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Minimal Realization in the Max Algebra is an Extended Linear Complementarity Problem

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Abstract: We demonstrate that the minimal state space realization problem in the max algebra can be transformed into an Extended Linear Complementarity Problem (ELCP). We use an algorithm that finds all solutions of an ELCP to find all equivalent minimal state space realizations of a single input single output (SISO) discrete event system. We also give a geometrical description of the set of all minimal realizations of a SISO max-linear discrete event system.

Keywords: discrete event systems, max algebra, state space models, minimal realization, linear complementarity problem.

1 Introduction

1.1 Overview

In this paper we consider discrete event systems, such as flexible manufacturing systems, subway traffic networks, parallel processing systems, etc. . Some of these systems can be described using the so called max algebra [1]. We shall show that the minimal state space realization problem in the max algebra can be transformed into an Extended Linear Complementarity Problem (ELCP). The ELCP is an extension of the well-known Linear Complementarity Problem, which is one of the fundamental problems of mathematical programming. In [5] we have developed an algorithm to find all solutions of an ELCP. We shall use this algorithm to find all equivalent minimal state space realizations of a single input single output (SISO) discrete event system and to give a geometrical insight in the structure of the set of all equivalent state space realizations.

Although there have been some attempts to solve this minimal realization problem [3, 8, 9] , this is – to the authors' knowledge – the first time it is solved entirely. And it is certainly the first time that a complete description of the set of all minimal realizations of a SISO max-linear discrete event system is given.

1.2 The max algebra

One of the mathematical tools used in this paper is the max algebra. In this introduction we only explain the notations we use to represent the max-algebraic operations and give

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some definitions and theorems that will be used in the remainder of this paper. A complete introduction to the max algebra can be found in [1].

In this paper we use the following notations: $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$. The neutral element for $\oplus$ in $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}, \oplus, \otimes$ is $\epsilon = -\infty$. The max-algebraic power is defined as follows: $a^{\otimes^k} = \underbrace{a \otimes a \otimes \ldots \otimes a}_{k \text{ times}}$ and is equal to ka in linear algebra.

E_n is the n by n identity matrix in $\mathbb{R}_{\max}$: $e_{ij} = 0$ if $i = j$ and $e_{ij} = \epsilon$ if $i \neq j$. The operations $\oplus$ and $\otimes$ are extended to matrices in the usual way.

We also use the extension of the max algebra $\mathbb{S}_{\max}$ that was introduced in [1, 7]. We shall restrict ourselves to the most important features of $\mathbb{S}_{\max}$. There are three kinds of elements in $\mathbb{S}_{\max}$: the positive elements ($\mathbb{S}_{\max}^{\oplus}$, this corresponds to $\mathbb{R}_{\max}$), the negative elements ($\mathbb{S}_{\max}^{\ominus}$) and the balanced elements ($\mathbb{S}_{\max}^{\bullet}$). The positive and the negative elements are called signed ($\mathbb{S}_{\max}^{\vee} = \mathbb{S}_{\max}^{\oplus} \cup \mathbb{S}_{\max}^{\ominus}$). The $\ominus$ operation in $\mathbb{S}_{\max}$ is defined as follows:

$$
\begin{aligned}
a \ominus b &= a \quad && \text{if } a > b\,,\\
a \ominus b &= \ominus b && \text{if } a < b\,,\\
a \ominus a &= a^{\bullet}\,. &&
\end{aligned}
$$

This extension allows us to "solve" equations that have no solutions in $\mathbb{R}_{\max}$. Unfortunately we then have to introduce balances (∇) instead of equalities. Informally an $\ominus$ sign in a balance means that the element should be at the other side: so $3 \ominus 3 \ \nabla \ 2$ means $3 \ \nabla \ 2 \oplus 3$. If both sides of a balance are signed we can replace the balance by an equality.

1.3 Some notations, definitions and theorems

To select submatrices we use the following notation: $A([i_1, i_2, \ldots, i_k], [j_1, j_2, \ldots, j_l])$ is the k by l matrix resulting from A by eliminating all rows except for rows $i_1, \ldots, i_k$ and all columns except for columns $j_1, \ldots, j_l$. $A(:, j)$ is the j-th column of A. We represent the set of all possible combinations of k different numbers out of the set $\{1, 2, \ldots, n\}$ as $\mathcal{C}_n^k$.

Definition 1.1 (Determinant) *Consider a matrix $A \in \mathbb{S}_{\max}^{n \times n}$. The determinant of A is defined as* $\det A = \bigoplus_{\sigma \in \mathcal{P}_n} sgn(\sigma) \otimes \bigotimes_{i=1}^{n} a_{i\sigma(i)}$ *, where $\mathcal{P}_n$ is the set of all permutations of* $\{1, \ldots, n\}$ *, and $sgn(\sigma) = 0$ if the permutation σ is even and $sgn(\sigma) = \ominus 0$ if the permutation is odd.*

Theorem 1.2 *Let $A \in \mathbb{S}_{\max}^{n \times n}$. The homogeneous linear balance $A \otimes x \ \nabla \ \epsilon$ has a non-trivial signed solution if and only if* $\det A \ \nabla \ \epsilon$.

Proof: See [7]. The proof given there is constructive so it can be used to find a solution. □

Definition 1.3 (Characteristic equation) *Let $A \in \mathbb{S}_{\max}^{n \times n}$. The characteristic equation of A is defined as* $\det(A \ominus \lambda \otimes E_n) \ \nabla \ \epsilon$.

Theorem 1.4 (Cayley-Hamilton) *In $\mathbb{S}_{\max}$ every square matrix satisfies its characteristic equation.*

2 The Extended Linear Complementarity Problem

2.1 Problem formulation

Consider the following problem:

Given two matrices $A \in \mathbb{R}^{p \times n}, B \in \mathbb{R}^{q \times n}$, two column vectors $c \in \mathbb{R}^p, d \in \mathbb{R}^q$ and m subsets ϕ_j of $\{1, 2, \ldots, p\}$, find a vector $x \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i = 0 \tag{1}$$

subject to $Ax \geqslant c$ and $Bx = d$,
or show that no such vector exists.

In [5] we have demonstrated that this problem is an extension of the Linear Complementarity Problem [2]. Therefore we call it the Extended Linear Complementarity Problem (ELCP). Equation (1) represents the *complementarity condition.* Since $Ax \geqslant c$, this condition is equivalent to

$$\prod_{i \in \phi_j} (Ax - c)_i = 0 \ , \quad \forall j \in \{1, 2, \ldots, m\} \ . \tag{2}$$

So we could say that each set ϕ_j corresponds to a subgroup of inequalities of $Ax \geqslant c$ and that in each group at least one inequality should hold with equality:

$$\forall j \in \{1, 2, \ldots, m\} : \exists i \in \phi_j \text{ such that } (Ax - c)_i = 0 \ .$$

We shall use this interpretation in Section 3 to demonstrate that a set of multivariate polynomial equations in the max algebra can be transformed into an ELCP.

In [5] we have made a thorough study of the solution set of the ELCP and developed an algorithm to find all its solutions. We shall now state the main results of that paper.

The ELCP algorithm results in 3 sets of rays: $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{inf}}$, $\mathcal{X}^{\text{fin}}$ and a set Λ of pairs $\left\{\mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}}\right\}$ where $\mathcal{X}_s^{\text{inf}}$ is a subset of $\mathcal{X}^{\text{inf}}$ and $\mathcal{X}_s^{\text{fin}}$ is a non-empty subset of $\mathcal{X}^{\text{fin}}$. The solution set of the ELCP is then characterized by the following theorems:

Theorem 2.1 *When $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{inf}}$, $\mathcal{X}^{\text{fin}}$ and Λ are given, then x is a solution of the ELCP if and only if there exists a pair $\left\{\mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}}\right\} \in \Lambda$ such that*

$$x = \sum_{x_k \in \mathcal{X}^{\text{cen}}} \lambda_k x_k + \sum_{x_k \in \mathcal{X}_s^{\text{inf}}} \kappa_k x_k + \sum_{x_k \in \mathcal{X}_s^{\text{fin}}} \mu_k x_k \ ,$$

with $\lambda_k \in \mathbb{R},\ \kappa_k, \mu_k \geqslant 0$ and $\sum_k \mu_k = 1$.

Theorem 2.2 *The general solution set of an ELCP consists of the union of (bounded and unbounded) polyhedra.*

3 Multivariate polynomial equations in the max algebra

Consider the following problem:

Given a set of integers $\{m_k\}$ and three sets of coefficients $\{a_{ki}\}$, $\{b_k\}$ and $\{c_{kij}\}$ with $i \in \{1, \ldots, m_k\}$, $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, p\}$, find a vector $x \in \mathbb{R}^n_{\max}$ that satisfies

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kij}}} = b_k \,, \qquad \text{for } k = 1, 2, \ldots, p \ . \tag{3}$$

Now we demonstrate that this problem can be transformed into an ELCP:
First we consider one equation of the form (3) :

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kij}}} = b_k \ .$$

In linear algebra this is equivalent to the set of linear inequalities

$$a_{ki} + c_{ki1}x_1 + c_{ki2}x_2 + \ldots + c_{kin}x_n \leqslant b_k \,, \qquad \text{for } i = 1, 2, \ldots, m_k \ ,$$

where at least one inequality should hold with equality. If we transfer the a_{ki}'s to the right hand side and if we define $d_{ki} = b_k - a_{ki}$, we get the following set of a linear inequalities:

$$c_{ki1}x_1 + c_{ki2}x_2 + \ldots + c_{kin}x_n \leqslant d_{ki} \,, \qquad \text{for } i = 1, 2, \ldots, m_k \ .$$

If we define p matrices C_k and p column vectors d_k such that $(C_k)_{ij} = c_{kij}$ and $(d_k)_i = d_{ki}$, then (3) leads to p groups of linear inequalities $C_k\, x \leqslant d_k$ with in each group at least one inequality that should hold with equality.

We put all C_k's in one large matrix $A = \begin{bmatrix} -C_1 \\ -C_2 \\ \vdots \\ -C_p \end{bmatrix}$ and all d_k's in one vector $c = \begin{bmatrix} -d_1 \\ -d_2 \\ \vdots \\ -d_p \end{bmatrix}$.

We also define p sets ϕ_j such that $\phi_j = \{s_j + 1, s_j + 2, \ldots, s_j + m_j\}$, for $j = 1, 2, \ldots, p$, where $s_1 = 0$ and $s_{j+1} = s_j + m_j$ for $j = 1, 2, \ldots, p - 1$. Our original problem (3) is then equivalent to the following ELCP:

Find a vector $x \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{p} \prod_{i \in \phi_j} (Ax - c)_i = 0 \tag{4}$$

subject to $Ax \;\; \geqslant \;\; c$.

This means that we can use the ELCP algorithm of [5] to find all solutions of problem (3).

4 Minimal state space realization

4.1 Realization and minimal realization

Suppose that we have a SISO discrete event system that can be described by an n-th order state space model:

$$
\begin{aligned}
x[k+1] &= A \otimes x[k] \ \oplus \ B \otimes u[k] && (5)\\
y[k] &= C \otimes x[k] && (6)
\end{aligned}
$$

with $A \in \mathbb{R}_{\max}^{n \times n}$, $B \in \mathbb{R}_{\max}^{n \times 1}$ and $C \in \mathbb{R}_{\max}^{1 \times n}$. u is the input, y is the output and x is the state vector. We define the unit impulse e as: $e[0] = 0$ and $e[k] = \epsilon$ if $k \neq 0$.
If we apply a unit impulse to the system and if we assume that the initial state $x[0]$ satisfies $x[0] = \epsilon$ or $A \otimes x[0] \leqslant B$, we get the impulse response as the output of the system:

$$
x[1] = B \, , \ x[2] = A \otimes B \, , \ \ldots \, , \ x[k] = A^{\otimes^{k-1}} \otimes B \quad \Rightarrow \quad y[k] = C \otimes A^{\otimes^{k-1}} \otimes B \ . \qquad (7)
$$

Let $g_k = C \otimes A^{\otimes^k} \otimes B$. The g_k's are called the *Markov parameters.*

Let us now reverse the process: suppose that A, B and C are unknown, and that we only know the Markov parameters (e.g. from experiments – where we assume that the system is max-linear and time-invariant and that there is no noise present). How can we construct A, B and C from the g_k's? This process is called realization. If we make the dimension of A minimal, we have a minimal realization.

In the next subsections we shall use the results of the previous sections to give a complete description of the set of all minimal realizations of a SISO max-linear discrete event system.

4.2 A lower bound for the minimal system order

Property 4.1 *Consider $A \in \mathbb{S}_{\max}^{n \times n}$, $B \in \mathbb{S}_{\max}^{n \times 1}$ and $C \in \mathbb{S}_{\max}^{1 \times n}$. If a_p is the coefficient of $A^{\otimes^{n-p}}$ in the characteristic equation of A then the Markov parameters satisfy*

$$
\bigoplus_{p=0}^{n} a_p \otimes g_{k+n-p} \ \nabla \ \epsilon \quad \textit{for } k = 0 \, , 1, \, 2, \, \ldots \ .
$$

Suppose that we have a system that can be described by (5) – (6), with unknown system matrices. If we want to find a minimal realization of this system the first question that has to be answered is that of the minimal system order.

Consider the semi-infinite Hankel matrix $H = \begin{bmatrix} g_0 & g_1 & \cdots \\ g_1 & g_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$.

As a direct consequence of Property 4.1 we have that the columns of H satisfy

$$
\bigoplus_{p=0}^{n} a_p \otimes H(:, k+n-p) \ \nabla \ \epsilon \quad \text{ for } k = 1, \, 2, \, \ldots \ . \qquad (8)
$$

If we combine this with Property 1.2 we find:

Property 4.2 *Let $H_{\text{sub},s} = H([i_1, i_2, \ldots, i_s], [j+1, j+2, \ldots, j+s])$ be an s by s square submatrix of the Hankel matrix H with arbitrary row indices and consecutive column indices. If $s > n$ then we have that $\det(H_{\text{sub},s}) \nabla \epsilon$.*

So the dimension of the largest square submatrix of H with consecutive column indices that has a non-balanced determinant will be less than or equal to n. We represent this dimension as $\mathrm{r_{cc}}(H)$.

Definition 4.3 (Consecutive column rank) *Consider $P \in \mathbb{S}_{\max}^{m\times n}$. The consecutive column rank of P, $\mathrm{r_{cc}}(P)$, is the dimension of the largest square submatrix of P with consecutive column indices, the determinant of which is not balanced:*

$$\begin{aligned}\mathrm{r_{cc}}(P) \;=\; \max\{&\dim(P_{\text{sub},s}) \,|\, P_{\text{sub},s} = P([i_1, i_2, \ldots, i_s], [j+1, j+2, \ldots, j+s]) \text{ with} \\ &0 \leqslant s \leqslant \min(m,n),\, 0 \leqslant j \leqslant n-s,\, \{i_1, i_2, \ldots, i_s\} \in \mathcal{C}_m^s \text{ and } \det(P_{\text{sub},s}) \not\nabla \epsilon\} \quad .\end{aligned}$$

We propose the following procedure to find a lower bound r for the minimal system order: First we construct a p by q Hankel matrix

$$H_{p,q} = \begin{bmatrix} g_0 & g_1 & \cdots & g_{q-1} \\ g_1 & g_2 & \cdots & g_q \\ \vdots & \vdots & \ddots & \vdots \\ g_{p-1} & g_p & \cdots & g_{p+q-2} \end{bmatrix}$$

with p and q large enough: $p, q \gg n$, where n is the real (but unknown) system order. Now we try to find n and $a_0, a_1, \ldots, a_n$ such that the columns of $H_{p,q}$ satisfy an equation of the form (8).
We start with r equal to $\mathrm{r_{cc}}(H_{p,q})$. Let

$$H_{\text{sub},r} = H_{p,q}([i_1, i_2, \ldots, i_r], [j+1, j+2, \ldots, j+r])$$

be an r by r submatrix of $H_{p,q}$ the determinant of which is not balanced: $\det H_{\text{sub},r} \not\nabla \epsilon$. If we add one arbitrary row and the $(j+r+1)$-st column to $H_{\text{sub},r}$ we get an $r+1$ by $r+1$ matrix $H_{\text{sub},r+1}$ that has a balanced determinant. So according to Theorem 1.2 the set of linear balances $H_{\text{sub},r+1} \otimes a \nabla \epsilon$ has a signed solution $a = [\, a_r \;\; a_{r-1} \;\; \ldots \;\; a_0 \,]^t$. We now look for a solution a that corresponds to the characteristic equation of a matrix with elements in $\mathbb{R}_{\max}$ (this should not necessarily be a signed solution). First of all we normalize a_0 to 0 and then we check if the necessary (and/or sufficient) conditions for the coefficients of the characteristic equation of a matrix with elements in $\mathbb{R}_{\max}$ (see [4]) are satisfied. If they are not satisfied we augment r and repeat the procedure.
We continue until we get the following stable relation among the columns of $H_{p,q}$:

$$H_{p,q}(:, k+r) \;\oplus\; a_1 \otimes H_{p,q}(:, k+r-1) \;\oplus\; \ldots \;\oplus\; a_r \otimes H_{p,q}(:, k) \;\nabla\; \epsilon \tag{9}$$

for $k \in \{1, \ldots, q-r\}$. Since we assume that the system can be described by (5)–(6) and that p, $q \gg n$, we can always find such a stable relationship by gradually augmenting r. The r that results from this procedure is a lower bound for the minimal system order, since it corresponds to the smallest number of terms in a relationship of the form (8) among the columns of $H_{p,q}$.

4.3 Determination of the system matrices

Now we have to find $A \in \mathbb{R}_{\max}^{r\times r}$, $B \in \mathbb{R}_{\max}^{r\times 1}$ and $C \in \mathbb{R}_{\max}^{1\times r}$ such that

$$C \otimes A^{\otimes^k} \otimes B = g_k\,, \qquad \text{for } k = 0, 1, 2, \ldots \ . \tag{10}$$

In practice it seems that we only have to take the transient behavior and the first cycles of the steady-state behavior into account. So we may limit ourselves to the first, say, N Markov parameters.

For $k = 0$ we get $\displaystyle\bigoplus_{i=1}^{r} c_i \otimes b_i = g_0$.

For $k > 0$ we have that (10) is equivalent to

$$\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{r} t_{kij} = g_k \ ,$$

with $\displaystyle t_{kij} = \bigoplus_{i_1=1}^{r} \ldots \bigoplus_{i_{k-1}=1}^{r} c_i \otimes a_{ii_1} \otimes a_{i_1 i_2} \otimes \ldots \otimes a_{i_{k-1}j} \otimes b_j$.

This can be rewritten as

$$\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{r} \bigoplus_{l=1}^{r^{k-1}} c_i \otimes \bigotimes_{u=1}^{r} \bigotimes_{v=1}^{r} a_{uv}{}^{\otimes^{\gamma_{kijluv}}} \otimes b_j = g_k \ ,$$

where γ_{kijluv} is the number of times that a_{uv} appears in the l-th subterm of term t_{kij}. If a_{uv} does not appear in that subterm we take $\gamma_{kijluv} = 0$ (since $a^{\otimes^0} = 0.a = 0$, the identity element for $\otimes$).

If we put all unknowns in one large vector x of size $r(r+2)$ we have to solve a set of multivariate polynomial equations of the following form:

$$\bigoplus_{i=1}^{r} \bigotimes_{j=1}^{r(r+2)} x_j{}^{\otimes^{\kappa_{0ij}}} = g_0$$

$$\bigoplus_{i=1}^{w_k} \bigotimes_{j=1}^{r(r+2)} x_j{}^{\otimes^{\kappa_{kij}}} = g_k\,, \qquad \text{for } k = 1\,, 2\,, \ldots, N-1 \ ,$$

and this can be transformed into an ELCP using the technique explained in Section 3. This means that in general all equivalent minimal state space realizations of a max-linear SISO system form a union of polyhedra in the x-space.

If we find a solution x we extract the elements of x and put them in the matrices $A\,, B$ and C. Then we have found a minimal realization. If we do not find a solution we have to augment r and start again. Since we assumed that the data were generated by a max-linear SISO system we shall eventually find a realization and it will be minimal.

5 Example

Example 5.1

Here we reconsider the example of [3, 9]. We start from a system with impulse response

$$\{g_k\} = 0, 1, 2, 3, 4, 5, 6, 8, 10, 12, 14, 16, 18, 20, 22, \ldots \ .$$

First we construct the Hankel matrix $H_{8,8}$. The consecutive column rank of $H_{8,8}$ is 2. The determinant of $H_{\mathrm{sub},2} = H_{8,8}([1,7],[1,2]) = \begin{bmatrix} 0 & 1 \\ 6 & 8 \end{bmatrix}$ is not balanced. We add one row and one column and then we look for a solution of the set of linear balances

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 6 & 8 & 10 \end{bmatrix} \otimes \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} \nabla \epsilon \ .$$

The solution $a_0 = 0,\ a_1 = \ominus 2,\ a_2 = 3$ satisfies the necessary and sufficient conditions for the coefficients of the characteristic polynomial of a 2 by 2 matrix with elements in $\mathbb{R}_{\max}$ (see [4]) and also corresponds to a stable relation among the columns of $H_{8,8}$:

$$H_{8,8}(:,k+2) \ \oplus \ 3 \otimes H_{8,8}(:,k) = 2 \otimes H_{8,8}(:,k+1) \quad \text{ for } k \in \{1,2,\ldots,6\} \ .$$

Let us take $N = 9$. Using the ELCP algorithm of [5] we find the rays of Table 1 and the pairs of subsets of Table 2. If we take $N > 9$ we get the same result, but if we take $N < 9$ some combinations of the rays lead to a partial realization of the given impulse response (i.e. they only fit the first N Markov parameters).
Any arbitrary minimal realization can now be expressed as

$$[\ a_{11}\ a_{12}\ a_{21}\ a_{22}\ b_1\ b_2\ c_1\ c_2\ ]^t \ = \ \lambda_1 x_1^{\mathrm{c}} + \lambda_2 x_2^{\mathrm{c}} + \kappa_1 x_{i_1}^{\mathrm{i}} + \kappa_2 x_{i_2}^{\mathrm{i}} + x_{j_1}^{\mathrm{f}} \ , \tag{11}$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$, $\kappa_1, \kappa_2 \geqslant 0$ and $x_{i_1}^{\mathrm{i}}, x_{i_2}^{\mathrm{i}} \in \mathcal{X}_s^{\mathrm{inf}}$, $x_{j_1}^{\mathrm{f}} \in \mathcal{X}_s^{\mathrm{fin}}$ for some $s \in \{1,2,\ldots,8\}$. Expression (11) shows that the set of all equivalent minimal state space realizations of the given impulse response is a union of 8 unbounded polyhedra. $\Box$

Remark: If we also want to include matrices with components equal to ϵ we have to take certain precautions, since they will be limit cases of (11) where some of the coefficients go to ∞ in a controlled way. However, since the max operation hides small numbers from larger numbers, it suffices in practice to replace negative elements that are large enough in absolute value by ϵ provided that there are no positive elements of the same order of magnitude. See [6] for a detailed explanation.

For another example, that does not satisfy the assumptions of [9], the interested reader is referred to [6], where also the proofs of the properties of Section 4 can be found.

6 Conclusions and further research

We have shown that a set of multivariate polynomial equations in the max algebra can be transformed into an Extended Linear Complementarity Problem (ELCP). This means that we can use the ELCP algorithm of [5] to solve such a problem. We have applied this technique to find all minimal state space realizations of a SISO discrete event system given its Markov parameters.

One of the main characteristics of our ELCP algorithm is that it finds all solutions. For the minimal realization problem this provides a geometrical insight in all equivalent (minimal) realizations of an impulse response. On the other hand this also leads to large computation times and storage space requirements if the number of variables and equations is large. Therefore it might be interesting to develop (heuristic) algorithms that only find one solution.

We also hope to extend the method presented here to find minimal state space realizations for multiple input multiple output (MIMO) systems. The only problem there is the determination of the minimal system order. Once this is found the same technique can be used to get a minimal realization.

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Set	$\mathcal{X}^{\mathrm{cen}}$		$\mathcal{X}^{\mathrm{inf}}$						$\mathcal{X}^{\mathrm{fin}}$	
Ray	x_1^{c}	x_2^{c}	x_1^{i}	x_2^{i}	x_3^{i}	x_4^{i}	x_5^{i}	x_6^{i}	x_1^{f}	x_2^{f}
a_{11}	0	0	0	0	0	0	0	0	2	1
a_{12}	0	1	0	0	0	0	0	0	0	0
a_{21}	0	-1	0	0	-1	-1	0	0	-2	-2
a_{22}	0	0	0	0	0	0	0	0	1	2
b_1	1	1	-1	0	0	1	-1	0	-2	-2
b_2	1	0	-1	-1	0	0	0	0	0	-6
c_1	-1	-1	1	0	0	-1	0	-1	-4	2
c_2	-1	0	0	0	0	0	0	0	0	0

Table 1: The rays for Example 5.1 .

s	$\mathcal{X}_s^{\mathrm{inf}}$	$\mathcal{X}_s^{\mathrm{fin}}$
1	$\{x_1^{\mathrm{i}}, x_2^{\mathrm{i}}\}$	$\{x_2^{\mathrm{f}}\}$
2	$\{x_1^{\mathrm{i}}, x_3^{\mathrm{i}}\}$	$\{x_2^{\mathrm{f}}\}$
3	$\{x_2^{\mathrm{i}}, x_4^{\mathrm{i}}\}$	$\{x_2^{\mathrm{f}}\}$
4	$\{x_3^{\mathrm{i}}, x_4^{\mathrm{i}}\}$	$\{x_2^{\mathrm{f}}\}$
5	$\{x_3^{\mathrm{i}}, x_4^{\mathrm{i}}\}$	$\{x_1^{\mathrm{f}}\}$
6	$\{x_3^{\mathrm{i}}, x_5^{\mathrm{i}}\}$	$\{x_1^{\mathrm{f}}\}$
7	$\{x_4^{\mathrm{i}}, x_6^{\mathrm{i}}\}$	$\{x_1^{\mathrm{f}}\}$
8	$\{x_5^{\mathrm{i}}, x_6^{\mathrm{i}}\}$	$\{x_1^{\mathrm{f}}\}$

Table 2: The pairs of subsets for Example 5.1 .