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Minimal Realization in the Max Algebra

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Abstract.

The main topic of this paper is the minimal realization problem in the max algebra, which is one of the modeling frameworks that can be used to model discrete event systems. First we determine necessary and for some cases also sufficient conditions for a polynomial to be the characteristic polynomial of a matrix in the max algebra. Then we show how a system of multivariate max-algebraic polynomial equalities can be transformed into an Extended Linear Complementarity Problem (ELCP). Finally we combine these results to find all equivalent minimal state space realizations of a single input single output (SISO) discrete event system. We also give a geometrical description of the set of all minimal realizations of a SISO max-linear discrete event system.

1 Introduction

1.1 Overview

In this paper we consider discrete event systems, examples of which are flexible manufacturing systems, subway traffic networks, parallel processing systems, telecommunication networks, ... There exists a wide range of frameworks to model and to analyze discrete event systems: Petri nets, generalized semi-Markov processes, formal languages, perturbation analysis, computer simulation and so on. In this paper we concentrate on a subclass of discrete event systems that can be described with the max algebra [1, 2, 5]. Although these systems lead to a non-linear description in linear algebra, this model becomes "linear" when we formulate it in the max algebra. The main operations of the max algebra are the maximum and the addition. There exist a lot of analogies between max algebra and linear algebra, e.g. the Cayley-Hamilton theorem, eigenvectors and eigenvalues, ... However, there is one major difference that prevents a straightforward translation of properties from linear algebra to max algebra: in general there exists no inverse element for the maximum operator. One

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of the main advantages of an analytic max-algebraic model of a discrete event system is that it allows us to derive some properties of the system (in particular the steady state behavior) fairly easily, whereas in some cases brute force simulation might require a lot of computation time. In this paper we only consider systems that can be described by a time-invariant state space model, limiting ourselves to deterministic systems in which the sequence of the events and the duration of the activities are fixed or can be determined in advance.

In order to analyze systems it is advantageous to have a compact description, i.e. a description with as few parameters as possible. For a system that can be described by a max-linear state space model this gives rise to the minimal state space realization problem. We shall address the minimal state space realization problem for max-algebraic single input single output (SISO) systems. Although there have been some attempts to solve this problem [6, 14, 17], our approach is – to the authors' knowledge – the first one to solve it entirely. And it is certainly the first time that a complete description of the set of all (minimal) realizations of a SISO max-linear discrete event system is given. Apart from further enhancing the max-algebraic system theory the solution of the minimal realization problem can also be seen as the first step towards identification of discrete event systems. Furthermore the technique presented in this paper can also be used to reduce the order of existing state space models.

The characteristic equation plays an important role in the solution of the minimal state space realization problem. Therefore we also make a study of the characteristic equation of a matrix in the max algebra. We show that the minimal state space realization problem in the max algebra can be transformed into a system of multivariate max-algebraic polynomial equations. This problem can in turn be transformed into an Extended Linear Complementarity Problem (ELCP). The ELCP is an extension of the well-known Linear Complementarity Problem, which is one of the fundamental problems of mathematical programming. In [10] we have developed an algorithm to find all solutions of an ELCP. We shall use this algorithm to find all equivalent minimal state space realizations of a SISO discrete event system and to give a geometrical insight in the structure of the set of all equivalent state space realizations.

This paper is organized as follows: In Section 1 we introduce the notations and some of the concepts and definitions that are used later on. In Section 2 we propose the Extended Linear Complementarity Problem and describe the general solution set of this problem. In Section 3 we determine necessary conditions for the coefficients of the characteristic polynomial of a matrix in the max algebra. We also state sufficient conditions for some cases. In Section 4 we show that a system of multivariate polynomial equations in the max algebra can be transformed into an ELCP. Then we derive a lower bound for the minimal system order of a SISO discrete event system in the max algebra. Finally we combine this with the results of the preceding sections to find all minimal state space realizations of a SISO discrete event system, given its impulse response. We also illustrate this procedure with a few examples.

1.2 Notations and definitions

If a is a vector then a_i or $(a)_i$ represents the ith component of a. If A is an m by n matrix then the entry on the ith row and on the jth column is denoted by a_{ij} or $(A)_{ij}$. To select submatrices of a matrix we use the following notation: $A([i_1, i_2, \ldots, i_k], [j_1, j_2, \ldots, j_l])$ is the k by l matrix resulting from A by eliminating all rows except for rows $i_1, i_2, \ldots, i_k$ and all columns except for columns $j_1, j_2, \ldots, j_l$. $A_(:, j)$ is the jth column of A. The transpose of A is A^T. We shall represent the set of all possible combinations of k different numbers out of the set $\{1, 2, \ldots, n\}$ as $\mathcal{C}_n^k$. $\mathcal{P}_n$ is the set of all possible permutations of the set $\{1, 2, \ldots, n\}$.

1.3 The max algebra

One of the mathematical tools used in this paper is the max algebra. In this introduction we only explain the notations we use to represent the max-algebraic operations. We also give some definitions and theorems that will be used in the remainder of this paper. A complete introduction to the max algebra can be found in [1, 5].

1.3.1 The max-algebraic operations

We use the following notations to represent the basic operations of the max algebra:

$$\begin{aligned} a \oplus b &= \max(a, b) \ , \\ a \otimes b &= a + b \ . \end{aligned}$$

The neutral element for $\oplus$ in $\mathbb{R}_\varepsilon = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is $\varepsilon = -\infty$. Since we use both linear algebra and max algebra in this paper, we always write the $\otimes$ sign explicitly to avoid confusion. The max-algebraic power is defined as follows: $a^{\otimes^k} = \underbrace{a \otimes a \otimes \ldots \otimes a}_{k \text{ times}}$ and is equal to ka in linear algebra. So $a^{\otimes^0} = 0 \cdot a = 0$. The inverse element of $a \neq \varepsilon$ is $a^{\otimes^{-1}} = -a$. The division is defined as follows:

$$\frac{a}{b} = a \otimes b^{\otimes^{-1}} \qquad \text{if } b \neq \varepsilon \ .$$

The operations $\oplus$ and $\otimes$ are extended to matrices in the usual way. So if $C = A \oplus B$ then $c_{ij} = a_{ij} \oplus b_{ij}$ and if $C = A \otimes B$ then $c_{ij} = \bigoplus_l a_{il} \otimes b_{lj}$.

ϵ_{mn} is the m by n zero matrix in the max algebra: $(\epsilon_{mn})_{ij} = \varepsilon$.

E_n is the n by n identity matrix in $\mathbb{R}_{\max}$: $\begin{aligned} (E_n)_{ij} &= 0 && \text{if } i = j \ , \\ (E_n)_{ij} &= \varepsilon && \text{if } i \neq j \ . \end{aligned}$

We also use the extension of the max algebra $\mathbb{S}_{\max}$ that was introduced in [1, 12]. $\mathbb{S}_{\max}$ is a kind of symmetrization of $\mathbb{R}_{\max}$. We shall restrict ourselves to the most important features of $\mathbb{S}_{\max}$. For a more formal derivation the interested reader is referred to [12].

First we define the $\ominus$ operation for $a, b \in \mathbb{R}_\varepsilon$:

$$\begin{aligned} a \ominus b &= a && \text{if } a > b \, , \\ a \ominus b &= \ominus b && \text{if } a < b \, , \\ a \ominus a &= a^\bullet \ . \end{aligned}$$

So we have to introduce two new kinds of elements ($\ominus b$ and $a^\bullet$). This leads to $\mathbb{S}_{\max}$, an extension of $\mathbb{R}_{\max}$ that contains three classes of elements:

- the max-positive elements: $\mathbb{S}^\oplus$, this corresponds to $\mathbb{R}_\varepsilon$,
- the max-negative elements: $\mathbb{S}^\ominus = \{\ominus a \,|\, a \in \mathbb{R}_\varepsilon\}$,
- the balanced elements: $\mathbb{S}^\bullet = \{a^\bullet \,|\, a \in \mathbb{R}_\varepsilon\}$.

The max-positive and the max-negative elements are called signed ($\mathbb{S}^\vee = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus$).

If $a \in \mathbb{S}$ then it can be written as $a = a^+ \ominus a^-$ where a^+ and a^- are as small as possible:

- if $a = b \in \mathbb{R}_\varepsilon$ then $a^+ = a$ and $a^- = \varepsilon$,
- if $a = \ominus b \in \mathbb{S}^\ominus$ then $a^+ = \varepsilon$ and $a^- = b$,
- if $a = b^\bullet \in \mathbb{S}^\bullet$ then $a^+ = a^- = b$.

a^+ is the max-positive part of a, a^- is the max-negative part of a and $|a|_\oplus = a^+ \oplus a^-$ is the max-absolute value of a.

Example 1.1 Let $a = 2^\bullet \in \mathbb{S}^\bullet$, then $a^+ = 2$, $a^- = 2$ and $|a|_\oplus = 2$.
For $b = \ominus(-1) \in \mathbb{S}^\ominus$ we have $b^+ = \varepsilon$, $b^- = -1$ and $|b|_\oplus = -1$.

This symmetrization allows us to "solve" equations that have no solutions in $\mathbb{R}_{\max}$. Unfortunately we then have to introduce balances (∇) instead of equalities. The main difference between balances and equalities is that a balance doesn't yield an equivalence relation since it is not transitive. Informally an $\ominus$ sign in a balance means that the element should be at the other side: so $4 \ominus 4 \nabla 2$ means $4 \nabla 4 \oplus 2$. If both sides of a balance are signed (max-positive or max-negative) we can replace the balance by an equality. So $4 \nabla 4 \oplus 2$ is equivalent to $4 = 4 \oplus 2$.

1.3.2 Some definitions and theorems

Definition 1.2 (Determinant) *Consider a matrix $A \in \mathbb{S}^{n \times n}$. The determinant of A is defined as*

$$\det A = \bigoplus_{\sigma \in \mathcal{P}_n} \operatorname{sgn}(\sigma) \otimes \bigotimes_{i=1}^{n} a_{i\sigma(i)} \ ,$$

where $\mathcal{P}_n$ is the set of all permutations of $\{1, \ldots, n\}$, and $\operatorname{sgn}(\sigma) = 0$ if the permutation σ is even and $\operatorname{sgn}(\sigma) = \ominus 0$ if the permutation is odd.

Definition 1.3 (Determinantal rank) *Let $A \in \mathbb{S}^{m \times n}$. The determinantal rank of A, $\mathrm{r}_{\det}(A)$, is defined as the dimension of the largest square submatrix of A the determinant of which is not balanced and not equal to ε.*

Theorem 1.4 *Let $A \in \mathbb{S}^{n \times n}$. The homogeneous linear balance $A \otimes x \nabla \varepsilon$ has a non-trivial signed solution if and only if $\det A \nabla \varepsilon$.*

Proof: See [12]. The proof given there is constructive so it can be used to find a solution.

Definition 1.5 (Characteristic equation) *Let $A \in \mathbb{S}^{n \times n}$. The characteristic equation of A is defined as $\det(A \ominus \lambda \otimes E_n) \nabla \varepsilon$.*

This leads to

$$\lambda^{\otimes^n} \ \oplus \ \bigoplus_{p=1}^{n} a_p \otimes \lambda^{\otimes^{n-p}} \ \nabla \ \varepsilon \ ,$$

which will be called a *monic* balance, since the coefficient of $\lambda^{\otimes^n}$ equals 0 (i.e. the identity element for $\otimes$).

If we define $\alpha_p = a_p^+$ and $\beta_p = a_p^-$ and if we move all terms with max-negative coefficients to the right hand side we get

$$\lambda^{\otimes^n} \;\oplus\; \bigoplus_{i=1}^{n} \alpha_i \otimes \lambda^{\otimes^{n-i}} \;\nabla\; \bigoplus_{j=1}^{n} \beta_j \otimes \lambda^{\otimes^{n-j}} \;\;,$$

with $\alpha_p, \beta_p \in \mathbb{R}_\varepsilon$. In [15] Olsder defines a variant of this equation using the dominant instead of the determinant. This leads to signed coefficients: $a_p^{\mathrm{Olsder}} \in \mathbb{S}^\vee$ or $\alpha_p^{\mathrm{Olsder}} \otimes \beta_p^{\mathrm{Olsder}} = \varepsilon$ with $a_p^{\mathrm{Olsder}} = a_p$ if $a_p \in \mathbb{S}^\vee$ and $\left| a_p^{\mathrm{Olsder}} \right|_\oplus \leqslant |a_p|_\oplus$ if $a_p \in \mathbb{S}^\bullet$.

Theorem 1.6 (Cayley-Hamilton) *In* $\mathbb{S}_{\max}$ *every square matrix satisfies its characteristic equation.*

Proof: See [13] and [15].

2 The Extended Linear Complementarity Problem

2.1 Problem formulation

Consider the following problem:

> Given two matrices $A \in \mathbb{R}^{p\times n}, B \in \mathbb{R}^{q\times n}$, two column vectors $c \in \mathbb{R}^p, d \in \mathbb{R}^q$ and m subsets ϕ_j of $\{1, 2, \ldots, p\}$, find a vector $x \in \mathbb{R}^n$ such that
>
> $$\sum_{j=1}^{m} \prod_{i\in\phi_j} (Ax - c)_i = 0 \tag{1}$$
>
> subject to $\begin{array}{rcl} Ax & \geqslant & c \\ Bx & = & d \end{array}$,
>
> or show that no such vector exists.

In [10] we have demonstrated that this problem is an extension of the Linear Complementarity Problem [4] and a unifying framework for other extensions of Linear Complementarity Problem such as the Vertical Linear Complementarity Problem of Cottle and Dantzig [3] and the Horizontal Linear Complementarity Problem of De Moor [7, 16]. Therefore we call it the Extended Linear Complementarity Problem (ELCP).

Equation (1) represents the *complementarity condition.* A possible interpretation of this condition is the following: since $Ax \geqslant c$, condition (1) is equivalent to

$$\prod_{i\in\phi_j} (Ax - c)_i = 0 \;, \quad \forall j \in \{1, 2, \ldots, m\} \;\;. \tag{2}$$

So we could say that each set ϕ_j corresponds to a subgroup of inequalities of $Ax \geqslant c$ and that in each group at least one inequality should hold with equality:

$$\forall j \in \{1, 2, \ldots, m\} \;:\; \exists i \in \phi_j \text{ such that } (Ax - c)_i = 0 \;\;.$$

We shall use this interpretation in Section 4 to demonstrate that a system of multivariate polynomial equations in the max algebra can be transformed into an ELCP.

In [10] we have made a thorough study of the solution set of the ELCP and developed an algorithm to find all its solutions. We shall now state the main results of that paper.
The ELCP algorithm results in 3 sets of rays $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{inf}}$, $\mathcal{X}^{\text{fin}}$ and a set Λ of pairs $\left\{\mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}}\right\}$ where $\mathcal{X}_s^{\text{inf}}$ is a subset of $\mathcal{X}^{\text{inf}}$ and $\mathcal{X}_s^{\text{fin}}$ is a non-empty subset of $\mathcal{X}^{\text{fin}}$. The solution set of the ELCP is then characterized by the following theorem:

Theorem 2.1 *When $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{inf}}$, $\mathcal{X}^{\text{fin}}$ and Λ are given, then x is a solution of the ELCP if and only if there exists a pair $\left\{\mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}}\right\} \in \Lambda$ such that*

$$x = \sum_{x_k \in \mathcal{X}^{\text{cen}}} \lambda_k x_k + \sum_{x_k \in \mathcal{X}_s^{\text{inf}}} \kappa_k x_k + \sum_{x_k \in \mathcal{X}_s^{\text{fin}}} \mu_k x_k \ , \tag{3}$$

with $\lambda_k \in \mathbb{R}$, $\kappa_k, \mu_k \geqslant 0$ and $\sum\limits_k \mu_k = 1$.

As a result we have that:

Corollary 2.2 *The general solution set of an ELCP consists of the union of faces a polyhedron.*

3 The characteristic equation of a max-positive matrix

The characteristic equation will play an important role in the determination of a lower bound for the minimal system order as will be shown in Section 5.3. Normally we are only interested in system matrices with max-positive entries. In this section we derive necessary (and for some cases also sufficient) conditions for a polynomial in $\mathbb{S}_{\max}$ to be generated by a matrix with entries in $\mathbb{R}_\varepsilon$, i.e. a max-positive matrix. If we have a monic polynomial in $\mathbb{S}_{\max}$ the results of this section will allow us to check whether the given polynomial can be the characteristic polynomial of a max-positive matrix.

3.1 The characteristic equation

Definition 3.1 (Principal submatrix) *Let $A \in \mathbb{S}^{n \times n}$ and let $\{i_1, i_2, \ldots, i_k\}$ be a combination of k elements out of $\{1, \ldots, n\}$. Then the matrix $A([i_1, i_2, \ldots, i_k], [i_1, i_2, \ldots, i_k])$ is a k by k principal submatrix of A. It can be obtained from A by deleting $n-k$ rows and columns.*

We represent the max-algebraic sum of the determinants of all k by k submatrices of A as $E_k(A)$:

$$E_k(A) = \bigoplus_{\varphi \in \mathcal{C}_n^k} \det A([i_1, i_2, \ldots, i_k], [i_1, i_2, \ldots, i_k])$$

where $\mathcal{C}_n^k$ is the set of all combinations of k numbers out of $\{1, \ldots, n\}$ and $\varphi = \{i_1, i_2, \ldots, i_k\}$.

Property 3.2 *If we represent the characteristic equation of $A \in \mathbb{S}^{n \times n}$ as $\bigoplus\limits_{p=0}^{n} a_p \otimes \lambda^{\otimes^{n-p}} \nabla \varepsilon$ then $a_p = (\ominus 0)^{\otimes^p} \otimes E_p(A)$.*

Example 3.3 Consider $A = \begin{bmatrix} 0 & 5 & 9 \\ 5 & 20 & 10 \\ 9 & 10 & 18 \end{bmatrix}$.

The characteristic equation of A is $\lambda^{\otimes^3} \ominus 20 \otimes \lambda^{\otimes^2} \oplus 38 \otimes \lambda \oplus 38^\bullet \nabla \varepsilon$.

Proposition 3.4 *In $\mathbb{S}_{\max}$ every monic nth order linear balance is the characteristic equation of an $n \times n$ matrix: the linear balance*

$$\lambda^{\otimes^n} \oplus a_1 \otimes \lambda^{\otimes^{n-1}} \oplus \ldots \oplus a_{n-1} \otimes \lambda \oplus a_n \nabla \varepsilon$$

is the characteristic equation of the matrix

$$A = \begin{bmatrix} \varepsilon & 0 & \varepsilon & \ldots & \varepsilon \\ \varepsilon & \varepsilon & 0 & \ldots & \varepsilon \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \varepsilon & \ldots & 0 \\ \ominus a_n & \ominus a_{n-1} & \ominus a_{n-2} & \ldots & \ominus a_1 \end{bmatrix} .$$

However, not every monic polynomial corresponds to the characteristic polynomial of a max-positive matrix as we shall see in the next subsection.

3.2 Properties of the characteristic polynomial of max-positive matrices

Property 3.5 *If $A \in \mathbb{R}_\varepsilon^{n\times n}$ then $a_1 \in \mathbb{S}^\ominus$.*

Proof: We know that $a_1 = \ominus \bigoplus_{i=1}^{n} a_{ii}$ with $a_{ii} \in \mathbb{R}_\varepsilon$ so $a_1 \in \mathbb{S}^\ominus$.

To prove the following property we first need a lemma involving permutations. The parity of a permutation can be determined in various ways. We use:

Property 3.6 *The parity of a permutation is equal to the parity of the number of its elementary cycles of even length.*

First consider a circular permutation σ_c of n elements:

$$\sigma_c(i_1) = i_2,\ \sigma_c(i_2) = i_3,\ \ldots, \sigma_c(i_{n-1}) = i_n,\ \sigma_c(i_n) = i_1 \ .$$

This permutation has a cycle of length n. If n is even, then $\sigma_c \in \mathcal{P}_n$ is odd because there is 1 cycle of even length. If n is odd, then $\sigma_c \in \mathcal{P}_n$ is even because there are 0 cycles of even length.
If a permutation of n numbers is not circular we can decompose it into r elementary cycles C_i of length l_i, with $r > 1$ and $\sum_{i=1}^{r} l_i = n$. Each cycle will be a circular permutation.

Lemma 3.7 *If $\sigma_{2k,\text{even}}$ $(k > 0)$ is an even permutation of $2k$ elements, then it can be decomposed into two even permutations of an odd number of elements or two odd permutations of an even number of elements:*

$$\sigma_{2k,\text{even}} = \sigma_{2l+1,\text{even}} \cup \sigma_{2k-2l-1,\text{even}} \quad \textit{or} \quad \sigma_{2k,\text{even}} = \sigma_{2m,\text{odd}} \cup \sigma_{2k-2m,\text{odd}} \ .$$

If $\sigma_{2k+1,\text{odd}}$ $(k > 0)$ is an odd permutation of $2k+1$ elements, then it can be decomposed into an even permutation of an odd number of elements and an odd permutation of an even number of elements:

$$\sigma_{2k+1,\text{odd}} = \sigma_{2p+1,\text{even}} \cup \sigma_{2k-2p,\text{odd}} . \ .$$

Proof:
First consider $\sigma_{2k,\text{even}}$. This is an even permutation of an even number of elements so it is not circular and it can be decomposed into elementary cycles. Suppose that there are c_{even} cycles of even length each having $n_{\text{even},i}$ elements and c_{odd} cycles of odd length each having $n_{\text{odd},j}$ elements. Let $n_{\text{tot,even}} = \sum_{i=1}^{c_{\text{even}}} n_{\text{even},i}$ and $n_{\text{tot,odd}} = \sum_{j=1}^{c_{\text{odd}}} n_{\text{odd},j}$. Since the parity of $\sigma_{2k,\text{even}}$ is even, c_{even} should also be even. $n_{\text{tot,even}}$ is always even. The total number of elements $n_{\text{tot}} = 2k$ is even, so we have that $n_{\text{tot,odd}}$ is also even and hence that c_{odd} is even. There are two cases: $c_{\text{even}} = 0$ and $c_{\text{even}} \neq 0$.
If $c_{\text{even}} = 0$ then $c_{\text{odd}} \neq 0$ because $2k \neq 0$. Take one cycle of odd length $2l+1$. This corresponds to an even permutation of $2l+1$ elements: $\sigma_{2l+1,\text{even}}$. The other cycles form a permutation with 0 cycles of even length, so it is an even permutation of the remaining $2k-2l-1$ elements: $\sigma_{2k-2l-1,\text{even}}$.
If $c_{\text{even}} \neq 0$ we take one cycle of even length $2m$. This corresponds to $\sigma_{2m,\text{odd}}$. The remaining cycles constitute a permutation with an odd number $(c_{\text{even}} - 1)$ of cycles of even length: $\sigma_{2k-2m,\text{odd}}$.
So we have proven that $\sigma_{2k,\text{even}}$ can be decomposed as $\sigma_{2l+1,\text{even}} \cup \sigma_{2k-2l-1,\text{even}}$ or as $\sigma_{2m,\text{odd}} \cup \sigma_{2k-2m,\text{odd}}$.

Now consider $\sigma_{2k+1,\text{odd}}$. This is an odd permutation of an odd number of elements so it is not circular and it can be decomposed into elementary cycles. Since the parity of $\sigma_{2k+1,\text{odd}}$ is odd, c_{even} should also be odd. $n_{\text{tot,even}}$ is always even, and since the total number of elements $n_{\text{tot}} = 2k+1$ is odd we have that $n_{\text{tot,odd}}$ is odd and hence that c_{odd} is odd. This means that $c_{\text{odd}} \neq 0$. So let us take one cycle of odd length $2p+1$. This corresponds to an even permutation of $2p+1$ elements: $\sigma_{2p+1,\text{even}}$.
The other cycles will then correspond to a permutation of with an odd number (c_{even}) of cycles of even length, so it is an odd permutation of $2k-2p$ elements: $\sigma_{2k-2p,\text{odd}}$.
So $\sigma_{2k+1,\text{odd}} = \sigma_{2p+1,\text{even}} \cup \sigma_{2k-2p,\text{odd}}$.

Now we give some properties of $a_p = (\ominus 0)^{\otimes^p} \otimes E_p(A) = a_p^+ \ominus a_p^-$. First we suppose that we don't simplify $\ominus$. This means that for $a = 3 \ominus 4$ we have $a^+ = 3$ and $a^- = 4$. Later we shall see how we have to adapt the properties to take simplification into account, because then we shall have that $a = 3 \ominus 4$ results in $a = \ominus 4$ or $a^+ = \varepsilon$ and $a^- = 4$.

Property 3.8 *Let $A \in \mathbb{R}_\varepsilon^{n \times n}$ and let $a_p = (\ominus 0)^{\otimes^p} \otimes E_p(A) = a_p^+ \ominus a_p^-$ (without simplifying $\ominus$). Then*

$$\forall p \in \{2, \ldots, n\} : \ a_p^+ \leqslant \bigoplus_{r=1}^{\lfloor \frac{p}{2} \rfloor} a_r^- \otimes a_{p-r}^- \ ,$$

where $\lfloor x \rfloor$ stands for the largest integer number less than or equal to x.

Proof: We know that

$$\begin{aligned} a_p &= (\ominus 0)^{\otimes^p} \otimes E_p(A) \\ &= (\ominus 0)^{\otimes^p} \bigoplus_{\varphi \in \mathcal{C}_n^p} \bigoplus_{\sigma \in \mathcal{P}_p} \operatorname{sgn}(\sigma) \otimes a_{i_1 i_{\sigma(1)}} \otimes a_{i_2 i_{\sigma(2)}} \otimes \ldots \otimes a_{i_p i_{\sigma(p)}} \ , \end{aligned}$$

with $\varphi = \{i_1, i_2, \ldots, i_p\}$.
If we extract the max-positive and the max-negative part of a_p (without simplifying $\ominus$), we find for $k > 0$:

$$a_{2k}^{+} = \bigoplus_{\varphi \in \mathcal{C}_n^{2k}} \bigoplus_{\sigma \in \mathcal{P}_{2k,\text{even}}} a_{i_1 i_{\sigma(1)}} \otimes a_{i_2 i_{\sigma(2)}} \otimes \ldots \otimes a_{i_{2k} i_{\sigma(2k)}} \tag{4}$$

$$a_{2k}^{-} = \bigoplus_{\varphi \in \mathcal{C}_n^{2k}} \bigoplus_{\sigma \in \mathcal{P}_{2k,\text{odd}}} a_{i_1 i_{\sigma(1)}} \otimes a_{i_2 i_{\sigma(2)}} \otimes \ldots \otimes a_{i_{2k} i_{\sigma(2k)}} \tag{5}$$

$$a_{2k+1}^{+} = \bigoplus_{\varphi \in \mathcal{C}_n^{2k+1}} \bigoplus_{\sigma \in \mathcal{P}_{2k+1,\text{odd}}} a_{i_1 i_{\sigma(1)}} \otimes a_{i_2 i_{\sigma(2)}} \otimes \ldots \otimes a_{i_{2k+1} i_{\sigma(2k+1)}} \tag{6}$$

$$a_{2k+1}^{-} = \bigoplus_{\varphi \in \mathcal{C}_n^{2k+1}} \bigoplus_{\sigma \in \mathcal{P}_{2k+1,\text{even}}} a_{i_1 i_{\sigma(1)}} \otimes a_{i_2 i_{\sigma(2)}} \otimes \ldots \otimes a_{i_{2k+1} i_{\sigma(2k+1)}} \ . \tag{7}$$

Let us first consider a_{2k}^{+}. The terms of a_{2k}^{+} are generated by even permutations of $2k$ elements. According to Lemma 3.7 such a permutation can be decomposed into two even permutations of odd lengths or two odd permutations of even lengths. So if we consider all possible concatenations of two even permutations of odd lengths (corresponding to $a_{2l+1}^{-} \otimes a_{2k-2l-1}^{-}$) or two odd permutations of even length ($a_{2m}^{-} \otimes a_{2k-2m}^{-}$), we are sure to have included all terms of a_{2k}^{+}. In other words $a_{2k}^{+} \leqslant \bigoplus_{r=1}^{2k-1} a_r^{-} \otimes a_{2k-r}^{-}$. Since $(a_r^{-} \otimes a_{2k-r}^{-}) \oplus (a_{2k-r}^{-} \otimes a_r^{-}) = a_r^{-} \otimes a_{2k-r}^{-}$ we find $a_{2k}^{+} \leqslant \bigoplus_{r=1}^{k} a_r^{-} \otimes a_{2k-r}^{-}$.
Now consider a_{2k+1}^{+}, the terms of which are generated by odd permutations of $2k+1$ elements. Lemma 3.7 also tells us that such a permutation can be decomposed into an odd permutation of an even number of elements and an even permutation of an odd number of elements. Using the same reasoning as for a_{2k}^{+}, we find that $a_{2k+1}^{+} \leqslant \bigoplus_{r=1}^{k} a_r^{-} \otimes a_{2k+1-r}^{-}$.

Combining the two inequalities leads to $a_p^{+} \leqslant \bigoplus_{r=1}^{\lfloor \frac{p}{2} \rfloor} a_r^{-} \otimes a_{p-r}^{-}$.

We don't have a similar expression for a_p^{-} because then some of the generating permutations are circular, and these cannot be decomposed into more than one elementary cycle.

Normally we simplify $\ominus$, by setting $a_p^{-} = \varepsilon$ if $a_p^{-} < a_p^{+}$ and $a_p^{+} = \varepsilon$ if $a_p^{-} > a_p^{+}$.
Therefore we shall from now on represent the characteristic equation of $A \in \mathbb{R}_{\varepsilon}^{n \times n}$ as

$$\lambda^{\otimes^n} \ \oplus \ \bigoplus_{i=2}^{n} \alpha_i \otimes \lambda^{\otimes^{n-i}} \ \nabla \ \beta_1 \otimes \lambda^{\otimes^{n-1}} \ \oplus \ \bigoplus_{j=2}^{n} \beta_j \otimes \lambda^{\otimes^{n-j}} \ ,$$

with $\begin{cases} \alpha_p = a_p^+, & \beta_p = \varepsilon & \text{if } a_p^+ > a_p^- \ , \\ \alpha_p = \varepsilon, & \beta_p = a_p^- & \text{if } a_p^+ < a_p^- \ , \\ \alpha_p = a_p^+, & \beta_p = a_p^- & \text{if } a_p^+ = a_p^- \ , \end{cases}$

where a_p^+ and a_p^- are defined as in (4)–(7), i.e. without simplification of $\ominus$.
So there are three possible cases: $\alpha_p = \varepsilon$, $\beta_p = \varepsilon$ or $\alpha_p = \beta_p$. We already have omitted α_1 because Property 3.5 leads to $\alpha_1 = a_1^+ = \varepsilon$.
We have that $\alpha_p \leqslant a_p^+$, $\beta_p \leqslant a_p^-$ and $|a_p|_\oplus = a_p^+ \oplus a_p^- = \alpha_p \oplus \beta_p$.

Property 3.9 $\forall i \in \{2,\ldots,n\}$: $\alpha_i \leqslant \bigoplus_{r=1}^{\lfloor \frac{i}{2} \rfloor} (\alpha_r \oplus \beta_r) \otimes (\alpha_{i-r} \oplus \beta_{i-r})$, *where* $\lfloor x \rfloor$ *stands for the largest integer number less than or equal to* x.

Proof: Using the fact that $a_i^- \leqslant |a_i|_\oplus$ Property 3.8 leads to

$$\begin{aligned} a_i^+ &\leqslant \bigoplus_{r=1}^{\lfloor \frac{i}{2} \rfloor} |a_r^-|_\oplus \otimes \left|a_{i-r}^-\right|_\oplus \\ &\leqslant \bigoplus_{r=1}^{\lfloor \frac{i}{2} \rfloor} (\alpha_r \oplus \beta_r) \otimes (\alpha_{i-r} \oplus \beta_{i-r}) \ . \end{aligned}$$

We also know that $\alpha_i \leqslant a_i^+$. So

$$\alpha_i \leqslant \bigoplus_{r=1}^{\lfloor \frac{i}{2} \rfloor} (\alpha_r \oplus \beta_r) \otimes (\alpha_{i-r} \oplus \beta_{i-r}) \ .$$

We even have a more stringent property which gives necessary conditions for the coefficients of an $\mathbb{S}_{\max}$ polynomial such that it is the characteristic polynomial of a max-positive matrix. We shall make extensive use of this property in Section 5 when we search a lower bound for the minimal system order of a discrete event system.

Property 3.10 (Necessary conditions)
$\forall i \in \{2,\ldots,n\}$ *at least one of the following statements is true:*

$$1) \quad \alpha_i \leqslant \bigoplus_{r=1}^{\lfloor \frac{i}{2} \rfloor} \beta_r \otimes \beta_{i-r}$$

$$2) \quad \alpha_i < \bigoplus_{r=2}^{\lfloor \frac{i}{2} \rfloor} \alpha_r \otimes \alpha_{i-r}$$

$$3) \quad \alpha_i < \bigoplus_{r=2}^{i-1} \alpha_r \otimes \beta_{i-r}$$

where $\lfloor x \rfloor$ *stands for the largest integer number less than or equal to* x.

Proof: Take an arbitrary $i \in \{2,\ldots,n\}$. Then according to Property 3.8 there exists an $s \leqslant \left\lfloor \frac{i}{2} \right\rfloor$ such that

$$\alpha_i \leqslant a_i^+ \leqslant a_s^- \otimes a_{i-s}^- \ .$$

We have that either $a_s^- = \beta_s$ or $a_s^- < \alpha_s$ and the same goes for a_{i-s}^-.
This means that at least one of the following inequalities holds:

$$1)\quad \alpha_i \leqslant \beta_s \otimes \beta_{i-s} \leqslant \bigoplus_{r=1}^{\lfloor \frac{i}{2} \rfloor} \beta_r \otimes \beta_{i-r}$$

$$2)\quad \alpha_i < \alpha_s \otimes \alpha_{i-s} \leqslant \bigoplus_{r=2}^{\lfloor \frac{i}{2} \rfloor} \alpha_r \otimes \alpha_{i-r}$$

$$3)\quad \alpha_i < \beta_s \otimes \alpha_{i-s} \ \oplus\ \alpha_s \otimes \beta_{i-s} \leqslant \bigoplus_{r=2}^{i-1} \alpha_r \otimes \beta_{i-r} \ .$$

In the last two max-algebraic sums we can start from $r = 2$ because $\alpha_1 = \varepsilon$.

3.3 Necessary and sufficient conditions for a polynomial to be the characteristic polynomial of a max-positive matrix

In the next subsections we shall case by case determine necessary and sufficient conditions for

$$\lambda^{\otimes^n} \ \oplus\ \bigoplus_{i=1}^{n} \alpha_i \otimes \lambda^{\otimes^{n-i}} \ \nabla\ \bigoplus_{j=1}^{n} \beta_j \otimes \lambda^{\otimes^{n-j}} \tag{8}$$

to be the characteristic equation of a max-positive matrix and indicate how such a matrix can be found (see [8] for proofs). Unfortunately we have currently only found necessary and sufficient conditions for 1×1 up to 4×4 matrices.
In all cases we have $\alpha_1 = \varepsilon$ as a necessary condition.

We also define
$$\begin{aligned} \kappa_{i,j} &= \frac{\alpha_j}{\beta_i} && \text{if} \quad \beta_i \neq \varepsilon \ , \\ &= \varepsilon && \text{if} \quad \beta_i = \varepsilon \ . \end{aligned}$$

3.3.1 The 1×1 case

The only necessary and also sufficient condition is $\alpha_1 = \varepsilon$.
The matrix $[\beta_1]$ has $\lambda \ \nabla\ \beta_1$ as its characteristic equation.

3.3.2 The 2×2 case

The necessary and also sufficient conditions are $\begin{cases} \alpha_1 = \varepsilon \\ \alpha_2 \leqslant \beta_1 \otimes \beta_1 \ . \end{cases}$

The matrix $\begin{bmatrix} \beta_1 & \beta_2 \\ 0 & \kappa_{1,2} \end{bmatrix}$ has $\lambda^{\otimes^2} \ \oplus\ \alpha_2 \ \nabla\ \beta_1 \otimes \lambda \ \oplus\ \beta_2$ as its characteristic equation.

3.3.3 The 3×3 case

The necessary and also sufficient conditions are $\begin{cases} \alpha_1 = \varepsilon \\ \alpha_2 \leqslant \beta_1 \otimes \beta_1 \\ \alpha_3 \leqslant \beta_1 \otimes \beta_2 \text{ or } \alpha_3 < \beta_1 \otimes \alpha_2 \ . \end{cases}$

The matrix $\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \\ 0 & \kappa_{1,2} & \kappa_{1,3} \\ \varepsilon & 0 & \varepsilon \end{bmatrix}$ has $\lambda^{\otimes^3} \oplus \alpha_2 \otimes \lambda \oplus \alpha_3 \nabla \beta_1 \otimes \lambda^{\otimes^2} \oplus \beta_2 \otimes \lambda \oplus \beta_3$ as its characteristic equation.

3.3.4 The 4×4 case

First we distinguish three possible cases:

Case A: $\alpha_4 \leqslant \beta_1 \otimes \beta_3$ or $\alpha_4 < \beta_1 \otimes \alpha_3$,

Case B: $\alpha_4 > \beta_1 \otimes \beta_3$ and $\alpha_4 \geqslant \beta_1 \otimes \alpha_3$ and $\alpha_4 \leqslant \beta_2 \otimes \beta_2$ and ($\beta_1 = \varepsilon$ or $\alpha_2 = \varepsilon$ or $\beta_4 = \alpha_4$) ,

Case C: $\alpha_4 > \beta_1 \otimes \beta_3$ and $\alpha_4 \geqslant \beta_1 \otimes \alpha_3$ and $\alpha_4 \leqslant \beta_2 \otimes \beta_2$ and $\alpha_2 = \beta_2 \neq \varepsilon$ and $\beta_4 = \varepsilon$.

If the coefficients don't fall into (exactly) one of these three cases, they cannot correspond to a max-positive matrix.

The necessary and sufficient conditions are:

$$\begin{cases} \alpha_1 = \varepsilon \\ \alpha_2 \leqslant \beta_1 \otimes \beta_1 \\ \alpha_3 \leqslant \beta_1 \otimes \beta_2 \text{ or } \alpha_3 < \beta_1 \otimes \alpha_2 \\ \text{for Case A: no extra conditions} \\ \text{for Case B: } \beta_1 \otimes \alpha_4 \leqslant \beta_2 \otimes \alpha_3 \text{ or } \beta_1 \otimes \alpha_4 < \beta_2 \otimes \beta_3 \\ \text{for Case C: } \beta_1 \otimes \alpha_3 = \beta_2 \otimes \alpha_2 \text{ and } \beta_1 \otimes \alpha_4 = \beta_2 \otimes \alpha_3 \ . \end{cases}$$

We find for Case A: $\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 0 & \kappa_{1,2} & \kappa_{1,3} & \kappa_{1,4} \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon \end{bmatrix}$, for Case B: $\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 0 & \kappa_{1,2} & \kappa_{1,3} & \varepsilon \\ \varepsilon & 0 & \varepsilon & \kappa_{2,4} \\ \varepsilon & \varepsilon & 0 & \varepsilon \end{bmatrix}$ and for Case C: $\begin{bmatrix} \beta_1 & \beta_2 & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \kappa_{2,3} & \kappa_{2,4} \\ \varepsilon & \varepsilon & 0 & \varepsilon \end{bmatrix}$.

4 Multivariate polynomial equations in the max algebra

Consider the following problem:

Given a set of integers $\{m_k\}$ and three sets of coefficients $\{a_{ki}\}$, $\{b_k\}$ and $\{c_{kij}\}$ with $i \in \{1, \ldots, m_k\}$, $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, p\}$, find a vector $x \in \mathbb{R}_\varepsilon^n$ that satisfies

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kij}}} = b_k \,, \qquad \text{for } k = 1, 2, \ldots, p \ , \tag{9}$$

or show that no such vector x exists.

We demonstrate that this problem can be transformed into an ELCP:
First we consider one equation of the form (9):

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kij}}} = b_k \quad .$$

In linear algebra this is equivalent to the system of linear inequalities

$$a_{ki} + \sum_{j=1}^{n} c_{kij}\, x_j \leqslant b_k \,, \qquad \text{for } i = 1, 2, \ldots, m_k \quad ,$$

where at least one inequality should hold with equality. If we transfer the a_{ki}'s to the right hand side and if we define $d_{ki} = b_k - a_{ki}$, we get

$$\sum_{j=1}^{n} c_{kij}\, x_j \leqslant d_{ki} \,, \qquad \text{for } i = 1, 2, \ldots, m_k \quad .$$

If we define p matrices C_k and p column vectors d_k such that $(C_k)_{ij} = c_{kij}$ and $(d_k)_i = d_{ki}$, then (9) leads to p groups of linear inequalities $C_k\, x \leqslant d_k$ with in each group at least one inequality that should hold with equality.

We put all C_k's in one large matrix $A = \begin{bmatrix} -C_1 \\ -C_2 \\ \vdots \\ -C_p \end{bmatrix}$ and all d_k's in one vector $c = \begin{bmatrix} -d_1 \\ -d_2 \\ \vdots \\ -d_p \end{bmatrix}$.

We also define p sets ϕ_j such that $\phi_j = \{s_j+1, s_j+2, \ldots, s_j+m_j\}$, for $j = 1, 2, \ldots, p$, where $s_1 = 0$ and $s_{j+1} = s_j + m_j$ for $j = 1, 2, \ldots, p-1$. Our original problem (9) is then equivalent to the following ELCP:

Find a vector $x \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{p} \prod_{i \in \phi_j} (Ax - c)_i = 0 \tag{10}$$

subject to $Ax \geqslant c$,

or show that no such vector x exists.

This means that we can use the ELCP algorithm of [10] to find all solutions of problem (9). In Section 5 we shall show that the minimal realization problem in the max algebra leads to a system of multivariate polynomial equations in the max algebra and can thus be solved using the technique of this section.
For other applications of the ELCP in the max algebra and in the max/min/plus algebra the interested reader is referred to [11].

Remark 4.1 To avoid problems arising from $0 \cdot (-\infty)$ and $-\infty + \infty$ we assume that all entries of x will be finite. Solutions with components equal to ε can be obtained by allowing some of the λ_k's or κ_k's in (3) to become infinite, but in a controlled way, since we only allow infinite components that are equal to $\varepsilon = -\infty$; components equal to $+\infty$ are not allowed.
The max operation hides small numbers from larger numbers. Therefore it suffices in practice to replace negative entries that are large enough in absolute value by ε provided that there are no positive entries of the same order of magnitude. This technique will be demonstrated in Example 6.2.

5 Minimal state space realization

In this section we shall use the results of the preceding sections to construct all equivalent minimal state space realizations of a max-linear discrete event system, given its impulse response. The procedure consists of two parts. First we determine a lower bound r for the minimal system order and we try to make it as tight as possible. In the second step we construct an ELCP that will yield the system matrices. This procedure will result in a compact description of the entire set of all possible state space realizations of the given impulse response.

5.1 Realization and minimal realization

Suppose that we have a single input single output (SISO) discrete event system that can be described by an nth order state space model:

$$x(k+1) \;=\; A \otimes x(k) \;\oplus\; B \otimes u(k) \tag{11}$$

$$y(k) \;=\; C \otimes x(k) \tag{12}$$

with $A \in \mathbb{R}_\varepsilon^{n \times n}$, $B \in \mathbb{R}_\varepsilon^{n \times 1}$ and $C \in \mathbb{R}_\varepsilon^{1 \times n}$. u is the input, y is the output and x is the state vector.

We define the unit impulse e as: $\begin{aligned} e(k) &= 0 \quad \text{if } k = 0\,, \\ &= \varepsilon \quad \text{otherwise}\;. \end{aligned}$

If we apply a unit impulse to the system and if we assume that the initial state $x(0)$ satisfies $x(0) = \varepsilon$ or $A \otimes x(0) \leqslant B$, we get the impulse response as the output of the system:

$$x(1) = B\,,\;\; x(2) = A \otimes B\,,\;\; \ldots\,,\;\; x(k) = A^{\otimes^{k-1}} \otimes B \quad \Rightarrow \quad y(k) = C \otimes A^{\otimes^{k-1}} \otimes B\;\;. \tag{13}$$

Let $g_k = C \otimes A^{\otimes^k} \otimes B$. The g_k's are called the *Markov parameters*.

Let us now reverse the process: suppose that A, B and C are unknown, and that we only know the Markov parameters (e.g. from experiments – where we assume that the system is max-linear and time-invariant and that there is no noise present). How can we construct A, B and C from the g_k's? This process is called realization. If we make the dimension of A minimal, we have a minimal realization. Although there have been some attempts to solve this problem [6, 14, 17] , this is – to the authors' knowledge – the first time it is solved entirely. It is certainly the first time that a complete description of the set of all minimal realizations of a SISO max-linear discrete event system is given.

5.2 Equivalent state space realizations

First we give some theorems on equivalent state space realizations. We shall again encounter these theorems when we look at the set of all equivalent state space realizations in Example 6.1.

Definition 5.1 (Equivalent state space realizations) *Two triples (A, B, C) and ($\tilde{A}$, $\tilde{B}$, $\tilde{C}$) are called equivalent if the corresponding state space models have the same impulse response, i.e.*

$$C \otimes A^{\otimes^k} \otimes B = \tilde{C} \otimes \tilde{A}^{\otimes^k} \otimes \tilde{B}\,, \qquad \forall k \geqslant 0\;\;.$$

Theorem 5.2 *Let $T \in \mathbb{R}_\varepsilon^{n\times n}$ be an invertible matrix. If the triple (A, B, C) is a realization of a max-linear time-invariant system then the triple $(T \otimes A \otimes T^{\otimes^{-1}}, T \otimes B, C \otimes T^{\otimes^{-1}})$ is an equivalent realization.*

Proof:

$$
\begin{aligned}
&(C \otimes T^{\otimes^{-1}}) \otimes \left(T \otimes A \otimes T^{\otimes^{-1}}\right)^{\otimes^k} \otimes (T \otimes B) \\
&\qquad = C \otimes T^{\otimes^{-1}} \otimes T \otimes A^{\otimes^k} \otimes T^{\otimes^{-1}} \otimes T \otimes B \\
&\qquad = C \otimes A^{\otimes^k} \otimes B \ .
\end{aligned}
$$

This theorem corresponds the max-algebraic equivalent of a similarity transformation. However, in contrast to linear algebra, the class of invertible matrices in $\mathbb{R}_{\max}$ is rather limited. It only consists of permuted diagonal matrices, i.e. matrices with only one non-ε entry on each row and on each column [5].

In contrast to linear systems, the entire set of nth equivalent state space realizations cannot be obtained solely by the similarity transformation of Theorem 5.2 as will be shown in Example 6.1.

Another way to construct equivalent state space realizations is the following:

Theorem 5.3 (*L*-transformation) *Let the triple (A, B, C) be a realization of a max-linear time-invariant system. Let $L \in \mathbb{R}_\varepsilon^{l\times n}$ be a common factor of A and C such that $A = \hat{A} \otimes L$ and $C = \hat{C} \otimes L$. Then the triple $(\tilde{A}, \tilde{B}, \tilde{C})$ with*

$$
\begin{aligned}
\tilde{A} &= L \otimes \hat{A}\,, && (14)\\
\tilde{B} &= L \otimes B\,, && (15)\\
\tilde{C} &= \hat{C}\ , && (16)
\end{aligned}
$$

is an equivalent realization.

Proof: See [11] or use a reasoning similar to the proof of Theorem 5.2.

We can also use a dual of this theorem:

Theorem 5.4 (*M*-transformation) *Let the triple (A, B, C) be a realization of a max-linear time-invariant system. Let $M \in \mathbb{R}_\varepsilon^{n\times l}$ be a common factor of A and B such that $A = M \otimes \hat{A}$ and $B = M \otimes \hat{B}$. Then the triple $(\tilde{A}, \tilde{B}, \tilde{C})$ with*

$$
\begin{aligned}
\tilde{A} &= \hat{A} \otimes M\,, && (17)\\
\tilde{B} &= \hat{B}\,, && (18)\\
\tilde{C} &= C \otimes M\ , && (19)
\end{aligned}
$$

is an equivalent realization.

So to get another equivalent state space model of a system with system matrices (A, B, C) all we have to do is find a decomposition

$$
\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} \hat{A} \\ \hat{C} \end{bmatrix} \otimes L \qquad \text{or} \qquad \begin{bmatrix} A & B \end{bmatrix} = M \otimes \begin{bmatrix} \hat{A} & \hat{B} \end{bmatrix}\ ,
$$

with $L \in \mathbb{R}_\varepsilon^{l \times n}$ or $M \in \mathbb{R}_\varepsilon^{n \times l}$. This matrix decomposition can be considered as a system of multivariate max-algebraic equalities [11] and can thus be solved using the techniques of Section 4.

The matrices L and M are not necessarily invertible (at least not in $\mathbb{R}_{\max}$) even if $l = n$, so in general L- and M- transformations are no similarity transformations.

But if we do the same operations in $\mathbb{S}_{\max}$, then L is invertible (provided that L is square and $\det L \not\nabla \varepsilon$). So $\hat{A} \nabla A \otimes L^{\otimes^{-1}}$ and then we have a similarity transformation since equations (14) – (16) can be transformed into

$$
\begin{aligned}
\tilde{A} &\quad \nabla \quad L \otimes A \otimes L^{\otimes^{-1}} , \\
\tilde{B} &\quad \nabla \quad L \otimes B , \\
\tilde{C} &\quad \nabla \quad C \otimes L^{\otimes^{-1}} .
\end{aligned}
$$

An analogous reasoning can be made for M-transformations.

In $\mathbb{R}_{\max}$ the M-transformation can be considered as the inverse of the L-transformation: if we can construct the triple (A_1, B_1, C_1) from the triple (A_2, B_2, C_2) with an L-transformation we can go back from (A_2, B_2, C_2) to (A_1, B_1, C_1) with an M-transformation with $M = L$ and with the same $\hat{A}$ as for the L-transformation.

However, as will be shown in Example 6.1 these L- and M-transformations don't yield the entire set of all equivalent state space realizations in one step.

Theorem 5.5 *If the triple (A, B, C) is a realization of a max-linear time-invariant system then the triple $(A, \alpha \otimes B, (-\alpha) \otimes C)$ with $\alpha \in \mathbb{R}$ is an equivalent realization.*

Proof: Apply Theorem 5.2 with $T = \alpha \otimes E_n$ and thus $T^{\otimes^{-1}} = \alpha^{\otimes^{-1}} \otimes E_n = (-\alpha) \otimes E_n$.

Theorem 5.6 *A state space realization of a max-linear time-invariant system is not isolated.*

Proof: If $B \neq \epsilon_{n1}$ and $C \neq \epsilon_{1n}$, then according to Theorem 5.5 we can find an equivalent realization in the neighborhood of (A, B, C): $(A, \delta \otimes B, (-\delta) \otimes C)$ with δ small enough.

If $B = \epsilon_{n1}$ or $C = \epsilon_{1n}$, the system is trivial since $\forall k \geqslant 0 : g_k = \varepsilon$ and so the A matrix is arbitrary. If $A \neq \epsilon_{nn}$ then $(\delta \otimes A, B, C)$ with δ small enough is an equivalent realization in the neighborhood of (A, B, C). If $A = \epsilon_{nn}$ then $((-\eta) \otimes E_n, B, C)$ with η large enough in absolute value is an equivalent realization in the neighborhood of (A, B, C).

5.3 A lower bound for the minimal system order

In this section we present a method to find a lower bound for the minimal system order. We shall use the following property:

Property 5.7 *Consider $A \in \mathbb{S}^{n \times n}$, $B \in \mathbb{S}^{n \times 1}$ and $C \in \mathbb{S}^{1 \times n}$. If A satisfies an equation of the form*

$$\bigoplus_{p=0}^{n} a_p \otimes A^{\otimes^{n-p}} \nabla \varepsilon \tag{20}$$

(e.g. its characteristic equation) then the Markov parameters satisfy

$$\bigoplus_{p=0}^{n} a_p \otimes g_{k+n-p} \nabla \varepsilon \quad \textit{for } k = 0\,, 1,\, 2,\, \ldots \ .$$

Suppose that we have a system that can be described by (11) − (12), with unknown system matrices. If we want to find a minimal realization of this system the first question that has to be answered is that of the minimal system order.

Consider the semi-infinite Hankel matrix $H = \begin{bmatrix} g_0 & g_1 & g_2 & \cdots \\ g_1 & g_2 & g_3 & \cdots \\ g_2 & g_3 & g_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$.

As a direct consequence of Theorem 1.6 and Property 5.7 we have that the columns of H satisfy

$$\bigoplus_{p=0}^{n} a_p \otimes H(:, k+n-p) \,\nabla\, \varepsilon \qquad \text{for } k = 1,\, 2,\, \ldots \tag{21}$$

where the coefficients a_p are the coefficients of the characteristic equation of the system matrix A. This leads to

Property 5.8 *Let $H_{\mathrm{sub},s} = H([i_1, i_2, \ldots, i_s], [j+1, j+2, \ldots, j+s])$ be an s by s square submatrix of the Hankel matrix H with arbitrary row indices and consecutive column indices. If $s > n$ then we have that* $\det(H_{\mathrm{sub},s}) \,\nabla\, \varepsilon$.

Proof: If A is an n by n matrix with entries in $\mathbb{R}_\varepsilon$ then according to Olsder's variant of the Cayley-Hamilton theorem [15], the coefficients in the characteristic equation of A are signed. This also means that the coefficients a_p in (21) are signed or that every balance of the form:

$$H([i_1, i_2, \ldots, i_s], [j+1, j+2, \ldots, j+s]) \otimes a \,\nabla\, \varepsilon$$

with $s > n, j \geqslant 0$ and $\{i_1, i_2, \ldots, i_s\} \in \mathcal{C}_\infty^s$ has a signed solution: if $s = n+1$ we get the coefficients of the characteristic equation as a solution and for $s > n+1$ we can always set some of the components of a equal to ε. Theorem 1.4 then leads to

$$\det\left(H([i_1, i_2, \ldots, i_s], [j+1, j+2, \ldots, j+s])\right) \,\nabla\, \varepsilon$$

for $s > n$.

So the dimension of the largest square submatrix of H with consecutive column indices that has a non-balanced determinant will be less than or equal to n. We represent this dimension as $\mathrm{r_{cc}}(H)$.

Definition 5.9 (Consecutive column rank) *Consider $P \in \mathbb{S}^{m \times n}$. The consecutive column rank of P, $\mathrm{r_{cc}}(P)$, is the dimension of the largest square submatrix of P with consecutive column indices, the determinant of which is not balanced:*

$$\begin{aligned} \mathrm{r_{cc}}(P) \;=\; \max\{\dim(P_{\mathrm{sub},s}) \,|\, & P_{\mathrm{sub},s} = P([i_1, i_2, \ldots, i_s], [j+1, j+2, \ldots, j+s]) \textit{ with} \\ & 0 \leqslant s \leqslant \min(m,n),\, 0 \leqslant j \leqslant n-s,\, \{i_1, i_2, \ldots, i_s\} \in \mathcal{C}_m^s \textit{ and } \det(P_{\mathrm{sub},s}) \,\not\nabla\, \varepsilon\} \;. \end{aligned}$$

We can define the consecutive row rank of P, $\mathrm{r_{cr}}(P)$, in an analogous way (in general $\mathrm{r_{cc}}(P) \neq \mathrm{r_{cr}}(P)$). But since we only consider symmetric matrices in this section, we only need the consecutive column rank: if $P = P^T$ then $\mathrm{r_{cc}}(P) = \mathrm{r_{cr}}(P)$. We have that $\mathrm{r_{cc}}(P) \leqslant \mathrm{r_{det}}(P)$.

To find a lower bound r for the minimal system order we shall search for a relation of the form (21) among the columns of H with a minimal number of terms. This number of terms will be a first estimate for the lower bound r. Since we know that the entries of the system matrix A belong to $\mathbb{R}_\varepsilon$ we shall search for coefficients a_p that correspond to a matrix with entries in $\mathbb{R}_\varepsilon$. So the coefficients should satisfy the necessary (and/or sufficient) conditions of Section 3.

This leads to the following procedure:

First we construct a p by q Hankel matrix

$$H_{p,q} = \begin{bmatrix} g_0 & g_1 & \cdots & g_{q-1} \\ g_1 & g_2 & \cdots & g_q \\ \vdots & \vdots & \ddots & \vdots \\ g_{p-1} & g_p & \cdots & g_{p+q-2} \end{bmatrix}$$

with p and q large enough: $p, q \gg n$, where n is the real (but unknown) system order. Now we try to find n and $a_0, a_1, \ldots, a_n$ such that the columns of $H_{p,q}$ satisfy an equation of the form (21).

We start with r equal to $\mathrm{r}_{\mathrm{cc}}(H_{p,q})$. Let

$$H_{\mathrm{sub},r} = H_{p,q}([i_1, i_2, \ldots, i_r], [j+1, j+2, \ldots, j+r])$$

be an r by r submatrix of $H_{p,q}$ the determinant of which is not balanced: $\det H_{\mathrm{sub},r} \not\nabla \varepsilon$. If we add one arbitrary row and the $(j+r+1)$-st column to $H_{\mathrm{sub},r}$ we get an $r+1$ by $r+1$ matrix $H_{\mathrm{sub},r+1}$ that has a balanced determinant. So according to Theorem 1.4 the system of linear balances

$$H_{\mathrm{sub},r+1} \otimes a \nabla \varepsilon$$

has a signed solution $a = \begin{bmatrix} a_r & a_{r-1} & \ldots & a_0 \end{bmatrix}^T$. We now look for a solution a that corresponds to the characteristic equation of a matrix with entries in $\mathbb{R}_\varepsilon$ (this should not necessarily be a signed solution; a signed solution would correspond to Olsder's variant of the characteristic equation). First of all we normalize a_0 to 0 and then we check if the necessary (and/or sufficient) conditions of Section 3 for the coefficients of the characteristic equation of a matrix with entries in $\mathbb{R}_\varepsilon$ are satisfied. If they are not satisfied we augment r and repeat the procedure. Note that even if the necessary conditions are satisfied we don't necessarily have coefficients that correspond to a matrix with entries in $\mathbb{R}_\varepsilon$.

However, we continue until we get the following stable relation among the columns of $H_{p,q}$:

$$H_{p,q}(:, k+r) \ \oplus \ a_1 \otimes H_{p,q}(:, k+r-1) \ \oplus \ \ldots \ \oplus \ a_r \otimes H_{p,q}(:, k) \nabla \varepsilon \tag{22}$$

for $k = 1, 2, \ldots, q-r$.

Since we assume that the system can be described by (11) – (12) and that p, $q \gg n$, we can always find such a stable relationship by gradually augmenting r. The r that results from this procedure is indeed a lower bound for the minimal system order, since it corresponds to the smallest number of terms in a relationship of form (21) among the columns of $H_{p,q}$.

5.4 Determination of the system matrices

Now we have to find $A \in \mathbb{R}_\varepsilon^{r\times r}$, $B \in \mathbb{R}_\varepsilon^{r\times 1}$ and $C \in \mathbb{R}_\varepsilon^{1\times r}$ such that

$$C \otimes A^{\otimes^k} \otimes B = g_k \,, \qquad \text{for } k = 0, 1, 2, \ldots \ . \tag{23}$$

In practice it seems that we only have to take the transient behavior and the first cycles of this steady-state behavior into account. So we may limit ourselves to the first, say, N Markov parameters.
For $k = 0$ we get

$$\bigoplus_{i=1}^{r} c_i \otimes b_i = g_0 \ .$$

For $k > 0$ we have that (23) is equivalent to

$$\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{r} t_{kij} = g_k \ ,$$

with

$$t_{kij} = \bigoplus_{i_1=1}^{r} \ldots \bigoplus_{i_{k-1}=1}^{r} c_i \otimes a_{ii_1} \otimes a_{i_1 i_2} \otimes \ldots \otimes a_{i_{k-1}j} \otimes b_j \ .$$

This can be rewritten as

$$\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{r} \bigoplus_{l=1}^{r^{k-1}} c_i \otimes \bigotimes_{u=1}^{r} \bigotimes_{v=1}^{r} {a_{uv}}^{\otimes^{\gamma_{kijluv}}} \otimes b_j = g_k \ ,$$

where γ_{kijluv} is the number of times that a_{uv} appears in the lth subterm of term t_{kij}. If a_{uv} doesn't appear in that subterm we take $\gamma_{kijluv} = 0$ since we have that $a^{\otimes^0} = 0.a = 0$, the identity element for $\otimes$. At first sight one could think that we are then left with r^{k+1} terms. However, some of these are the same and can thus be left out. If we use the fact that $x \otimes y \leqslant x \otimes x \oplus y \otimes y$ we can again remove many redundant terms. Then we are left with, say, w_k terms where $w_k \leqslant r^{k+1}$.
If we put all unknowns in one large vector x of size $r(r+2)$ we have to solve a system of multivariate polynomial equations of the following form:

$$\bigoplus_{i=1}^{r} \bigotimes_{j=1}^{r(r+2)} {x_j}^{\otimes^{\kappa_{0ij}}} = g_0$$

$$\bigoplus_{i=1}^{w_k} \bigotimes_{j=1}^{r(r+2)} {x_j}^{\otimes^{\kappa_{kij}}} = g_k \,, \qquad \text{for } k = 1\,, 2\,, \ldots, N-1 \ ,$$

and this can be transformed into an ELCP using the technique explained in Section 4. This leads to the following theorem:

Theorem 5.10 *In general all equivalent minimal state space realizations of a max-linear SISO system form a union of polyhedra in the x-space, where x is the vector obtained by putting the elements of the system matrices (A, B, C) in one large vector.*

If we find a solution x we extract the elements of x and put them in the matrices $A\,,B$ and C. Then we have found a minimal realization. If we don't find a solution we augment r, construct a new ELCP and try to solve it. Since we assumed that the data were generated by a max-linear SISO system we shall eventually find a realization and it will be minimal. The results of this Section 5.2 allow us to state that if the system can be described by (11) – (12) then the solution set of the ELCP will contain either no solutions (if r is smaller than the minimal system order) or infinitely many (if r is larger than or equal to the minimal system order).

By transforming the problem to linear algebra we have assumed that all components of A, B and C are finite. If we also want include matrices with components equal to ε we have to apply the procedure that was explained in Remark 4.1. This technique will be demonstrated in Example 6.2.

5.5 Computational complexity and algorithmic aspects

The execution time and the storage space requirement of the ELCP algorithm depend on the number of equations and variables. For the minimal realization problem the number of equations and variables becomes very large as the system order rises or as the number of Markov parameters that should be considered grows. Therefore the ELCP algorithm in its present form is not suited for large systems or for systems with a long and complex transient behavior. Moreover, we are not always interested in finding all minimal realizations. In [9] we have presented a heuristic algorithm that is relatively fast and that in most cases will find a minimal realization.

Since the method to solve the ELCP is an iterative process where in each step a new equation is taken into account, we can make use of the special structure of our problem to speed up the algorithm. To each Markov parameter there corresponds a group of linear inequalities. After each group we can test whether the impulse response of the solution up to that group matches the desired impulse response. If this is the case we don't have to take the other groups into account, since they will automatically be satisfied. This means that we can start with a small N and gradually take more and more groups into account. We don't have to start all over again for each new group since we can simply continue with the rays of the previous groups.

There are still some open problems. It is e.g. not clear how to determine the minimal subset of Markov parameters that is needed and how to select them. Since we have one group of inequalities for each Markov parameter that we take into consideration and since the computational complexity grows with the number of inequalities, it is important to use as few Markov parameters as possible. But if we take N too small we can get solutions with an impulse response that doesn't coincide entirely with the desired impulse response (only the first N Markov parameters are exactly the same). On the other hand we shall show in Example 6.2 that it is not always necessary to consider the entire set $\{g_0, g_1, \ldots, g_{N-1}\}$ to find all solutions with the desired impulse response: sometimes a subset of $\{g_0, g_1, \ldots, g_{N-1}\}$ suffices.

6 Examples

In this section we illustrate the procedure of the previous section with a few examples.

Example 6.1

Consider the system of Figure 1. This production system consists of 3 processing units P_1,

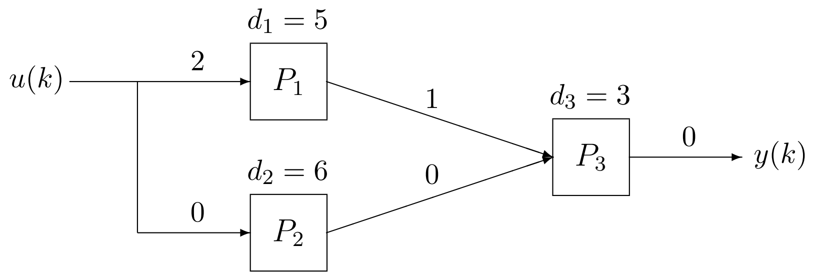

Figure 1: The production system of Example 6.1.

P_2 and P_3. Raw material is fed to P_1 and P_2, processed and sent to P_3 where assembly takes place. The processing times for P_1, P_2 and P_3 are $d_1 = 5$, $d_2 = 6$ and $d_3 = 3$ time units respectively. We assume that the raw material needs 2 time units to get from the input source to P_1 and that it takes 1 time unit for the finished product of processing unit P_1 to reach P_3. The other transportation times are assumed to be negligible. Between the processing units there are buffers with a capacity that is large enough to ensure that no buffer overflow will occur.
Define:

- $u(k)$: time instant at which raw material is fed to the system for the $k+1$st time,
- $x_i(k)$: time instant at which the ith processing unit starts working for the kth time,
- $y(k)$: time instant at which the kth finished product leaves the system.

A processing unit can only start working on a new product if it has finished processing the previous one. If we assume that each processing unit starts working as soon as all parts are available we get the following evolution equations for the system:

$$\begin{aligned}
x_1(k+1) &= \max(\, x_1(k) + 5,\, u(k) + 2\,) \\
x_2(k+1) &= \max(\, x_2(k) + 6,\, u(k)\,) \\
x_3(k+1) &= \max(\, x_1(k+1) + 5 + 1,\, x_2(k+1) + 6,\, x_3(k) + 3\,) \\
&= \max(\, x_1(k) + 11,\, x_2(k) + 12,\, x_3(k) + 3,\, u(k) + 8\,) \\
y(k) &= x_3(k) + 3 \ ,
\end{aligned}$$

or in max-algebraic matrix notation:

$$\begin{aligned}
x(k+1) &= \begin{bmatrix} 5 & \varepsilon & \varepsilon \\ \varepsilon & 6 & \varepsilon \\ 11 & 12 & 3 \end{bmatrix} \otimes x(k) \ \oplus \ \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} \otimes u(k) \\
y(k) &= \begin{bmatrix} \varepsilon & \varepsilon & 3 \end{bmatrix} \otimes x(k) \ ,
\end{aligned}$$

where $x(k) = [\, x_1(k) \;\; x_2(k) \;\; x_3(k) \,]^T$. Now we are going to construct all equivalent minimal state space realizations of this system starting from its impulse response, which is given by

$$\{g_k\} = 11,\ 16,\ 21,\ 27,\ 33,\ 39,\ 45,\ 51,\ 57,\ 63,\ 69,\ 75 \ldots \ .$$

First we construct the Hankel matrix

$$H_{6,6} = \begin{bmatrix} 11 & 16 & 21 & 27 & 33 & 39 \\ 16 & 21 & 27 & 33 & 39 & 45 \\ 21 & 27 & 33 & 39 & 45 & 51 \\ 27 & 33 & 39 & 45 & 51 & 57 \\ 33 & 39 & 45 & 51 & 57 & 63 \\ 39 & 45 & 51 & 57 & 63 & 69 \end{bmatrix} .$$

The consecutive column rank of $H_{6,6}$ is 2. The determinant of $H_{\mathrm{sub},2} = H_{6,6}([1,3],[1,2]) = \begin{bmatrix} 11 & 16 \\ 21 & 27 \end{bmatrix}$ is not balanced. We add one row and one column and then we look for a solution of the system of linear balances

$$\begin{bmatrix} 11 & 16 & 21 \\ 16 & 21 & 27 \\ 21 & 27 & 33 \end{bmatrix} \otimes \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} \nabla\ \varepsilon \ .$$

The solution $a_0 = 0$, $a_1 = \ominus 6$, $a_2 = 11$ satisfies the necessary and sufficient conditions for the coefficients of the characteristic polynomial of a 2 by 2 matrix with elements in $\mathbb{R}_\varepsilon$ since $\alpha_1 = \varepsilon$ and $\alpha_2 = 11 \leqslant 12 = 6 \otimes 6 = \beta_1 \otimes \beta_1$. This solution also corresponds to a stable relation among the columns of $H_{6,6}$:

$$H_{6,6}(:, k+2) \ \oplus\ 11 \otimes H_{6,6}(:, k) = 6 \otimes H_{6,6}(:, k+1) \qquad \text{for } k = 1,\ 2,\ 3,\ 4 \ .$$

Let's take $N = 5$. Using the ELCP algorithm of [10] we find the rays of Table 1 and the pairs of subsets of Table 2. If we take $N > 5$ we get the same result, but if we take $N < 5$ some combinations of the rays lead to a partial realization of the given impulse response (i.e. they only fit the first N Markov parameters).

Set	$\mathcal{X}^{\mathrm{cen}}$		$\mathcal{X}^{\mathrm{inf}}$						$\mathcal{X}^{\mathrm{fin}}$	
Ray	x_1^{c}	x_2^{c}	x_1^{i}	x_2^{i}	x_3^{i}	x_4^{i}	x_5^{i}	x_6^{i}	x_1^{f}	x_2^{f}
a_{11}	0	0	0	0	0	0	0	0	6	5
a_{12}	1	0	0	0	-1	0	0	-1	10	10
a_{21}	-1	0	0	0	0	0	0	0	0	0
a_{22}	0	0	0	0	0	0	0	0	5	6
b_1	1	-1	1	0	0	0	-1	-1	4	6
b_2	0	-1	0	0	0	0	0	0	0	0
c_1	-1	1	-1	0	0	-1	0	1	5	5
c_2	0	1	-1	-1	0	0	0	0	11	9

Table 1: The rays for Example 6.1 .

s	$\mathcal{X}_s^{\rm inf}$	$\mathcal{X}_s^{\rm fin}$	s	$\mathcal{X}_s^{\rm inf}$	$\mathcal{X}_s^{\rm fin}$
1	$\{x_1^{\rm i}, x_2^{\rm i}\}$	$\{x_2^{\rm f}\}$	5	$\{x_3^{\rm i}, x_6^{\rm i}\}$	$\{x_1^{\rm f}\}$
2	$\{x_1^{\rm i}, x_3^{\rm i}\}$	$\{x_2^{\rm f}\}$	6	$\{x_3^{\rm i}, x_6^{\rm i}\}$	$\{x_2^{\rm f}\}$
3	$\{x_2^{\rm i}, x_6^{\rm i}\}$	$\{x_2^{\rm f}\}$	7	$\{x_4^{\rm i}, x_5^{\rm i}\}$	$\{x_1^{\rm f}\}$
4	$\{x_3^{\rm i}, x_4^{\rm i}\}$	$\{x_1^{\rm f}\}$	8	$\{x_5^{\rm i}, x_6^{\rm i}\}$	$\{x_1^{\rm f}\}$

Table 2: The pairs of subsets for Example 6.1 .

Any arbitrary minimal realization can now be expressed as

$$\begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{bmatrix} = \lambda_1 x_1^{\rm c} + \lambda_2 x_2^{\rm c} + \kappa_1 x_{i_1}^{\rm i} + \kappa_2 x_{i_2}^{\rm i} + x_{j_1}^{\rm f} \ , \tag{24}$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$, $\kappa_1, \kappa_2 \geqslant 0$ and $x_{i_1}^{\rm i}, x_{i_2}^{\rm i} \in \mathcal{X}_s^{\rm inf}$, $x_{j_1}^{\rm f} \in \mathcal{X}_s^{\rm fin}$ with $s \in \{1, 2, \ldots, 8\}$. Expression (24) shows that the set of all equivalent minimal state space realizations of the given impulse response is a union of 8 unbounded polyhedra.

We shall now give an interpretation of this solution set in terms of the theorems on state space transformations of Section 5.2:
Ray $x_1^{\rm c}$ corresponds to a similarity transformation with $T_1 = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 0 \end{bmatrix}$. Ray $x_2^{\rm c}$ corresponds to the invariance of Theorem 5.5 or to a similarity transformation with $T_2 = \begin{bmatrix} -1 & \varepsilon \\ \varepsilon & -1 \end{bmatrix}$. Ray $x_2^{\rm f}$ can be obtained from $x_1^{\rm f}$ by a similarity transformation with $T_3 = \begin{bmatrix} \varepsilon & -4 \\ 6 & \varepsilon \end{bmatrix}$. So we could say that the set of combinations of the central rays and one of the finite rays:

$$\mathcal{S} = \left\{ x \;\middle|\; x = \lambda_1 x_1^{\rm c} + \lambda_2 x_2^{\rm c} + x_1^{\rm f} \text{ or } x = \lambda_1 x_1^{\rm c} + \lambda_2 x_2^{\rm c} + x_2^{\rm f} \text{ with } \lambda_1, \lambda_2 \in \mathbb{R} \right\} \ ,$$

corresponds to an entire class of 2nd order state space realizations that are linked by a similarity transformation. But in this way we can't construct the entire set of all possible 2nd order realizations since e.g. $x = x_1^{\rm f} + x_3^{\rm i}$ doesn't belong to $\mathcal{S}$. However, $x_1^{\rm f} + x_3^{\rm i}$ can be obtained from ray $x_1^{\rm f}$ by an L-transformation with e.g. $L = \begin{bmatrix} 0 & 4 \\ -6 & 0 \end{bmatrix}$, $\hat{A} = \begin{bmatrix} 6 & 9 \\ 0 & 5 \end{bmatrix}$ and $\hat{C} = \begin{bmatrix} 5 & 11 \end{bmatrix}$. The realization $x_1^{\rm f} + x_3^{\rm i}$ can be obtained from $x_1^{\rm f} + x_6^{\rm i}$ with an L-transformation and $x_1^{\rm f} + x_6^{\rm i}$ can be obtained from $x_1^{\rm f} + x_4^{\rm i}$ with an M-transformation, but it is impossible to transform $x_1^{\rm f} + x_4^{\rm i}$ into $x_1^{\rm f} + x_3^{\rm i}$ by an L- or an M-transformation. So starting from an

arbitrary realization, we can't get the set of all equivalent 2nd order state space realizations in one step by applying L- or M-transformations.
It is also impossible to find an L- or M-transformation that transforms the original 3rd order state space description into a 2nd order model.

Example 6.2

We start from the system (A, B, C) with

$$A = \begin{bmatrix} 3 & 1 & 0 \\ \varepsilon & 3 & 2 \\ 0 & 5 & \varepsilon \end{bmatrix} , \quad B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & \varepsilon & \varepsilon \end{bmatrix} . \tag{25}$$

The impulse response of this system is

$$\{g_k\} = 0,\ 3,\ 6,\ 9,\ 13,\ 16,\ 20,\ 23,\ 27,\ 30,\ 34,\ 37,\ \ldots\ .$$

Since there are two different alternating increments in steady state (3 and 4), we can't use the technique of [17], where only impulse responses that exhibit a uniformly up-terrace behavior are considered, i.e. impulse responses that consist of m sequences of length n_i such that

$$g_{j+1} - g_j = c_i\ , \qquad \text{for } j = t_i,\ t_i + 1,\ \ldots,\ t_i + n_i - 1 \ \text{ and for } i = 1,\ 2,\ \ldots,\ m\ ,$$

with $c_{i+1} > c_i$, $t_1 = 0$, $t_{i+1} = t_i + n_i$ and $n_m = +\infty$.
First we construct the Hankel matrix

$$H_{8,8} = \begin{bmatrix} 0 & 3 & 6 & 9 & 13 & 16 & 20 & 23 \\ 3 & 6 & 9 & 13 & 16 & 20 & 23 & 27 \\ 6 & 9 & 13 & 16 & 20 & 23 & 27 & 30 \\ 9 & 13 & 16 & 20 & 23 & 27 & 30 & 34 \\ 13 & 16 & 20 & 23 & 27 & 30 & 34 & 37 \\ 16 & 20 & 23 & 27 & 30 & 34 & 37 & 41 \\ 20 & 23 & 27 & 30 & 34 & 37 & 41 & 44 \\ 23 & 27 & 30 & 34 & 37 & 41 & 44 & 48 \end{bmatrix}$$

which has consecutive column rank 3. A 3 by 3 submatrix of $H_{8,8}$ the determinant of which is not balanced is $H_{\text{sub},3} = H_{8,8}([1,3,4],[1,2,3]) = \begin{bmatrix} 0 & 3 & 6 \\ 6 & 9 & 13 \\ 9 & 13 & 16 \end{bmatrix}$.
The system of linear balances

$$\begin{bmatrix} 0 & 3 & 6 & 9 \\ 3 & 6 & 9 & 13 \\ 6 & 9 & 13 & 16 \\ 9 & 13 & 16 & 20 \end{bmatrix} \otimes \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} \ \nabla\ \varepsilon$$

has a solution

$$a_0 = 0,\ a_1 = \ominus 3,\ a_2 = \ominus 7,\ a_3 = 10\ , \tag{26}$$

that satisfies the necessary and sufficient conditions of Section 3:

$$\begin{cases} \alpha_1 = \varepsilon \\ \alpha_2 = \varepsilon \leqslant 6 = 3 \otimes 3 = \beta_1 \otimes \beta_1 \\ \alpha_3 = 10 \leqslant 10 = 3 \otimes 7 = \beta_1 \otimes \beta_2 \ . \end{cases}$$

This solution also corresponds to a stable relation among the columns of $H_{8,8}$:

$$H_{8,8}(:, k+3) \ \oplus \ 10 \otimes H_{8,8}(:, k) = 3 \otimes H_{8,8}(:, k+2) \ \oplus \ 7 \otimes H_{8,8}(:, k+1)$$

for $k = 1, 2, \ldots, 5$. If we take $N = 7$ the ELCP algorithm yields the rays of Table 3 and the pairs of Table 4. So the set of all equivalent minimal realizations of system (25) consists of the union of 54 unbounded polyhedra.

If we take $N > 7$ we get the same results but for $N < 7$ some solutions only yield a partial realization of the given impulse response (i.e. they only fit the first N Markov parameters). The subset $\{g_0, g_1, g_3, g_4, g_5, g_6\}$ of $\{g_0, \ldots, g_7\}$ also leads to the same solutions as $\{g_0, \ldots, g_7\}$. So it is not always necessary to consider the entire set $\{g_0, \ldots, g_{N-1}\}$ to find all realizations of the given impulse response.

The original matrices (25) correspond to the following combination of the central rays and the rays of the pair $\left\{\mathcal{X}_{35}^{\text{inf}}, \mathcal{X}_{35}^{\text{fin}}\right\}$:

$$x_2^{\text{c}} + 2x_3^{\text{c}} + (\eta+2)x_2^{\text{i}} + (\eta+8)x_6^{\text{i}} + 6x_9^{\text{i}} + (\eta+7)x_{13}^{\text{i}} + \eta x_{15}^{\text{i}} + x_5^{\text{f}}$$

for η large enough. Then we get

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -(\eta+4) & 3 & 2 \\ 0 & 5 & -(\eta+4) \end{bmatrix} \ , \ \ B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \ , \ \ C = \begin{bmatrix} 0 & -(\eta+4) & -(\eta+4) \end{bmatrix} \ ,$$

and as explained in Remark 4.1 we can replace $-(\eta+4)$ by $\varepsilon = -\infty$ for η large enough since there are no positive components of the same order of magnitude as η. In fact for $\eta \to +\infty$ we would exactly get the matrices of (25).

In [8] we have solved the same example by constructing a matrix A such that the coefficients of its characteristic equation were equal to (26). There we found

$$A = \begin{bmatrix} 3 & \varepsilon & \varepsilon \\ 0 & \varepsilon & 7 \\ \varepsilon & 0 & \varepsilon \end{bmatrix} , \ B = \begin{bmatrix} 0 \\ -3 \\ \varepsilon \end{bmatrix} , \ C = \begin{bmatrix} 0 & 2 & \varepsilon \end{bmatrix} \ ,$$

which corresponds to the pair $\left\{\mathcal{X}_{52}^{\text{inf}}, \mathcal{X}_{52}^{\text{fin}}\right\}$:

$$-(\eta+7)x_2^{\text{c}} + 7x_3^{\text{c}} + (\eta+12)x_5^{\text{i}} + (\eta+10)x_8^{\text{i}} + (\eta+4)x_9^{\text{i}} + (\eta+16)x_{10}^{\text{i}} +$$
$$(\eta+10)x_{13}^{\text{i}} + \eta x_{14}^{\text{i}} + (\eta+12)x_{15}^{\text{i}} + x_5^{\text{f}}$$

with η large enough.

7 Conclusions and future research

First we have examined necessary (and/or sufficient) conditions for the coefficients of the characteristic polynomial of a matrix in the max algebra. We have shown that a system of multivariate polynomial equations in the max algebra can be transformed into an Extended Linear Complementarity Problem (ELCP). This means that we can use the ELCP algorithm of [10] to solve such a problem. Finally we have combined the previous results to find all minimal state space realizations of a single input single output discrete event system given its Markov parameters and illustrated the procedure with a few examples.

One of the main characteristics of the ELCP algorithm that was used in this paper is that it finds all solutions. For the minimal realization problem this provides a geometrical insight in all equivalent (minimal) realizations of an impulse response. On the other hand this also leads to large computation times and storage space requirements if the number of variables and equations is large. Therefore it might be interesting to develop (heuristic) algorithms that only find one solution as we have done for the minimal realization problem in [9].
Among the set of all possible realizations we could also try to find certain "privileged" realizations such as balanced realizations.
We hope to extend the method presented here to find minimal state space realizations for multiple input multiple output (MIMO) systems. The only problem there is the determination of the minimal system order. Once this is found the same technique can be used to get a minimal realization. In the future we shall therefore look for methods to get a estimate of the minimal system order of a MIMO system.

References

[1] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat, *Synchronization and Linearity.* New York: John Wiley & Sons, 1992.

[2] G. Cohen, D. Dubois, J.P. Quadrat, and M. Viot, "A linear-system-theoretic view of discrete-event processes and its use for performance evaluation in manufacturing," *IEEE Transactions on Automatic Control*, vol. 30, no. 3, pp. 210–220, Mar. 1985.

[3] R.W. Cottle and G.B. Dantzig, "A generalization of the linear complementarity problem," *Journal of Combinatorial Theory*, vol. 8, no. 1, pp. 79–90, Jan. 1970.

[4] R.W. Cottle, J.S. Pang, and R.E. Stone, *The Linear Complementarity Problem.* Boston, Massachusetts: Academic Press, 1992.

[5] R.A. Cuninghame-Green, *Minimax Algebra*, vol. 166 of *Lecture Notes in Economics and Mathematical Systems.* Berlin, Germany: Springer-Verlag, 1979.

[6] R.A. Cuninghame-Green, "Algebraic realization of discrete dynamic systems," in *Proceedings of the 1991 IFAC Workshop on Discrete Event System Theory and Applications in Manufacturing and Social Phenomena*, Shenyang, China, pp. 11–15, June 1991.

[7] B. De Moor, L. Vandenberghe, and J. Vandewalle, "The generalized linear complementarity problem and an algorithm to find all its solutions," *Mathematical Programming*, vol. 57, pp. 415–426, 1992.

[8] B. De Schutter and B. De Moor, "Minimal state space realization of SISO systems in the max algebra," Tech. rep. 93-57, ESAT-SISTA, K.U.Leuven, Leuven, Belgium, Oct. 1993.

[9] B. De Schutter and B. De Moor, "The characteristic equation and minimal state space realization of SISO systems in the max algebra," in *11th International Conference on*

Analysis and Optimization of Systems (Sophia-Antipolis, France, June 1994) (G. Cohen and J.P. Quadrat, eds.), vol. 199 of *Lecture Notes in Control and Information Sciences*, pp. 273–282, Springer, 1994.

[10] B. De Schutter and B. De Moor, "The extended linear complementarity problem," *Mathematical Programming*, vol. 71, no. 3, pp. 289–325, Dec. 1995.

[11] B. De Schutter and B. De Moor, "A method to find all solutions of a system of multivariate polynomial equalities and inequalities in the max algebra," *Discrete Event Dynamic Systems: Theory and Applications*, vol. 6, no. 2, pp. 115–138, Mar. 1996.

[12] S. Gaubert, *Théorie des Systèmes Linéaires dans les Dioïdes*. PhD thesis, Ecole Nationale Supérieure des Mines de Paris, France, July 1992.

[13] P. Moller, "Théorème de Cayley-Hamilton dans les dioïdes et application à l'étude des systèmes à événements discrets," in *Proceedings of the 7th International Conference on Analysis and Optimization of Systems* (Antibes, France), vol. 83 of *Lecture Notes in Control and Information Sciences*, pp. 215–226, Berlin, Germany: Springer-Verlag, 1986.

[14] G.J. Olsder and R.E. de Vries, "On an analogy of minimal realizations in conventional and discrete-event dynamic systems," in *Algèbres Exotiques et Systèmes à Evénements Discrets, CNRS/CNET/INRIA Seminar*, CNET, Issy-les-Moulineaux, France, pp. 193–213, June 1987.

[15] G.J. Olsder and C. Roos, "Cramer and Cayley-Hamilton in the max algebra," *Linear Algebra and Its Applications*, vol. 101, pp. 87–108, 1988.

[16] L. Vandenberghe, B. De Moor, and J. Vandewalle, "The generalized linear complementarity problem applied to the complete analysis of resistive piecewise-linear circuits," *IEEE Transactions on Circuits and Systems*, vol. 36, no. 11, pp. 1382–1391, Nov. 1989.

[17] L. Wang and X. Xu, "On minimal realization of SISO DEDS over max algebra," in *Proceedings of the 2nd European Control Conference*, Groningen, The Netherlands, pp. 535–540, June 1993.

Set	$\mathcal{X}^{\text{cen}}$			$\mathcal{X}^{\text{inf}}$								
Ray	x_1^{c}	x_2^{c}	x_3^{c}	x_1^{i}	x_2^{i}	x_3^{i}	x_4^{i}	x_5^{i}	x_6^{i}	x_7^{i}	x_8^{i}	x_9^{i}
a_{11}	0	0	0	0	0	0	0	0	0	-1	0	0
a_{12}	0	1	0	0	0	0	0	0	0	0	0	0
a_{13}	0	1	1	0	0	0	0	1	0	0	0	0
a_{21}	0	-1	0	0	0	0	0	-1	-1	0	0	0
a_{22}	0	0	0	0	0	0	0	0	0	0	-1	0
a_{23}	0	0	1	0	0	0	0	0	0	0	0	0
a_{31}	0	-1	-1	0	0	0	0	-1	0	0	0	-1
a_{32}	0	0	-1	0	0	0	0	0	0	0	0	0
a_{33}	0	0	0	0	0	0	0	0	0	0	0	0
b_1	-1	0	0	-1	0	-1	0	0	0	0	0	0
b_2	-1	-1	0	-1	0	0	-1	-1	0	0	0	0
b_3	-1	-1	-1	-1	0	0	0	-1	0	0	0	0
c_1	1	0	0	0	0	0	0	0	0	0	0	0
c_2	1	1	0	1	-1	0	0	1	0	0	0	0
c_3	1	1	1	1	0	0	0	1	0	0	0	0

Set	$\mathcal{X}^{\text{inf}}$						$\mathcal{X}^{\text{fin}}$					
Ray	x_{10}^{i}	x_{11}^{i}	x_{12}^{i}	x_{13}^{i}	x_{14}^{i}	x_{15}^{i}	x_1^{f}	x_2^{f}	x_3^{f}	x_4^{f}	x_5^{f}	x_6^{f}
a_{11}	0	0	0	0	0	0	3	3	3	3	3	3
a_{12}	0	0	0	0	0	0	0	0	0	0	0	0
a_{13}	-1	1	0	0	0	0	-4	-4	-2	-3	-3	-2
a_{21}	0	0	0	0	0	0	6	7	6	7	5	5
a_{22}	0	0	0	0	0	0	3	3	3	3	3	3
a_{23}	0	0	0	0	0	0	0	0	0	0	0	0
a_{31}	0	-1	0	0	0	0	9	9	9	9	9	9
a_{32}	0	-1	-1	0	0	0	7	6	5	5	7	6
a_{33}	0	0	0	-1	0	0	3	3	3	3	3	3
b_1	0	0	0	0	0	0	0	-1	-2	-2	0	-1
b_2	0	0	0	0	0	0	2	2	2	2	2	2
b_3	0	-1	0	0	-1	0	6	6	4	5	5	4
c_1	0	0	0	0	0	0	0	0	0	0	0	0
c_2	0	0	0	0	0	0	-4	-4	-2	-3	-3	-2
c_3	0	1	0	0	0	-1	-7	-6	-5	-5	-7	-6

Table 3: The rays for Example 6.2 .

s	$\mathcal{X}_s^{\mathrm{inf}}$	$\mathcal{X}_s^{\mathrm{fin}}$
1	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}\}$	$\{x^{\mathrm{f}}_{2}\}$
2	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_{4}\}$
3	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_{4}\}$
4	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}\}$	$\{x^{\mathrm{f}}_{2}\}$
5	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_{4}\}$
6	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_{3}\}$
7	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{13}\}$	$\{x^{\mathrm{f}}_{3}\}$
8	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{13}\}$	$\{x^{\mathrm{f}}_{3}\}$
9	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{6}\}$
10	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{13}\}$	$\{x^{\mathrm{f}}_{3}\}$
11	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{3}\}$
12	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_{3}\}$
13	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}\}$	$\{x^{\mathrm{f}}_{4}\}$
14	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_{4}\}$
15	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_{4}\}$
16	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{3}\}$
17	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_{4}\}$
18	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{13}\}$	$\{x^{\mathrm{f}}_{3}\}$
19	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{6}\}$
20	$\{x^{\mathrm{i}}_{1}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_{4}\}$
21	$\{x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_{2}\}$
22	$\{x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}\}$	$\{x^{\mathrm{f}}_{1}\}$
23	$\{x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{13}\}$	$\{x^{\mathrm{f}}_{1}\}$
24	$\{x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{13}\}$	$\{x^{\mathrm{f}}_{1}\}$
25	$\{x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{5}\}$
26	$\{x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{13}\}$	$\{x^{\mathrm{f}}_{1}\}$
27	$\{x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_{1}\}$
28	$\{x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{1}\}$
29	$\{x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_{2}\}$
30	$\{x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_{2}\}$
31	$\{x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_{2}\}$
32	$\{x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_{2}\}$
33	$\{x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{1}\}$
34	$\{x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{13}\}$	$\{x^{\mathrm{f}}_{1}\}$
35	$\{x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{5}\}$
36	$\{x^{\mathrm{i}}_{2}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_{2}\}$
37	$\{x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{13}\}$	$\{x^{\mathrm{f}}_{3}\}$
38	$\{x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{6}\}$
39	$\{x^{\mathrm{i}}_{3}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_{4}\}$
40	$\{x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{13}\}$	$\{x^{\mathrm{f}}_{1}\}$
41	$\{x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{5}\}$
42	$\{x^{\mathrm{i}}_{4}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}\}$	$\{x^{\mathrm{f}}_{2}\}$
43	$\{x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_{6}\}$
44	$\{x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{6}\}$
45	$\{x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{6}\}$
46	$\{x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}\}$	$\{x^{\mathrm{f}}_{5}\}$
47	$\{x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{5}\}$
48	$\{x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{5}\}$
49	$\{x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{5}\}$
50	$\{x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{6}\}$
51	$\{x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{6}\}$
52	$\{x^{\mathrm{i}}_{5}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{5}\}$
53	$\{x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{7}, x^{\mathrm{i}}_{11}, x^{\mathrm{i}}_{12}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{6}\}$
54	$\{x^{\mathrm{i}}_{6}, x^{\mathrm{i}}_{8}, x^{\mathrm{i}}_{9}, x^{\mathrm{i}}_{10}, x^{\mathrm{i}}_{13}, x^{\mathrm{i}}_{14}, x^{\mathrm{i}}_{15}\}$	$\{x^{\mathrm{f}}_{5}\}$

Table 4: The pairs of subsets for Example 6.2 .