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Minimal Realization in the Max Algebra is an Extended Linear Complementarity Problem*

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Abstract

In this paper we demonstrate that the minimal state space realization problem in the max algebra can be transformed into an Extended Linear Complementarity Problem (ELCP). We use an algorithm that finds all solutions of an ELCP to find all equivalent minimal state space realizations of a single input single output (SISO) discrete event system. We also give a geometrical description of the set of all minimal realizations of a SISO max-linear discrete event system.

1 Introduction

1.1 Overview

In this paper we consider discrete event systems, such as flexible manufacturing systems, subway traffic networks, parallel processing systems, telecommunication networks, etc. . Some of these systems can be described using the so called max algebra [1, 3]. We shall show that the minimal state space realization problem in the max algebra can be transformed into an Extended Linear Complementarity Problem (ELCP). The ELCP is an extension of the well-known Linear Complementarity Problem, which is one of the fundamental problems of mathematical programming. In [7] we have developed an algorithm to find all solutions of an ELCP. We shall use this algorithm to find all equivalent minimal state space realizations of a single input single output discrete event system and to give a geometrical insight in the structure of the set of all equivalent state space realizations.

Although there have been some attempts to solve this minimal realization problem [4, 11, 13] , this is – to the authors' knowledge – the first time it is solved entirely. And it is certainly the first time that a complete description of the set of all minimal realizations of a SISO max-linear discrete event system is given.

In Section 1 of this paper we introduce the notations and some of the concepts and definitions that are used later on.

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In Section 2 we propose the Extended Linear Complementarity Problem (ELCP) and describe the general solution set of this problem.

In Section 3 we show that a set of multivariate polynomial equations in the max algebra can be transformed into an ELCP.

Then we derive a lower bound for the minimal state space order of a single input single output (SISO) discrete event system in the max algebra. Finally we combine this with the results of the preceding sections to find all minimal state space realizations of a SISO discrete event system, given its Markov parameters.

We also illustrate this procedure with a few examples.

1.2 Notations and definitions

If a is a vector then a_i represents the i-th element of a. If A is an m by n matrix then the element on the i-th row and on the j-th column is denoted by a_{ij}. A^t is the transpose of A.

To select submatrices of a matrix we use the following notation:
$A([i_1, i_2, \ldots, i_k], [j_1, j_2, \ldots, j_l])$ is the k by l matrix resulting from A by eliminating all rows except for rows $i_1, i_2, \ldots, i_k$ and all columns except for columns $j_1, j_2, \ldots, j_l$. $A(i, :)$ is the i-th row of A and $A(:, j)$ is the j-th column of A.

Definition 1.1 (Polyhedron) *A polyhedron is the solution set of a finite system of linear inequalities.*

We shall represent the set of all possible combinations of k different numbers out of the set $\{1, 2, \ldots, n\}$ as $\mathcal{C}_n^k$. $\mathcal{P}_n$ is the set of all possible permutations of the set $\{1, 2, \ldots, n\}$.

1.3 The max algebra

One of the mathematical tools used in this paper is the max algebra. In this introduction we only explain the notations we use to represent the max-algebraic operations and give some definitions and theorems that will be used in the remainder of this paper. A complete introduction to the max algebra can be found in [1, 3].

1.3.1 The max-algebraic operations

In this paper we use the following notations: $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$. The neutral element for $\oplus$ in $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is $\varepsilon = -\infty$. Since we use both linear algebra and max algebra in this report, we always write the $\otimes$ sign explicitly to avoid confusion. The max-algebraic power is defined as follows: $a^{\otimes^k} = \underbrace{a \otimes a \otimes \ldots \otimes a}_{k \text{ times}}$ and is equal to ka in linear algebra.

E_n is the n by n identity matrix in $\mathbb{R}_{\max}$: $e_{ij} = 0$ if $i = j$ and $e_{ij} = \varepsilon$ if $i \neq j$. The operations $\oplus$ and $\otimes$ are extended to matrices in the usual way. $A^{\otimes^k} = \underbrace{A \otimes A \otimes \ldots \otimes A}_{k \text{ times}}$.

We also use the extension of the max algebra $\mathbb{S}_{\max}$ that was introduced in [1, 9]. $\mathbb{S}_{\max}$ is a kind of symmetrization of $\mathbb{R}_{\max}$. We shall restrict ourselves to the most important features of $\mathbb{S}_{\max}$. For a more formal derivation the interested reader is referred to [9].

There are three kinds of elements in $\mathbb{S}_{\max}$: the positive elements ($\mathbb{S}_{\max}^{\oplus}$, this corresponds to $\mathbb{R}_{\max}$), the negative elements ($\mathbb{S}_{\max}^{\ominus}$) and the balanced elements ($\mathbb{S}_{\max}^{\bullet}$). The positive and the negative elements are called signed ($\mathbb{S}_{\max}^{\vee} = \mathbb{S}_{\max}^{\oplus} \cup \mathbb{S}_{\max}^{\ominus}$). The $\ominus$ operation in $\mathbb{S}_{\max}$ is defined as follows:

$$\begin{aligned} a \ominus b &= a \quad && \text{if } a > b\,, \\ a \ominus b &= \ominus b \quad && \text{if } a < b\,, \\ a \ominus a &= a^{\bullet}\ . \end{aligned}$$

If $a \in \mathbb{S}_{\max}$ then it can be written as $a = a^{+} \ominus a^{-}$ where a^{+} is the positive part of a, a^{-} is the negative part of a and $|a| = a^{+} \oplus a^{-}$ is the absolute value of a. There are three possible cases: if $a \in \mathbb{S}_{\max}^{\oplus}$ then $a^{+} = a$ and $a^{-} = \varepsilon$, if $a \in \mathbb{S}_{\max}^{\ominus}$ then $a^{+} = \varepsilon$ and $a^{-} = \ominus a$ and if $a \in \mathbb{S}_{\max}^{\bullet}$ then $a^{+} = a^{-} = |a|$.

Example 1.2 Let $a = 3^{\bullet} \in \mathbb{S}_{\max}^{\bullet}$, then $a^{+} = 3$, $a^{-} = 3$ and $|a| = 3$.
For $b = \ominus 2 \in \mathbb{S}_{\max}^{\ominus}$ we have $b^{+} = \varepsilon$, $b^{-} = 2$ and $|b| = 2$.

This symmetrization allows us to 'solve' equations that have no solutions in $\mathbb{R}_{\max}$. Unfortunately we then have to introduce balances (∇) instead of equalities. Informally an $\ominus$ sign in a balance means that the element should be at the other side: so $3 \ominus 3 \ \nabla\ 2$ means $3 \ \nabla\ 2 \oplus 3$. If both sides of a balance are signed (positive or negative) we can replace the balance by an equality.

1.3.2 Some definitions and theorems

Definition 1.3 (Determinant) *Consider a matrix $A \in \mathbb{S}_{\max}^{n \times n}$. The determinant of A is defined as*

$$\det A = \bigoplus_{\sigma \in \mathcal{P}_n} \operatorname{sgn}(\sigma) \otimes \bigotimes_{i=1}^{n} a_{i\sigma(i)}$$

where $\mathcal{P}_n$ is the set of all permutations of $\{1, \ldots, n\}$, and $\operatorname{sgn}(\sigma) = 0$ if the permutation σ is even and $\operatorname{sgn}(\sigma) = \ominus 0$ if the permutation is odd.

Definition 1.4 (Determinantal rank) *Let $A \in \mathbb{S}_{\max}^{m \times n}$. The determinantal rank of A, $\mathrm{r}_{\det}(A)$, is defined as the dimension of the largest square submatrix of A the determinant of which is not balanced and not equal to ε.*

Theorem 1.5 *Let $A \in \mathbb{S}_{\max}^{n \times n}$. The homogeneous linear balance $A \otimes x \ \nabla\ \varepsilon$ has a non-trivial signed solution if and only if $\det A \ \nabla\ \varepsilon$.*

Proof: See [9]. The proof given there is constructive so it can be used to find a solution.

Definition 1.6 (Characteristic equation) *Let $A \in \mathbb{S}_{\max}^{n \times n}$. The characteristic equation of A is defined as $\det(A \ominus \lambda \otimes E_n) \ \nabla\ \varepsilon$.*

This leads to

$$\lambda^{\otimes^{n}} \oplus \bigoplus_{p=1}^{n} a_p \otimes \lambda^{\otimes^{n-p}} \ \nabla\ \varepsilon\ .$$

If we define $\alpha_p = a_p^{+}$ and $\beta_p = a_p^{-}$ and if we move all terms with negative coefficients to the right hand side we get

$$\lambda^{\otimes^{n}} \oplus \bigoplus_{i=1}^{n} \alpha_i \otimes \lambda^{\otimes^{n-i}} \ \nabla\ \bigoplus_{j=1}^{n} \beta_j \otimes \lambda^{\otimes^{n-j}}\ ,$$

with $\alpha_p, \beta_p \in \mathbb{R}_{\max}$. In [12] Olsder defines a variant of this equation using the dominant instead of the determinant. This leads to signed coefficients: $a_p^{\text{Olsder}} \in \mathbb{S}_{\max}^{\vee}$ or $\alpha_p^{\text{Olsder}} \otimes \beta_p^{\text{Olsder}} = \varepsilon$ with $a_p^{\text{Olsder}} = a_p$ if $a_p \in \mathbb{S}_{\max}^{\vee}$ and $|a_p^{\text{Olsder}}| \leqslant |a_p|$ if $a_p \in \mathbb{S}_{\max}^{\bullet}$.

Theorem 1.7 (Cayley-Hamilton) *In $\mathbb{S}_{\max}$ every square matrix satisfies its characteristic equation.*

Proof: See [10] and [12].

2 The Extended Linear Complementarity Problem

2.1 Problem formulation

Consider the following problem:

> Given two matrices $A \in \mathbb{R}^{p \times n}, B \in \mathbb{R}^{q \times n}$, two column vectors $c \in \mathbb{R}^p, d \in \mathbb{R}^q$ and m subsets ϕ_j of $\{1, 2, \ldots, p\}$, find a vector $x \in \mathbb{R}^n$ such that
>
> $$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i = 0 \qquad (1)$$
>
> $$\text{subject to} \quad \begin{array}{rcl} Ax & \geqslant & c \\ Bx & = & d \end{array} ,$$
>
> or show that no such vector exists.

In [7] we have demonstrated that this problem is an extension of the Linear Complementarity Problem [2]. Therefore we call it the Extended Linear Complementarity Problem (ELCP). Equation (1) represents the *complementarity condition*. One possible interpretation of this condition is the following: since $Ax \geqslant c$, condition (1) is equivalent to

$$\prod_{i \in \phi_j} (Ax - c)_i = 0 \ , \quad \forall j \in \{1, 2, \ldots, m\} \ . \qquad (2)$$

So we could say that each set ϕ_j corresponds to a subgroup of inequalities of $Ax \geqslant c$ and that in each group at least one inequality should hold with equality:

$$\forall j \in \{1, 2, \ldots, m\} \ : \ \exists i \in \phi_j \text{ such that } (Ax - c)_i = 0 \ .$$

We'll use this interpretation in Section 3 to demonstrate that a set of multivariate polynomial equations in the max algebra can be transformed into an ELCP.

In [7] we have made a thorough study of the solution set of the ELCP and developed an algorithm to find all its solutions. We shall now state the main results of that paper.

The ELCP algorithm results in 3 sets of rays $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{inf}}$, $\mathcal{X}^{\text{fin}}$ and a set Λ of pairs $\left\{\mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}}\right\}$ where $\mathcal{X}_s^{\text{inf}}$ is a subset of $\mathcal{X}^{\text{inf}}$ and $\mathcal{X}_s^{\text{fin}}$ is a non-empty subset of $\mathcal{X}^{\text{fin}}$. The solution set of the ELCP is then characterized by the following theorem:

Theorem 2.1 *When $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{inf}}$, $\mathcal{X}^{\text{fin}}$ and Λ are given, then x is a solution of the ELCP if and only if there exists a pair $\left\{\mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}}\right\} \in \Lambda$ such that*

$$x = \sum_{x_k \in \mathcal{X}^{\text{cen}}} \lambda_k x_k + \sum_{x_k \in \mathcal{X}_s^{\text{inf}}} \kappa_k x_k + \sum_{x_k \in \mathcal{X}_s^{\text{fin}}} \mu_k x_k \ ,$$

with $\lambda_k \in \mathbb{R}$, $\kappa_k, \mu_k \geqslant 0$ and $\sum_k \mu_k = 1$.

As a result we have that:

Corollary 2.2 *The general solution set of an ELCP consists of the union of faces of a polyhedron.*

3 Multivariate polynomial equations in the max algebra

Consider the following problem:

Given a set of integers $\{m_k\}$ and three sets of coefficients $\{a_{ki}\}$, $\{b_k\}$ and $\{c_{kij}\}$ with $i \in \{1, \ldots, m_k\}$, $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, p\}$, find a vector $x \in \mathbb{R}^n_{\max}$ that satisfies

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} = b_k \ , \qquad \text{for } k = 1, 2, \ldots, p \ , \tag{3}$$

or show that no such vector x exists.

Now we demonstrate that this problem can be transformed into an ELCP:
First we consider one equation of the form (3):

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} = b_k \ .$$

In linear algebra this is equivalent to the set of linear inequalities

$$a_{ki} + c_{ki1}x_1 + c_{ki2}x_2 + \ldots + c_{kin}x_n \leqslant b_k \ , \qquad \text{for } i = 1, 2, \ldots, m_k \ ,$$

where at least one inequality should hold with equality. If we transfer the a_{ki}'s to the right hand side and if we define $d_{ki} = b_k - a_{ki}$, we get the following set of a linear inequalities:

$$c_{ki1}x_1 + c_{ki2}x_2 + \ldots + c_{kin}x_n \leqslant d_{ki} \ , \qquad \text{for } i = 1, 2, \ldots, m_k \ .$$

If we define p matrices C_k and p column vectors d_k such that $(C_k)_{ij} = c_{kij}$ and $(d_k)_i = d_{ki}$, then (3) leads to p groups of linear inequalities $C_k\, x \leqslant d_k$ with in each group at least one inequality that should hold with equality.

We put all C_k's in one large matrix $A = \begin{bmatrix} -C_1 \\ -C_2 \\ \vdots \\ -C_p \end{bmatrix}$ and all d_k's in one vector $c = \begin{bmatrix} -d_1 \\ -d_2 \\ \vdots \\ -d_p \end{bmatrix}$.

We also define p sets ϕ_j such that $\phi_j = \{s_j + 1, s_j + 2, \ldots, s_j + m_j\}$, for $j = 1, 2, \ldots, p$, where $s_1 = 0$ and $s_{j+1} = s_j + m_j$ for $j = 1, 2, \ldots, p - 1$. Our original problem (3) is then equivalent to the following ELCP:

Find a vector $x \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{p} \prod_{i \in \phi_j} (Ax - c)_i = 0 \tag{4}$$

subject to $Ax \;\geqslant\; c$,
or show that no such vector x exists.

This means that we can use the ELCP algorithm of [7] to find all solutions of problem (3).

For other applications of the ELCP in the max algebra and in the max/min/plus algebra the interested reader is referred to [8].

4 Minimal state space realization

4.1 Realization and minimal realization

Suppose that we have a single input single output (SISO) discrete event system that can be described by an n-th order state space model:

$$x[k+1] \;=\; A \otimes x[k] \;\oplus\; B \otimes u[k] \tag{5}$$

$$y[k] \;=\; C \otimes x[k] \tag{6}$$

with $A \in \mathbb{R}_{\max}^{n \times n}$, $B \in \mathbb{R}_{\max}^{n \times 1}$ and $C \in \mathbb{R}_{\max}^{1 \times n}$. u is the input, y is the output and x is the state vector.

We define the unit impulse e as: $e[k] \;=\; 0$ if $k = 0$,
$\;=\; \varepsilon$ otherwise .

If we apply a unit impulse to the system and if we assume that the initial state $x[0]$ satisfies $x[0] = \varepsilon$ or $A \otimes x[0] \leqslant B$, we get the impulse response as the output of the system:

$$x[1] = B\,,\;\; x[2] = A \otimes B\,,\;\; \ldots\,,\;\; x[k] = A^{\otimes^{k-1}} \otimes B \quad \Rightarrow \quad y[k] = C \otimes A^{\otimes^{k-1}} \otimes B\;\; . \tag{7}$$

Let $g_k = C \otimes A^{\otimes^k} \otimes B$. The g_k's are called the *Markov parameters*.

Let us now reverse the process: suppose that A, B and C are unknown, and that we only know the Markov parameters (e.g. from experiments – where we assume that the system is max-linear and time-invariant and that there is no noise present). How can we construct A, B and C from the g_k's? This process is called realization. If we make the dimension of A minimal, we have a minimal realization. Although there have been some attempts to solve this problem [4, 11, 13] , this is – to the authors' knowledge – the first time it is solved entirely. It is certainly the first time that a complete description of the set of all minimal realizations of a SISO max-linear discrete event system is given.

4.2 A lower bound for the minimal system order

In this section we present a method to find a lower bound for the minimal system order. Although we already more or less presented this technique in [6], we now explicitly prove that we indeed find a lower bound for the system order. We use the following property:

Property 4.1 *Consider $A \in \mathbb{S}_{\max}^{n\times n}$, $B \in \mathbb{S}_{\max}^{n\times 1}$ and $C \in \mathbb{S}_{\max}^{1\times n}$. If A satisfies an equation of the form*

$$\bigoplus_{p=0}^{n} a_p \otimes A^{\otimes^{n-p}} \nabla \varepsilon \qquad (8)$$

(e.g. its characteristic equation) then the Markov parameters satisfy

$$\bigoplus_{p=0}^{n} a_p \otimes g_{k+n-p} \nabla \varepsilon \quad \text{for } k = 0\,, 1,\ 2,\ \ldots\ .$$

Proof: Left multiplication of (8) by $C \otimes A^{\otimes^k}$ and right multiplication by B leads to $\bigoplus_{p=0}^{n} a_p \otimes C \otimes A^{\otimes^{k+n-p}} \otimes B \nabla \varepsilon$. Since $g_k = C \otimes A^{\otimes^k} \otimes B$ we find that $\bigoplus_{p=0}^{n} a_p \otimes g_{k+n-p} \nabla \varepsilon$.

Suppose that we have a system that can be described by (5) and (6), with unknown system matrices. If we want to find a minimal realization of this system the first question that has to be answered is that of the minimal system order.

Consider the semi-infinite Hankel matrix $H = \begin{bmatrix} g_0 & g_1 & g_2 & \cdots \\ g_1 & g_2 & g_3 & \cdots \\ g_2 & g_3 & g_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$.

As a direct consequence of Theorem 1.7 and Property 4.1 we have that the columns of H satisfy

$$\bigoplus_{p=0}^{n} a_p \otimes H(:, k+n-p) \nabla \varepsilon \qquad \text{for } k = 1,\ 2,\ \ldots \qquad (9)$$

where the coefficients a_p are the coefficients of the characteristic equation of the system matrix A. This leads to

Property 4.2 *Let $H_{\text{sub},s} = H([i_1, i_2, \ldots, i_s], [j+1, j+2, \ldots, j+s])$ be an s by s square submatrix of the Hankel matrix H with arbitrary row indices and consecutive column indices. If $s > n$ then we have that $\det(H_{\text{sub},s}) \nabla \varepsilon$.*

Proof: If A is an n by n matrix with elements in $\mathbb{R}_{\max}$ then according to Olsder's variant of the Cayley-Hamilton theorem [12], the coefficients in the characteristic equation of A are signed. This also means that the coefficients a_p in (9) are signed or that every balance of the form:

$$H([i_1, i_2, \ldots, i_s], [j+1, j+2, \ldots, j+s]) \otimes a \nabla \varepsilon$$

with $s > n, j \geqslant 0$ and $\{i_1, i_2, \ldots, i_s\} \in \mathcal{C}_\infty^s$ has a signed solution: if $s = n+1$ we get the coefficients of the characteristic equation as a solution and for $s > n+1$ we can always set some of the components of a equal to ε. Theorem 1.5 then leads to

$$\det\left(H([i_1, i_2, \ldots, i_s], [j+1, j+2, \ldots, j+s])\right) \nabla \varepsilon$$

for $s > n$.

So the dimension of the largest square submatrix of H with consecutive column indices that has a non-balanced determinant will be less than or equal to n. We represent this dimension as $\mathrm{r_{cc}}(H)$.

Definition 4.3 (Consecutive column rank) *Consider $P \in \mathbb{S}_{\max}^{m \times n}$. The consecutive column rank of P, $\mathrm{r_{cc}}(P)$, is the dimension of the largest square submatrix of P with consecutive column indices, the determinant of which is not balanced:*

$$\mathrm{r_{cc}}(P) \;=\; \max \{ \dim(P_{\mathrm{sub},s}) \,|\, P_{\mathrm{sub},s} = P([i_1, i_2, \ldots, i_s], [j+1, j+2, \ldots, j+s]) \text{ with } \\ 0 \leqslant s \leqslant \min(m,n),\, 0 \leqslant j \leqslant n-s,\, \{i_1, i_2, \ldots, i_s\} \in \mathcal{C}_m^s \text{ and } \det(P_{\mathrm{sub},s}) \,\not\nabla\, \varepsilon \} \;.$$

We can define the consecutive row rank of P, $\mathrm{r_{cr}}(P)$, in an analogous way (in general $\mathrm{r_{cc}}(P) \neq \mathrm{r_{cr}}(P)$). But since we only consider symmetric matrices in this section, we only need the consecutive column rank: if $H = H^t$ then $\mathrm{r_{cc}}(P) = \mathrm{r_{cr}}(P)$. We have that $\mathrm{r_{cc}}(P) \leqslant \mathrm{r_{det}}(P)$.

To find a lower bound r for the minimal system order we shall search for a relation of the form (9) among the columns of H with a minimal number of terms. This number of terms will be a first estimate for the lower bound r. Since we know that the elements of the system matrix A belong to $\mathbb{R}_{\max}$ we shall search for coefficients a_p that correspond to a matrix with elements in $\mathbb{R}_{\max}$. See [6] for necessary (and sufficient) conditions for these coefficients.
This leads to the following procedure:
First we construct a p by q Hankel matrix

$$H_{p,q} = \begin{bmatrix} g_0 & g_1 & g_2 & \cdots & g_{q-1} \\ g_1 & g_2 & g_3 & \cdots & g_q \\ g_2 & g_3 & g_4 & \cdots & g_{q+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{p-1} & g_p & g_{p+1} & \cdots & g_{p+q-2} \end{bmatrix}$$

with p and q large enough: $p, q \gg n$, where n is the real (but unknown) system order. Now we try to find n and $a_0, a_1, \ldots, a_n$ such that the columns of $H_{p,q}$ satisfy an equation of the form (9).
We start with r equal to $\mathrm{r_{cc}}(H_{p,q})$. Let

$$H_{\mathrm{sub},r} = H_{p,q}([i_1, i_2, \ldots, i_r], [j+1, j+2, \ldots, j+r])$$

be an r by r submatrix of $H_{p,q}$ the determinant of which is not balanced: $\det H_{\mathrm{sub},r} \,\not\nabla\, \varepsilon$. If we add one arbitrary row and the $(j+r+1)$-st column to $H_{\mathrm{sub},r}$ we get an $r+1$ by $r+1$ matrix $H_{\mathrm{sub},r+1}$ that has a balanced determinant. So according to Theorem 1.5 the set of linear balances

$$H_{\mathrm{sub},r+1} \otimes a \;\nabla\; \varepsilon$$

has a signed solution $a = \begin{bmatrix} a_r & a_{r-1} & \ldots & a_0 \end{bmatrix}^t$. We now look for a solution a that corresponds to the characteristic equation of a matrix with elements in $\mathbb{R}_{\max}$ (this should not necessarily be a signed solution; a signed solution would correspond to Olsder's variant of the characteristic equation). First of all we normalize a_0 to 0 and then we check if the necessary (and sufficient) conditions for the coefficients of the characteristic equation of a matrix with elements in $\mathbb{R}_{\max}$ (see [6]) are satisfied. If they are not satisfied we augment r and repeat the

procedure.
We continue until we get the following stable relation among the columns of $H_{p,q}$:

$$H_{p,q}(:,k+r) \;\oplus\; a_1 \otimes H_{p,q}(:,k+r-1) \;\oplus\; \ldots \;\oplus\; a_r \otimes H_{p,q}(:,k) \;\nabla\; \varepsilon \tag{10}$$

for $k \in \{1,\ldots,q-r\}$. Since we assume that the system can be described by (5) and (6) and that $p,\, q \gg n$, we can always find such a stable relationship by gradually augmenting r. The r that results from this procedure is indeed a lower bound for the minimal system order, since it corresponds to the smallest number of terms in a relationship of form (9) among the columns of $H_{p,q}$.

4.3 Determination of the system matrices

Now we have to find $A \in \mathbb{R}_{\max}^{r\times r}$, $B \in \mathbb{R}_{\max}^{r\times 1}$ and $C \in \mathbb{R}_{\max}^{1\times r}$ such that

$$C \otimes A^{\otimes^k} \otimes B = g_k\,, \qquad \text{for } k = 0,1,2,\ldots \;. \tag{11}$$

In practice it seems that we only have to take the transient behavior and the first cycles of this steady-state behavior into account. So we may limit ourselves to the first, say, N Markov parameters.
For $k = 0$ we get

$$\bigoplus_{i=1}^{r} c_i \otimes b_i = g_0 \;\; .$$

For $k > 0$ we have that (11) is equivalent to

$$\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{r} t_{kij} = g_k \;\; ,$$

with

$$t_{kij} = \bigoplus_{i_1=1}^{r} \ldots \bigoplus_{i_{k-1}=1}^{r} c_i \otimes a_{ii_1} \otimes a_{i_1 i_2} \otimes \ldots \otimes a_{i_{k-1}j} \otimes b_j \;\; .$$

This can be rewritten as

$$\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{r} \bigoplus_{l=1}^{r^{k-1}} c_i \otimes \bigotimes_{u=1}^{r} \bigotimes_{v=1}^{r} {a_{uv}}^{\otimes^{\gamma_{kijluv}}} \otimes b_j = g_k \;\; ,$$

where γ_{kijluv} is the number of times that a_{uv} appears in the l-th subterm of term t_{kij}. If a_{uv} doesn't appear in that subterm we take $\gamma_{kijluv} = 0$ since we have that $a^{\otimes^0} = 0.a = 0$, the identity element for $\otimes$. At first sight one could think that we are then left with r^{k+1} terms. However, some of these are the same and can thus be left out. If we use the fact that $\forall x, y \in \mathbb{R}_{\max} : x \otimes y \leqslant x \otimes x \oplus y \otimes y$ we can again remove many redundant terms. Then we are left with, say, w_k terms where $w_k \leqslant r^{k+1}$.
If we put all unknowns in one large vector x of size $r(r+2)$ we have to solve a set of multivariate polynomial equations of the following form:

$$\bigoplus_{i=1}^{r} \bigotimes_{j=1}^{r(r+2)} x_j{}^{\otimes^{\kappa_{0ij}}} = g_0$$

$$\bigoplus_{i=1}^{w_k} \bigotimes_{j=1}^{r(r+2)} x_j{}^{\otimes^{\kappa_{kij}}} = g_k \,, \qquad \text{for } k = 1\,,2\,,\ldots, N-1 \ ,$$

and this can be transformed into an ELCP using the technique explained in Section 3. This means that in general all equivalent minimal state space realizations of a max-linear SISO system form a union of polyhedra in the x-space.
If we find a solution x we extract the elements of x and put them in the matrices $A\,, B$ and C. Then we have found a minimal realization. If we don't find a solution we have to augment r and start again. Since we assumed that the data were generated by a max-linear SISO system we shall eventually find a realization and it will be minimal.

Remark 4.4 By transforming the problem to linear algebra we have assumed that all components of A, B and C are finite. If we also want to include matrices with components equal to ε we have to take certain precautions. Normally they can be obtained by allowing some of the λ_k's or κ_k's to become infinite in a controlled way, since we only allow infinite components that are equal to $\varepsilon = -\infty$; components equal to ∞ are not allowed.
Since the max operation hides small numbers from larger numbers it suffices in practice to replace negative elements that are large enough in absolute value by ε provided that there are no positive elements of the same order of magnitude. This technique will be demonstrated in Example 5.1.

4.4 Computational complexity and algorithmic aspects

The execution time and the storage space requirement of the ELCP algorithm depend on the number of equations and variables. For the minimal realization problem the number of equations and variables becomes very large as the system order rises or as the number of Markov parameters that should be considered grows. Therefore the ELCP algorithm in its present form is not suited for large systems or for systems with a long and complex transient behavior.
Moreover, we are not always interested in finding all minimal realizations. In [6] we have developed a heuristic algorithm that is relatively fast and that will in most cases find a minimal realization.

Since the method to solve the ELCP is an iterative process where in each step a new equation is taken into account, we can make use of the special structure of our problem to speed up the algorithm. To each Markov parameter there corresponds a group of linear inequalities. After each group we can test whether the impulse response of the solution up to that group matches the desired impulse response. If this is the case we don't have to take the other groups into account, since they will automatically be satisfied. This means that we can start with a small N and gradually take more and more groups into account. We don't have to start all over again for each new group since we can simply continue with the rays of the previous groups.

There are still some open problems. It is e.g. not clear how to determine the minimal subset of Markov parameters that is needed and how to select them. In Example 5.1 of the next section the equations for the subset $\{g_0, g_1, g_6, g_7, g_8\}$ will lead to the same (right) solutions as $\{g_0, g_1, \ldots, g_8\}$ whereas the subset $\{g_0, g_1, \ldots, g_7\}$ yields some solutions with an impulse response that doesn't coincide entirely with the desired impulse response (only the first 8 Markov parameters are exactly the same). Since we have one group of inequalities for each Markov parameter that we take into consideration and since the computational complexity grows with the number of inequalities, it is important to use as few Markov parameters as possible. The example above shows that it is not necessary to consider the entire set $\{g_0, g_1, \ldots, g_{N-1}\}$ to find all solutions with the desired impulse response.

5 Examples

We now illustrate the procedure to find all minimal realizations with a few examples.

Example 5.1

Here we reconsider the example of [4, 13]. We start from a system with system matrices

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 0 \\ -3 & 1 & 2 \end{bmatrix} , \quad B = \begin{bmatrix} 0 \\ \varepsilon \\ \varepsilon \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & \varepsilon & \varepsilon \end{bmatrix} \ .$$

Now we are going to construct the system matrices from the impulse response of the system. This impulse response is given by

$$\{g_k\} = 0\,,1\,,2\,,3\,,4\,,5\,,6\,,8\,,10\,,12\,,14\,,\ldots \ .$$

First we construct the Hankel matrix

$$H_{8,8} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 8 & 10 \\ 2 & 3 & 4 & 5 & 6 & 8 & 10 & 12 \\ 3 & 4 & 5 & 6 & 8 & 10 & 12 & 14 \\ 4 & 5 & 6 & 8 & 10 & 12 & 14 & 16 \\ 5 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\ 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 \\ 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 \end{bmatrix} .$$

The consecutive column rank of $H_{8,8}$ is 2. The determinant of $H_{\text{sub},2} = H_{8,8}([1,7],[1,2]) = \begin{bmatrix} 0 & 1 \\ 6 & 8 \end{bmatrix}$ is not balanced. We add one row and one column and then we look for a solution of the set of linear balances

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 6 & 8 & 10 \end{bmatrix} \otimes \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} \nabla \, \varepsilon \ .$$

The solution $a_0 = 0$, $a_1 = \ominus 2$, $a_2 = 3$ satisfies the necessary and sufficient conditions for the coefficients of the characteristic polynomial of a 2 by 2 matrix with elements in $\mathbb{R}_{\max}$ (see [6]) since $\alpha_1 = \varepsilon$ and $\alpha_2 = 3 \leqslant 4 = 2 \otimes 2 = \beta_1 \otimes \beta_1$. This solution also corresponds to a stable relation among the columns of $H_{8,8}$:

$$H_{8,8}(:,k+2) \ \oplus \ 3 \otimes H_{8,8}(:,k) = 2 \otimes H_{8,8}(:,k+1) \qquad \text{for } k \in \{1,2,\ldots,6\} \ .$$

Set	$\mathcal{X}^{\text{cen}}$		$\mathcal{X}^{\text{inf}}$						$\mathcal{X}^{\text{fin}}$	
Ray	x_1^{c}	x_2^{c}	x_1^{i}	x_2^{i}	x_3^{i}	x_4^{i}	x_5^{i}	x_6^{i}	x_1^{f}	x_2^{f}
a_{11}	0	0	0	0	0	0	0	0	2	1
a_{12}	0	1	0	0	0	0	0	0	0	0
a_{21}	0	-1	0	0	-1	-1	0	0	-2	-2
a_{22}	0	0	0	0	0	0	0	0	1	2
b_1	1	1	-1	0	0	1	-1	0	-2	-2
b_2	1	0	-1	-1	0	0	0	0	0	-6
c_1	-1	-1	1	0	0	-1	0	-1	-4	2
c_2	-1	0	0	0	0	0	0	0	0	0

Table 1: The rays for Example 5.1 .

s	$\mathcal{X}_s^{\text{inf}}$	$\mathcal{X}_s^{\text{fin}}$
1	$\{x_1^{\text{i}}, x_2^{\text{i}}\}$	$\{x_2^{\text{f}}\}$
2	$\{x_1^{\text{i}}, x_3^{\text{i}}\}$	$\{x_2^{\text{f}}\}$
3	$\{x_2^{\text{i}}, x_4^{\text{i}}\}$	$\{x_2^{\text{f}}\}$
4	$\{x_3^{\text{i}}, x_4^{\text{i}}\}$	$\{x_2^{\text{f}}\}$
5	$\{x_3^{\text{i}}, x_4^{\text{i}}\}$	$\{x_1^{\text{f}}\}$
6	$\{x_3^{\text{i}}, x_5^{\text{i}}\}$	$\{x_1^{\text{f}}\}$
7	$\{x_4^{\text{i}}, x_6^{\text{i}}\}$	$\{x_1^{\text{f}}\}$
8	$\{x_5^{\text{i}}, x_6^{\text{i}}\}$	$\{x_1^{\text{f}}\}$

Table 2: The pairs of subsets for Example 5.1 .

Let's take $N = 9$. Using the ELCP algorithm of [7] we find the rays of Table 1 and the pairs of subsets of Table 2. If we take $N > 9$ we get the same result, but if we take $N < 9$ some combinations of the rays lead to a partial realization of the given impulse response (i.e. they only fit the first N Markov parameters).

Any arbitrary minimal realization can now be expressed as

$$\begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{bmatrix} = \lambda_1 x_1^{\text{c}} + \lambda_2 x_2^{\text{c}} + \kappa_1 x_{i_1}^{\text{i}} + \kappa_2 x_{i_2}^{\text{i}} + x_{j_1}^{\text{f}} \ , \tag{12}$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$, $\kappa_1, \kappa_2 \geqslant 0$ and $x^{\mathrm{i}}_{i_1}, x^{\mathrm{i}}_{i_2} \in \mathcal{X}^{\mathrm{inf}}_s$, $x^{\mathrm{f}}_{j_1} \in \mathcal{X}^{\mathrm{fin}}_s$ for some $s \in \{1, 2, \ldots, 8\}$. Expression (12) shows that the set of all equivalent minimal state space realizations of the given impulse response is a union of 8 unbounded polyhedra.
The result of [13]

$$A = \begin{bmatrix} 1 & 0 \\ \varepsilon & 2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ -4 \end{bmatrix}, \; C = \begin{bmatrix} 0 & \varepsilon \end{bmatrix} \tag{13}$$

corresponds to the following combination of the central solutions and the solutions of the pair $\{\mathcal{X}^{\mathrm{inf}}_2, \mathcal{X}^{\mathrm{fin}}_2\}$:

$$(\eta + 2)x^{\mathrm{c}}_1 + \eta x^{\mathrm{i}}_1 + \eta x^{\mathrm{i}}_3 + x^{\mathrm{f}}_2$$

for η large enough. Then we get

$$A = \begin{bmatrix} 1 & 0 \\ -(\eta+2) & 2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ -4 \end{bmatrix}, \; C = \begin{bmatrix} 0 & -(\eta+2) \end{bmatrix}$$

and as explained in Remark 4.4 we can replace $-(\eta + 2)$ by $\varepsilon = -\infty$ for η large enough since there are no positive components of the same order of magnitude as η. In fact for $\eta \to +\infty$ we would exactly get solution (13).

We now give another example that doesn't satisfy the assumptions of [13], where only impulse responses that exhibit a uniformly up-terrace behavior are considered, i.e. impulse responses that consist of m sequences of length n_i such that

$$g_{j+1} - g_j = c_i \, , \qquad \text{for } j = t_i,\, t_i + 1,\, \ldots,\, t_i + n_i - 1 \;\text{ and for } i = 1,\, 2,\, \ldots,\, m \;\; ,$$

with $c_{i+1} > c_i$, $t_1 = 0$, $t_{i+1} = t_i + n_i$ and $n_m = +\infty$.

Example 5.2

We start from the system (A, B, C) with

$$A = \begin{bmatrix} 3 & 1 & 0 \\ \varepsilon & 3 & 2 \\ 0 & 5 & \varepsilon \end{bmatrix} \; , \; B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \; \text{ and } \; C = \begin{bmatrix} 0 & \varepsilon & \varepsilon \end{bmatrix} \; . \tag{14}$$

The impulse response of this system is: $0\,,3\,,6\,,9\,,13\,,16\,,20\,,23\,,27\,,\ldots$. Since there are two different alternating increments in steady state (3 and 4), we can't use the technique of [13]. First we construct the Hankel matrix

$$H_{8,8} = \begin{bmatrix} 0 & 3 & 6 & 9 & 13 & 16 & 20 & 23 \\ 3 & 6 & 9 & 13 & 16 & 20 & 23 & 27 \\ 6 & 9 & 13 & 16 & 20 & 23 & 27 & 30 \\ 9 & 13 & 16 & 20 & 23 & 27 & 30 & 34 \\ 13 & 16 & 20 & 23 & 27 & 30 & 34 & 37 \\ 16 & 20 & 23 & 27 & 30 & 34 & 37 & 41 \\ 20 & 23 & 27 & 30 & 34 & 37 & 41 & 44 \\ 23 & 27 & 30 & 34 & 37 & 41 & 44 & 48 \end{bmatrix}$$

which has consecutive column rank 3. A 3 by 3 submatrix of $H_{8,8}$ the determinant of which is not balanced is $H_{\text{sub},3} = H_{8,8}([1,3,4],[1,2,3]) = \begin{bmatrix} 0 & 3 & 6 \\ 6 & 9 & 13 \\ 9 & 13 & 16 \end{bmatrix}$.

The set of linear balances

$$\begin{bmatrix} 0 & 3 & 6 & 9 \\ 3 & 6 & 9 & 13 \\ 6 & 9 & 13 & 16 \\ 9 & 13 & 16 & 20 \end{bmatrix} \otimes \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} \nabla \, \varepsilon$$

has a solution

$$a_0 = 0,\ a_1 = \ominus 3,\ a_2 = \ominus 7,\ a_3 = 10 \ , \tag{15}$$

that satisfies the necessary and sufficient conditions of [6] :

$$\begin{cases} \alpha_1 = \varepsilon \\ \alpha_2 = \varepsilon \leqslant 6 = 3 \otimes 3 = \beta_1 \otimes \beta_1 \\ \alpha_3 = 10 \leqslant 10 = 3 \otimes 7 = \beta_1 \otimes \beta_2 \ . \end{cases}$$

This solution also corresponds to a stable relation among the columns of $H_{8,8}$:

$$H_{8,8}(:,k+3) \ \oplus \ 10 \otimes H_{8,8}(:,k) = 3 \otimes H_{8,8}(:,k+2) \ \oplus \ 7 \otimes H_{8,8}(:,k+1)$$

for $k \in \{1,2,\ldots,5\}$.
If we take $N = 7$ the ELCP algorithm yields the rays of Table 3 and the pairs of Table 4. So the set of all equivalent minimal realizations of system (14) consists of the union of 54 unbounded polyhedra.
If we take $N > 7$ we get the same results but for $N < 7$ some solutions only yield a partial realization of the given impulse response (i.e. they only fit the first N Markov parameters).

The original matrices (14) can be found as combination of the central rays and the rays of the pair $\{\mathcal{X}_{12}^{\text{inf}}, \mathcal{X}_{12}^{\text{fin}}\}$:

$$(\eta+4)x_1^{\text{c}} + 5x_2^{\text{c}} + \eta x_1^{\text{i}} + (\eta+7)x_3^{\text{i}} + 6x_7^{\text{i}} + (\eta+8)x_{10}^{\text{i}} + (\eta+2)x_{14}^{\text{i}} + x_2^{\text{f}}$$

for η large enough.
In [5] we solved the same example by constructing a matrix A such that the coefficients of its characteristic equation were equal to (15). There we found

$$A = \begin{bmatrix} 3 & \varepsilon & \varepsilon \\ 0 & \varepsilon & 7 \\ \varepsilon & 0 & \varepsilon \end{bmatrix},\ B = \begin{bmatrix} 0 \\ -3 \\ \varepsilon \end{bmatrix},\ C = \begin{bmatrix} 0 & 2 & \varepsilon \end{bmatrix} \ ,$$

which corresponds to the pair $\{\mathcal{X}_8^{\text{inf}}, \mathcal{X}_8^{\text{fin}}\}$:

$$(\eta+7)x_1^{\text{c}} + (\eta+7)x_3^{\text{c}} + (\eta+12)x_1^{\text{i}} + \eta x_2^{\text{i}} + (\eta+10)x_3^{\text{i}} + (\eta+16)x_6^{\text{i}} + \\ (\eta+4)x_7^{\text{i}} + (\eta+10)x_8^{\text{i}} + (\eta+12)x_{11}^{\text{i}} + x_2^{\text{f}}$$

with η large enough.

6 Conclusions and further research

We have shown that a set of multivariate polynomial equations in the max algebra can be transformed into an Extended Linear Complementarity Problem (ELCP). This means that we can use the ELCP algorithm of [7] to solve such a problem. We have applied this technique to find all minimal state space realizations of a single input single output discrete event system given its Markov parameters and illustrated the procedure with a few examples.

One of the main characteristics of the ELCP algorithm that was used in this paper is that it finds all solutions. For the minimal realization problem this provides a geometrical insight in all equivalent (minimal) realizations of an impulse response. On the other hand this also leads to large computation times and storage space requirements if the number of variables and equations is large. Therefore it might be interesting to develop (heuristic) algorithms that only find one solution as we have done for the minimal realization problem in [6].
Among the set of all possible realizations we could also try to find privileged realizations such as balanced or canonical realizations.
We hope to extend the method presented here to find minimal state space realizations for multiple input multiple output (MIMO) systems. The only problem there is the determination of the minimal system order. Once this is found the same technique can be used to get a minimal realization. In the future we shall therefore look for methods to get a estimate of the minimal system order of a MIMO system.

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Set	$\mathcal{X}^{\text{cen}}$			$\mathcal{X}^{\text{inf}}$								
Ray	x^{c}_1	x^{c}_2	x^{c}_3	x^{i}_1	x^{i}_2	x^{i}_3	x^{i}_4	x^{i}_5	x^{i}_6	x^{i}_7	x^{i}_8	x^{i}_9
a_{11}	0	0	0	0	0	0	0	0	0	0	0	-1
a_{12}	0	1	-1	0	0	0	1	0	1	0	0	0
a_{13}	0	0	-1	0	0	0	0	0	0	0	0	0
a_{21}	0	-1	1	0	0	0	-1	0	-1	0	0	0
a_{22}	0	0	0	0	0	0	0	0	0	0	-1	0
a_{23}	0	-1	0	0	0	0	-1	-1	0	0	0	0
a_{31}	0	0	1	0	0	0	0	0	-1	-1	0	0
a_{32}	0	1	0	0	0	0	0	0	0	0	0	0
a_{33}	0	0	0	0	0	-1	0	0	0	0	0	0
b_1	1	0	-1	-1	0	0	0	0	1	0	0	0
b_2	1	-1	0	-1	0	0	-1	0	0	0	0	0
b_3	1	0	0	-1	-1	0	0	0	0	0	0	0
c_1	-1	0	1	1	0	0	0	0	-1	0	0	0
c_2	-1	1	0	1	0	0	1	0	0	0	0	0
c_3	-1	0	0	0	0	0	0	0	0	0	0	0

Set	$\mathcal{X}^{\text{inf}}$						$\mathcal{X}^{\text{fin}}$					
Ray	x^{i}_{10}	x^{i}_{11}	x^{i}_{12}	x^{i}_{13}	x^{i}_{14}	x^{i}_{15}	x^{f}_1	x^{f}_2	x^{f}_3	x^{f}_4	x^{f}_5	x^{f}_6
a_{11}	0	0	0	0	0	0	3	3	3	3	3	3
a_{12}	0	-1	0	0	0	0	-4	-4	-2	-3	-2	-3
a_{13}	0	0	0	0	0	0	0	0	0	0	0	0
a_{21}	-1	0	0	0	0	0	9	9	9	9	9	9
a_{22}	0	0	0	0	0	0	3	3	3	3	3	3
a_{23}	0	0	0	0	0	0	6	7	5	5	6	7
a_{31}	0	0	0	0	0	0	7	6	6	7	5	5
a_{32}	0	0	0	0	0	0	0	0	0	0	0	0
a_{33}	0	0	0	0	0	0	3	3	3	3	3	3
b_1	0	0	0	-1	0	0	-5	-4	-4	-5	-3	-3
b_2	0	0	-1	0	0	0	2	2	2	2	2	2
b_3	0	0	0	0	0	0	-2	-2	0	-1	0	-1
c_1	0	0	0	0	0	-1	4	4	2	3	2	3
c_2	0	0	0	0	-1	0	-2	-3	-3	-2	-4	-4
c_3	0	0	0	0	0	0	0	0	0	0	0	0

Table 3: The rays for Example 5.2 .

s	$\mathcal{X}_s^{\mathrm{inf}}$	$\mathcal{X}_s^{\mathrm{fin}}$
1	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{10}\}$	$\{x^{\mathrm{f}}_1\}$
2	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{11}\}$	$\{x^{\mathrm{f}}_1\}$
3	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_4\}$
4	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}\}$	$\{x^{\mathrm{f}}_1\}$
5	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_4, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_1\}$
6	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_5, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{13}\}$	$\{x^{\mathrm{f}}_1\}$
7	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{10}\}$	$\{x^{\mathrm{f}}_2\}$
8	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{11}\}$	$\{x^{\mathrm{f}}_2\}$
9	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_6\}$
10	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}\}$	$\{x^{\mathrm{f}}_2\}$
11	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_6, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_2\}$
12	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_2\}$
13	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_2\}$
14	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_1\}$
15	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_2, x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}\}$	$\{x^{\mathrm{f}}_1\}$
16	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_2, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}\}$	$\{x^{\mathrm{f}}_2\}$
17	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_3, x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}\}$	$\{x^{\mathrm{f}}_1\}$
18	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_3, x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_4\}$
19	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_3, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}\}$	$\{x^{\mathrm{f}}_2\}$
20	$\{x^{\mathrm{i}}_1, x^{\mathrm{i}}_3, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_6\}$
21	$\{x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}\}$	$\{x^{\mathrm{f}}_1\}$
22	$\{x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_4\}$
23	$\{x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}\}$	$\{x^{\mathrm{f}}_2\}$
24	$\{x^{\mathrm{i}}_2, x^{\mathrm{i}}_3, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_6\}$
25	$\{x^{\mathrm{i}}_3, x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{13}\}$	$\{x^{\mathrm{f}}_4\}$
26	$\{x^{\mathrm{i}}_3, x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_4\}$
27	$\{x^{\mathrm{i}}_3, x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_4\}$
28	$\{x^{\mathrm{i}}_3, x^{\mathrm{i}}_4, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_4\}$
29	$\{x^{\mathrm{i}}_3, x^{\mathrm{i}}_5, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_4\}$
30	$\{x^{\mathrm{i}}_3, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_6\}$
31	$\{x^{\mathrm{i}}_3, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_6\}$
32	$\{x^{\mathrm{i}}_3, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_6\}$
33	$\{x^{\mathrm{i}}_3, x^{\mathrm{i}}_6, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_6\}$
34	$\{x^{\mathrm{i}}_3, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_6\}$
35	$\{x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_5\}$
36	$\{x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{13}\}$	$\{x^{\mathrm{f}}_3\}$
37	$\{x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_5\}$
38	$\{x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_3\}$
39	$\{x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_5\}$
40	$\{x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_3\}$
41	$\{x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_5\}$
42	$\{x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_3\}$
43	$\{x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_6, x^{\mathrm{i}}_8, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_5\}$
44	$\{x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_6, x^{\mathrm{i}}_8, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_3\}$
45	$\{x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_3\}$
46	$\{x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_5\}$
47	$\{x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_8, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_3\}$
48	$\{x^{\mathrm{i}}_4, x^{\mathrm{i}}_5, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_4\}$
49	$\{x^{\mathrm{i}}_4, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_5\}$
50	$\{x^{\mathrm{i}}_4, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_3\}$
51	$\{x^{\mathrm{i}}_5, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_5\}$
52	$\{x^{\mathrm{i}}_5, x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_3\}$
53	$\{x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_9, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_5\}$
54	$\{x^{\mathrm{i}}_6, x^{\mathrm{i}}_7, x^{\mathrm{i}}_8, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_6\}$

Table 4: The pairs of subsets for Example 5.2.