically a computationally hard problem:

Theorem 4.16 *The general ELCP is an NP-hard problem.*

Proof: The decision problem that corresponds to the ELCP belongs to NP: a nondeterministic algorithm can guess a vector x and then check in polynomial time whether x satisfies the complementarity condition and the system of linear equalities and inequalities. Chung [4] has proved that the decision problem that corresponds to the LCP is in general an NP-complete problem. Since the LCP is a special case of the ELCP, the decision problem that corresponds to the ELCP is also NP-complete. This means that in general the ELCP is NP-hard. □

So the ELCP can probably not be solved in polynomial time (unless the class P would coincide with the class NP). The interested reader is referred to [14] for an extensive treatment of NP-completeness.

5 Example: application of the ELCP in the max algebra

The formulation of the ELCP arose from our research on discrete event systems. Normally the behavior of discrete event systems is highly nonlinear. However, when the order of the events is known or fixed some of these systems can be described by a *linear* description in the max algebra [2].

The basic operations of the max algebra are the maximum (represented by $\oplus$) and the addition (represented by $\otimes$):

$$x \oplus y = \max\{x, y\} \tag{16}$$

$$x \otimes y = x + y \ . \tag{17}$$

The max-algebraic power is defined as follows:

$$x^{\otimes^a} = a \cdot x \ . \tag{18}$$

Many important problems in the max algebra can be reformulated as a set of multivariate polynomial equalities and inequalities in the max algebra:

> Given a set of integers $\{m_k\}$ and three sets of real numbers $\{a_{ki}\}$, $\{b_k\}$ and $\{c_{kij}\}$ for $i = 1, 2, \ldots, m_k$; for $j = 1, 2, \ldots, n$ and for $k = 1, 2, \ldots, p_1 + p_2$, find a vector $x \in \mathbb{R}^n$ that satisfies
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} = b_k \qquad \text{for } k = 1, 2, \ldots, p_1 \ ,$$
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} \leqslant b_k \qquad \text{for } k = p_1 + 1, p_1 + 2, \ldots, p_1 + p_2 \ ,$$
>
> or show that no such vector x exists.

Note that the exponents can be negative or real. In [12] we have proved the following theorem:

Theorem 5.1 *A set of multivariate polynomial equalities and inequalities in the max algebra is equivalent to an ELCP.*

We shall illustrate this by an example:

Example 5.2
Consider the following set of multivariate polynomial equalities and inequalities:

$$(-8) \otimes x_1^{\otimes^4} \otimes x_3 \otimes x_5^{\otimes^{-2}} \otimes x_6 \ \oplus \ 9 \otimes x_2^{\otimes^4} \otimes x_4^{\otimes^{-2}} \otimes x_6^{\otimes^{-2}} \ \oplus \ 10 \otimes x_1 \otimes x_2^{\otimes^{-2}} \otimes x_5^{\otimes^{-4}} = 3 \tag{19}$$

$$5 \otimes x_1^{\otimes^{-2}} \otimes x_2^{\otimes^4} \otimes x_3^{\otimes^{-7}} \otimes x_4^{\otimes^2} \otimes x_5^{\otimes^2} \otimes x_6^{\otimes^{-3}} \ \oplus \ 4 \otimes x_2^{\otimes^{-1}} \otimes x_4^{\otimes^3} \otimes x_5^{\otimes^{-2}} = 14 \tag{20}$$

$$(-3) \otimes x_1^{\otimes^{-1}} \otimes x_3^{\otimes^3} \otimes x_4^{\otimes^{-2}} \leqslant 8 \ . \tag{21}$$

Using definitions (17) and (18) we find that the first "term" of (19) is equivalent to

$$(-8) + 4x_1 + x_3 + (-2)x_5 + x_6 \ .$$

The other terms of (19) can be transformed to linear algebra in a similar way. Each term has to be smaller than 3 and at least one of them has to be equal to 3. Hence we get a group of three inequalities in which at least one inequality should hold with equality. If we also include (20) and (21) we get the following ELCP:

Given

$$A = \begin{bmatrix} -4 & 0 & -1 & 0 & 2 & -1 \\ 0 & -4 & 0 & 2 & 0 & 2 \\ -1 & 2 & 0 & 0 & 4 & 0 \\ 2 & -4 & 7 & -2 & -2 & 3 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 1 & 0 & -3 & 2 & 0 & 0 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} -11 \\ 6 \\ 7 \\ -9 \\ -10 \\ -11 \end{bmatrix} ,$$

find a vector $x \in \mathbb{R}^6$ such that

$$(Ax-c)_1\,(Ax-c)_2\,(Ax-c)_3 \;+\; (Ax-c)_4\,(Ax-c)_5 \;=\; 0$$

subject to $Ax \geqslant c$.

Using a similar approach we can also transform an ELCP into a set of multivariate max-algebraic polynomial equalities and inequalities.
If we apply the ELCP algorithm to this ELCP we get the rays of Table 1 and the pairs of subsets of Table 2. Any arbitrary solution of the set of multivariate polynomial equalities and inequalities can now be expressed as

$$x \;=\; \lambda_1 x_1^{\mathrm{c}} + \sum_{x_k^{\mathrm{i}} \in \mathcal{X}_s^{\mathrm{inf}}} \kappa_k x_k^{\mathrm{i}} + \sum_{x_k^{\mathrm{f}} \in \mathcal{X}_s^{\mathrm{fin}}} \mu_k x_k^{\mathrm{f}}$$

for some $s \in \{1, \ldots, 6\}$ with $\lambda_1 \in \mathbb{R}$, $\kappa_k \geqslant 0$, $\mu_k \geqslant 0$ and $\sum_k \mu_k = 1$. □

Many problems in the max algebra such as matrix decompositions, transformation of state space models, construction of matrices with a given characteristic polynomial, minimal state space realization and so on, can be reformulated as a set of multivariate max-algebraic polynomial equalities and inequalities [11, 12]. These problems are equivalent to an ELCP and can thus be solved using the ELCP algorithm. In general their solution set consists of the union of a set of faces of a polyhedron.

6 Conclusions and Future Research

In this paper we have proposed the Extended Linear Complementarity Problem (ELCP) and established a link between the ELCP and other Linear Complementarity Problems. We have shown that the ELCP can be considered as a unifying framework for the LCP and its generalizations. Furthermore, we have made a thorough study of the general solution set of the ELCP and developed an algorithm to find all its solutions.

Since our algorithm yields all solutions, it provides a geometric insight in the solution set of an ELCP and other problems that can be reduced to an ELCP. On the other hand, this also leads to large computation times and storage space requirements if the number of variables and (in)equalities is large. However, we are not always interested in obtaining all solutions of an ELCP. Therefore our further research efforts will concentrate on the development of (heuristic) algorithms that yield only one solution (as we have already done for the minimal state space realization problem in the max algebra [10]).

Although we have shown that in general the ELCP is NP-hard, it may also be interesting to determine which subclasses of the ELCP can be solved with polynomial time algorithms.

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Set	$\mathcal{X}^{\text{cen}}$	$\mathcal{X}^{\text{inf}}$					$\mathcal{X}^{\text{fin}}$		
Ray	x_1^{c}	x_1^{i}	x_2^{i}	x_3^{i}	x_4^{i}	x_5^{i}	x_1^{f}	x_2^{f}	x_3^{f}
x_1	-108	12	-4	0	36	0	-15	3	-15
x_2	136	-16	8	-2	-56	-2	6	2	22
x_3	-48	0	0	0	0	0	0	0	0
x_4	-18	-6	2	0	6	0	2	5	2
x_5	-95	11	-1	1	37	1	-5	1.5	-13
x_6	290	-26	14	-2	-70	2	13	2	45

Table 1: The central, infinite and finite rays of the ELCP of Example 5.2.

s	$\mathcal{X}_s^{\text{inf}}$	$\mathcal{X}_s^{\text{fin}}$
1	$\{x_1^{\text{i}}, x_2^{\text{i}}, x_4^{\text{i}}\}$	$\{x_2^{\text{f}}\}$
2	$\{x_1^{\text{i}}, x_2^{\text{i}}\}$	$\{x_1^{\text{f}}, x_2^{\text{f}}\}$
3	$\{x_1^{\text{i}}, x_3^{\text{i}}, x_4^{\text{i}}\}$	$\{x_1^{\text{f}}, x_2^{\text{f}}\}$
4	$\{x_2^{\text{i}}, x_4^{\text{i}}, x_5^{\text{i}}\}$	$\{x_2^{\text{f}}, x_3^{\text{f}}\}$
5	$\{x_2^{\text{i}}\}$	$\{x_1^{\text{f}}, x_2^{\text{f}}, x_3^{\text{f}}\}$
6	$\{x_3^{\text{i}}, x_4^{\text{i}}, x_5^{\text{i}}\}$	$\{x_1^{\text{f}}, x_2^{\text{f}}, x_3^{\text{f}}\}$

Table 2: The pairs of cross-complementary subsets of the ELCP of Example 5.2.

The Extended Linear Complementarity Problem: Addendum

Bart De Schutter and Bart De Moor

A Some worked examples of the ELCP algorithm

In this appendix we give two worked examples that illustrate our ELCP algorithm. In the first example we solve a simple ELCP problem with Algorithms 1 and 2. However for this small sized ELCP problem we do not get enough extreme rays to demonstrate Algorithm 3. Therefore we give a second example in which we show how to determine all maximal sets of cross-complementary extreme rays.

Example A.1: determination of the central and extreme rays

Consider the following ELCP:

Given

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & -2 \\ 1 & -1 & 0 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} ,$$

find $u \in \mathbb{R}^4$ such that

$$(Pu)_1 \, (Pu)_2 + (Pu)_2 \, (Pu)_3 \; = \; 0 \tag{22}$$

$$\text{subject to} \quad \begin{aligned} Pu & \geqslant 0 \\ Qu & = 0 \; . \end{aligned}$$

Since all inequalities of $Pu \geqslant 0$ appear in the complementarity condition we do not have to split $Pu \geqslant 0$.
First we process the inequalities of $Pu \geqslant 0$:

k=0

Initialization:

$$c_{0,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} , \; c_{0,2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} , \; c_{0,3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} , \; c_{0,4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \; .$$

k=1

First we calculate the residues:

$$\text{res}\,(c_{0,1}) = 1 \, , \; \text{res}\,(c_{0,2}) = 0 \, , \; \text{res}\,(c_{0,3}) = 1 \, , \; \text{res}\,(c_{0,4}) = 0 \; .$$

Hence $\mathcal{C}^+ = \{c_{0,1}, c_{0,3}\}$ and $\mathcal{C}^0 = \{c_{0,2}, c_{0,4}\}$. Since the set $\mathcal{C}^+$ is not empty we go to Case 3 of Algorithm 1. First we put the elements of $\mathcal{C}^0$ in $\mathcal{C}$: $c_{1,1} = c_{0,2}$ and $c_{1,2} = c_{0,4}$. Since $\mathcal{C}^-$ is empty we do not have to transfer rays from $\mathcal{C}^-$ to $\mathcal{C}^+$. We set $c = c_{0,1}$ and put it in $\mathcal{E}$: $e_{1,1} = c_{0,1}$. Because no group of inequalities has been processed entirely yet, condition (6) is void. So c satisfies the partial complementarity condition by definition. Finally we combine $c = c_{0,1}$ and $c_{0,3}$ and put the result in $\mathcal{C}$:

$$c_{1,3} = \operatorname{res}(c_{0,3})\, c_{0,1} - \operatorname{res}(c_{0,1})\, c_{0,3} = c_{0,1} - c_{0,3} \ .$$

So we find

$$c_{1,1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ c_{1,2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},\ c_{1,3} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix},\ e_{1,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \ .$$

<u>k=2</u>

We have that

$$\operatorname{res}(c_{1,1}) = 1\,,\ \operatorname{res}(c_{1,2}) = -2\,,\ \operatorname{res}(c_{1,3}) = 0\,,\ \operatorname{res}(e_{1,1}) = 1 \ .$$

So $\mathcal{C}^+ = \{c_{1,1}\}\,, \mathcal{C}^0 = \{c_{1,3}\}\,, \mathcal{C}^- = \{c_{1,2}\}$ and $\mathcal{E}^+ = \{e_{1,1}\}$. Since $\mathcal{C}^+$ is not empty we go again to Case 3. We put $c_{1,3}$ in $\mathcal{C}$: $c_{2,1} = c_{1,3}$ and transfer $-c_{1,2}$ to $\mathcal{C}^+$. The ray $c = c_{1,1}$ satisfies the partial complementarity condition $(Pu)_1\,(Pu)_2 = 0$, since $P_{1:2,.}c_{1,1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence we put it in $\mathcal{E}$: $e_{2,1} = c_{1,1}$. We combine $c = c_{1,1}$ and $-c_{1,2}$ and transfer the result to $\mathcal{C}$:

$$c_{2,2} = \operatorname{res}(-c_{1,2})\, c_{1,1} - \operatorname{res}(c_{1,1})\,(-c_{1,2}) = 2c_{1,1} + c_{1,2} \ .$$

The combination of $c = c_{1,1}$ and $e_{1,1}$ satisfies the partial complementarity condition, so we also put it to $\mathcal{E}$:

$$e_{2,2} = \operatorname{res}(c_{1,1})\, e_{1,1} - \operatorname{res}(e_{1,1})\, c_{1,1} = e_{1,1} - c_{1,1} \ .$$

This yields

$$c_{2,1} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix},\ c_{2,2} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix},\ e_{2,1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ e_{2,2} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \ .$$

<u>k=3</u>

Using the same procedure as in the previous steps we find

$$c_{3,1} = c_{2,1} + c_{2,2}\,,\ \ e_{3,1} = c_{2,1}\,,\ \ e_{3,2} = e_{2,1} + c_{2,1}\,,\ \ e_{3,3} = e_{2,2} - 2c_{2,1} \ ,$$

and thus

$$c_{3,1} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \; e_{3,1} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \; e_{3,2} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \; e_{3,3} = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \; .$$

We do not have to reject any extreme rays since they all satisfy complementarity condition (22).

Since we did not encounter any redundant inequalities, we have that $P_{\text{nec}} = P$. Now we take the equality $Qu = 0$ into account:

k=1

We have that

$$\text{res}\,(c_{3,1}) = 0\,, \; \text{res}\,(e_{3,1}) = -1\,, \; \text{res}\,(e_{3,2}) = -1\,, \; \text{res}\,(e_{3,3}) = 2 \; .$$

So $\mathcal{C}^0 = \{c_{3,1}\}\,, \mathcal{E}^+ = \{e_{3,3}\}$ and $\mathcal{E}^- = \{e_{3,1}, e_{3,2}\}$. Since $\mathcal{C}^+ = \mathcal{C}^- = \emptyset$ we go to Case 2 of Algorithm 2. All elements of $\mathcal{C}$ stay in $\mathcal{C}$:

$$c_{4,1} = c_{3,1} = \begin{bmatrix} 1 & 2 & -1 & 1 \end{bmatrix}^T \; .$$

Now we have to determine which pairs of rays are adjacent. We find the following zero index sets:

$$\mathcal{I}_0(e_{3,1}) = \{1,2\}\,, \; \mathcal{I}_0(e_{3,2}) = \{1,3\}\,, \; \mathcal{I}_0(e_{3,3}) = \{2,3\} \; .$$

If we consider Adjacency Test 1 then a necessary condition for two extreme rays to be adjacent is that their zero index sets have $n - t - 2 = 4 - 1 - 2 = 1$ common element. This means that all possible combinations of two different extreme rays pass Adjacency Test 1. Note that Adjacency Test 2 is still necessary and sufficient since we have not rejected any extreme rays. As a consequence we can conclude that the rays $e_{3,3}$ and $e_{3,1}$ are adjacent because $\mathcal{I}_0(e_{3,3}) \cap \mathcal{I}_0(e_{3,1}) = \{2\} \not\subset \{1,3\} = \mathcal{I}_0(e_{3,2})$. If we combine them we get a new extreme ray that satisfies the complementarity condition, so we put it in $\mathcal{E}$:

$$e_{4,1} = 2e_{3,1} + e_{3,3} = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^T \; .$$

Since the combination of the adjacent rays $e_{3,3}$ and $e_{3,2}$:

$$2e_{3,2} + e_{3,3} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$$

does not satisfy the complementarity condition, we have to reject it.

Now we have $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 2 & -1 & 1 \end{bmatrix}^T \right\}$ and $\mathcal{E} = \left\{ \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^T \right\}$. Hence every combination of the form

$$u = \lambda \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} + \kappa \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{with } \lambda \in \mathbb{R} \text{ and } \kappa \geqslant 0$$

is a solution of the ELCP.

Example A.2: determination of the cross-complementary sets

Since the determination of the cross-complementary sets of an ELCP essentially reduces to the determination of the cross-complementary sets of a GLCP, we demonstrate Algorithm 3 for a GLCP. Suppose that we have the following GLCP:

Given $Z = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 \end{bmatrix}$, find $u \in \mathbb{R}^5$ such that

$$u_2\, u_3\, u_4 + u_3\, u_5 \ = \ 0 \tag{23}$$

subject to $u \geqslant 0$ and $Zu = 0$.

The extreme rays of the solution set of this GLCP are

$$e_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ e_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ e_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \ .$$

Now we use Algorithm 3 to determine which nonnegative combinations of the extreme rays are also solutions of the GLCP.

- First we have to transform every ray into its binary equivalent. But since the rays are already binary we can leave them as they are. The binary complementarity condition is

$$[(u_2 = 0) \vee (u_3 = 0) \vee (u_4 = 0)] \wedge [(u_3 = 0) \vee (u_5 = 0)] \ . \tag{24}$$

- Next we construct the cross-complementarity matrix, i.e. we determine which pairs of extreme rays are cross-complementary.
  The rays e_1 and e_2 are cross-complementary since

  $$e_1 \vee e_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \end{bmatrix}^T$$

  satisfies the binary complementarity condition.
  However e_2 and e_4 are not cross-complementary since

  $$e_2 \vee e_4 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \end{bmatrix}^T$$

  does not satisfy condition (24).
  We check all other combinations of two different extreme rays and put the results in the cross-complementarity matrix. To make the subsequent steps easier to follow, we represent this cross-complementarity matrix by its graph $\mathcal{G}$ (see Figure 2). An edge between two vertices of $\mathcal{G}$ indicates that the corresponding extreme rays are cross-complementary.

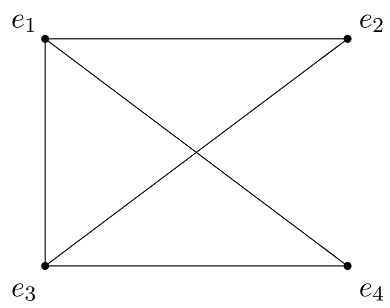

Figure 2: The cross-complementarity graph $\mathcal{G}$ for Example A.2.

- In Algorithm 3 we keep track of our progress by using lists of vertices to be investigated for each depth d, where d is the number of elements of the set that we are currently investigating. These lists are stored in the matrix *vertices* where the dth row contains the vertices for depth d. The column index for the first vertex of the list for depth d is $start(d)$, and the column index for the last vertex is $last(d)$. The combination we are currently investigating consists of the first vertex of each depth. In our explanation $\mathcal{N}_d$ will represent the lists of vertices for depth d. So $\mathcal{N}_d = \bigcup_{j=start(d)}^{last(d)} \{vertices(d,j)\}$.
- We start with a list of vertices for depth 1: $\mathcal{N}_1 = \{1, 2, 3, 4\}$.
  Vertex 1 is the first vertex in the list, so now we look for other vertices of $\mathcal{N}_1$ that are connected by an edge to vertex 1. Since the vertices 2, 3 and 4 satisfy this condition, we get $\mathcal{N}_2 = \{2, 3, 4\}$.
  The first vertices of each list are 1 and 2 so the set we are currently investigating is $\{1, 2\}$. Now we try to expand this set. The only other vertex in list $\mathcal{N}_2$ that is connected to both vertex 1 and vertex 2 is vertex 3. So we check whether the set $\{e_1, e_2, e_3\}$ is cross-complementary. This is not the case since

  $$e_1 \vee e_2 \vee e_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \end{bmatrix}^T$$

  does not satisfy condition (24). So we do not find any vertices for depth 3. This means that the current set $\{e_1, e_2\}$ is a maximal cross-complementary set. We put it in the list Γ: $\Gamma = \{\,\{e_1, e_2\}\,\}$.
- We return to the previous point where a choice was made: we remove vertex 2 from $\mathcal{N}_2$: $\mathcal{N}_2 = \{3, 4\}$.
  Now we are investigating the set $\{e_1, e_3\}$. Since vertex 4 is connected to both vertex 1 and vertex 3 we check whether

  $$e_1 \vee e_3 \vee e_4 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \end{bmatrix}^T$$

  satisfies the complementarity condition. Since this is the case, we add vertex 4 to the list of depth 3: $\mathcal{N}_3 = \{4\}$. Because there is only one vertex in $\mathcal{N}_3$ we cannot augment the depth. Hence we have again found a maximal cross-complementary set that should be added to the list Γ: $\Gamma = \{ \{e_1, e_2\}, \{e_1, e_3, e_4\} \}$.

- We return to previous point where a choice was made: we go again to depth 2 and we remove vertex 3 from list $\mathcal{N}_2$: $\mathcal{N}_2 = \{4\}$.
  The current set is $\{1, 4\}$. There are no more vertices left for the next depth so we have a maximal cross-complementary set: $\{e_1, e_4\}$. But since this set is a subset of the set $\{e_1, e_3, e_4\}$, that is already in the list Γ, we do not add it to Γ.
- Next we go back to depth 1, remove vertex 1 from $\mathcal{N}_1$ and so on.

Finally we get

$$\Gamma = \{ \{e_1, e_2\}, \{e_1, e_3, e_4\}, \{e_2, e_3\} \} \quad .$$

This means that every combination of the form

$$u = \kappa_1 e_1 + \kappa_2 e_2$$

or

$$u = \kappa_1 e_1 + \kappa_3 e_3 + \kappa_4 e_4$$

or

$$u = \kappa_2 e_2 + \kappa_3 e_3$$

with $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \geqslant 0$ is a solution of the GLCP.

Remark: We see again that finding all maximal sets of cross-complementary extreme rays is not the same as finding all maximal cliques of the cross-complementarity graph $\mathcal{G}$ since $\{e_1, e_2, e_3\}$ is a maximal clique but not a cross-complementary set.