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On state space realizations of impulse responses of max-linear systems*

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Abstract.

The topic of this paper is state space realization of impulse responses in the max-plus algebra, which is one of the modeling frameworks that can be used to model discrete event systems. We use the fact that a system of multivariate max-algebraic polynomial equalities and inequalities can be transformed into an Extended Linear Complementarity Problem to find all equivalent fixed order state space realizations of a multiple-input multiple-output max-linear discrete event system starting from its impulse response.

1 Introduction

1.1 Overview

Typical examples of which are flexible manufacturing systems, railroad traffic networks, logistic systems, parallel processing systems and telecommunication networks are typical examples of *discrete event systems*. There exists a wide range of frameworks to model and to analyze discrete event systems: Petri nets, generalized semi-Markov processes, formal languages, perturbation analysis, computer simulation, queueing theory, and so on. Although the description of discrete event systems is non-linear in conventional algebra, there exists a subclass of discrete event systems — the so-called *max-linear* discrete event systems — for which the description becomes "linear" when we formulate it in the max-plus algebra [1, 2]. In this paper we consider max-linear systems that can be described by a max-algebraic time-invariant state space model.

The basic operations of the max-plus algebra are maximization and addition. Although there are many analogies between max-plus algebra and linear algebra (many properties of linear algebra also hold when we replace addition by maximization, and multiplication by addition), there are also some major differences that prevent a straightforward translation of properties and algorithms from linear algebra to max-plus algebra. As a result many problems that can be solved rather easily in linear system theory are not that easy to solve in max-algebraic system theory. In this paper we address such a problem: state space realization of impulse responses.

This paper is organized as follows. In the remainder of this section we give a concise introduction to the max-plus algebra. In Section 2 we discuss the problem of solving a system of multivariate max-algebraic polynomial equalities and inequalities. In Section 3 we use the results of Section 2 to solve the (partial) state space realization problem for multiple-input multiple-output max-linear discrete event systems. We conclude with a worked example.

1.2 The max-plus algebra

In this section we give a short introduction to the max-plus algebra and state the definitions, theorems and properties we shall need in the remainder of this paper. For a more complete overview of the max-plus algebra the interested reader is referred to [1, 3]. The basic max-algebraic operations are defined as follows:

$$a \oplus b = \max(a, b)$$
$$a \otimes b = a + b$$

where $a, b \in \mathbb{R} \cup \{-\infty\}$. Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The structure $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ is called the *max-plus algebra*. Note that the zero element for $\oplus$ in $\mathbb{R}_\varepsilon$ is ε: $\forall a \in \mathbb{R}_\varepsilon : a \oplus \varepsilon = a = \varepsilon \oplus a$. Let $r \in \mathbb{R}$. The rth max-algebraic power of $a \in \mathbb{R}$ is denoted by $a^{\otimes^r}$ and corresponds to ra in linear algebra. So $a^{\otimes^0} = 0$. If $a \neq \varepsilon$ then $a^{\otimes^{-1}} = -a$ is the inverse element of a w.r.t. $\otimes$. There is no inverse element for ε since ε is absorbing for $\otimes$. If $r > 0$ then $\varepsilon^{\otimes^r} = \varepsilon$. If $r \leqslant 0$ then $\varepsilon^{\otimes^r}$ is not defined.

The basic max-algebraic operations are extended to matrices as follows. If A and B are m by n matrices with entries in $\mathbb{R}_\varepsilon$ then $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$ for all i, j. If $A \in \mathbb{R}_\varepsilon^{m \times p}$ and $B \in \mathbb{R}_\varepsilon^{p \times n}$ then $(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj}$ for all i, j.

The matrix E_n is the n by n max-algebraic identity matrix: $e_{ij} = 0$ if $i = j$, and $e_{ij} = \varepsilon$ if $i \neq j$. The m by n zero matrix in the max-plus algebra is denoted by $\mathcal{E}_{m \times n}$: $(\mathcal{E}_{m \times n})_{ij} = \varepsilon$ for all i, j. The off-diagonal entries of a max-algebraic diagonal matrix $D \in \mathbb{R}_\varepsilon^{m \times n}$ are equal to ε: $d_{ij} = \varepsilon$ for all i, j with $i \neq j$. If we permute the rows or the columns of the max-algebraic identity matrix, we obtain a max-algebraic permutation matrix.

Let $k \in \mathbb{N}$. The kth max-algebraic power of a matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ is defined recursively as follows:

$$A^{\otimes^k} = A^{\otimes^{k-1}} \otimes A \qquad \text{if } k > 0,$$
$$A^{\otimes^0} = E_n .$$

One of the major differences between conventional algebra and max-plus algebra is that in general there do not exist inverse elements w.r.t. $\oplus$ in $\mathbb{R}_{\max}$. This also means that in general matrices are not invertible either.

Proposition 1.1 *A matrix $T \in \mathbb{R}_\varepsilon^{n \times n}$ is invertible in the max-plus algebra (or max-invertible for short) if and only if it can be factorized as $T = D \otimes P$ where $D \in \mathbb{R}_\varepsilon^{n \times n}$ is a max-algebraic diagonal matrix with non-ε diagonal entries and $P \in \mathbb{R}_\varepsilon^{n \times n}$ is a max-algebraic permutation matrix.*

Proof: See [3]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

If D is a square max-algebraic diagonal matrix with non-ε diagonal entries then its max-algebraic inverse $D^{\otimes^{-1}}$ is a max-algebraic diagonal matrix with $(D^{\otimes^{-1}})_{ii} = -d_{ii}$ for all i. If P is a permutation matrix then $P^{\otimes^{-1}} = P^T$. If $T = D \otimes P$ then $T^{\otimes^{-1}} = P^{\otimes^{-1}} \otimes D^{\otimes^{-1}}$.

2   Systems of multivariate max-algebraic polynomial equalities and inequalities

In this section we consider systems of multivariate max-algebraic polynomial equalities and inequalities, which can be seen as a generalized framework for many important max-algebraic problems such as computing max-algebraic matrix decompositions, transformation of max-linear state space models, state space realization of impulse responses of max-linear discrete event systems, construction of matrices with a given max-algebraic characteristic polynomial and so on [9, 12]. In the first instance, the ELCP technique that we use to solve systems of multivariate max-algebraic polynomial equalities and inequalities only yields finite solutions and can only be used if all the right-hand side coefficients are finite. In this section we also discuss how solutions with infinite components can be retrieved and how one should deal with right-hand side coefficients that are not finite.
Consider the following problem:

Given a set of integers $\{m_k\}$ and three sets of real numbers $\{a_{ki}\}$, $\{b_k\}$ and $\{c_{kij}\}$ with $k = 1, 2, \ldots, p_1 + p_2$, $i = 1, 2, \ldots, m_k$ and $j = 1, 2, \ldots, n$, find $x \in \mathbb{R}^n$ such that

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes c_{kij}} = b_k \quad \text{for } k = 1, 2, \ldots, p_1 \tag{1}$$

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes c_{kij}} \leqslant b_k \quad \text{for } k = p_1 + 1, p_1 + 2, \ldots, p_1 + p_2 \tag{2}$$

or show that no such x exists.

We call $(1)-(2)$ a system of *multivariate max-algebraic polynomial equalities and inequalities*. Note that the exponents can be negative or real.

In [12] we have shown that this problem is equivalent to an Extended Linear Complementarity Problem (ELCP) [8]. This leads to an algorithm that yields the entire solution set of problem $(1)-(2)$. This solution set can be described in terms of linear algebra concepts as follows: in general it consists of the union of faces of a polyhedron $\mathcal{P}$ and is defined by three sets of vectors $\mathcal{X}^{\mathrm{cen}}$, $\mathcal{X}^{\mathrm{ext}}$, $\mathcal{X}^{\mathrm{fin}}$ and a set Λ. These sets can be characterized as follows:

- $\mathcal{X}^{\mathrm{cen}}$ is the set of "central generators" of $\mathcal{P}$. It is a basis for the linear subspace $\mathcal{L}(\mathcal{P})$ associated with the largest affine subspace of $\mathcal{P}$.

- There exists a pointed polyhedron $\mathcal{P}_{\mathrm{red}}$ such that $\mathcal{P} = \mathcal{P}_{\mathrm{red}} + \mathcal{L}(\mathcal{P})$ and such that the elements of $\mathcal{X}^{\mathrm{ext}}$ are extreme generators or "vertices at infinity" of $\mathcal{P}_{\mathrm{red}}$ and such that the elements of $\mathcal{X}^{\mathrm{fin}}$ are finite points of $\mathcal{P}_{\mathrm{red}}$.

- Λ is a set of pairs $\left\{\mathcal{X}_s^{\mathrm{ext}}, \mathcal{X}_s^{\mathrm{fin}}\right\}$ with $\mathcal{X}_s^{\mathrm{ext}} \subset \mathcal{X}^{\mathrm{ext}}$, $\mathcal{X}_s^{\mathrm{fin}} \neq \emptyset$ and $\mathcal{X}_s^{\mathrm{fin}} \subset \mathcal{X}^{\mathrm{fin}}$. Each pair determines a face $\mathcal{F}_s$ of the polyhedron $\mathcal{P}$ that belongs to the solution set: $\mathcal{X}_s^{\mathrm{ext}}$ contains the extreme generators of $\mathcal{F}_s$, if any, and $\mathcal{X}_s^{\mathrm{fin}}$ contains the finite vertices of $\mathcal{F}_s$. We call Λ the set of *pairs of maximal cross-complementary subsets* of $\mathcal{X}^{\mathrm{ext}}$ and $\mathcal{X}^{\mathrm{fin}}$.

The solution set of problem $(1)-(2)$ is characterized by the following theorems:

Theorem 2.1 *When $\mathcal{X}^{\mathrm{cen}}$, $\mathcal{X}^{\mathrm{ext}}$, $\mathcal{X}^{\mathrm{fin}}$ and Λ are given, then x is a (finite) solution of the system of multivariate max-algebraic polynomial equalities and inequalities if and only if there exists a pair $\left\{\mathcal{X}_s^{\mathrm{ext}}, \mathcal{X}_s^{\mathrm{fin}}\right\} \in \Lambda$ such that*

$$x = \sum_{x_k \in \mathcal{X}^{\mathrm{cen}}} \lambda_k x_k + \sum_{x_k \in \mathcal{X}_s^{\mathrm{ext}}} \kappa_k x_k + \sum_{x_k \in \mathcal{X}_s^{\mathrm{fin}}} \mu_k x_k \tag{3}$$

with $\lambda_k \in \mathbb{R}$, $\kappa_k, \mu_k \geqslant 0$ and $\displaystyle\sum_k \mu_k = 1$.

Theorem 2.2 *The general solution set of a system of multivariate max-algebraic polynomial equalities and inequalities consists of the union of faces of a polyhedron.*

Proposition 2.3 *Let $\mathcal{S}$ be a system of multivariate max-algebraic polynomial equalities and inequalities with finite right-hand sides. If there exists a solution x of $\mathcal{S}$, then there also exists a solution $\tilde{x}$ of $\mathcal{S}$ with finite components.*

Proof: If all the components of x are finite, we set $\tilde{x} = x$ and then $\tilde{x}$ is a finite solution of $\mathcal{S}$.

From now on we assume that x has at least one component that is equal to ε. Suppose that $\mathcal{S}$ is defined by $(1)-(2)$. Define $\Psi = \{\, j \,|\, x_j = \varepsilon \,\}$ and $\Psi^{\mathrm{c}} = \{1, 2, \ldots, n\} \setminus \Psi$. Since negative max-algebraic powers of ε are not defined, x can only be a solution of $\mathcal{S}$ if $c_{kij} \geqslant 0$ for all k, i and all $j \in \Psi$.

Now we define $\tilde{x} \in \mathbb{R}^n$ such that

$$\tilde{x}_j = \begin{cases} x_j & \text{if } j \in \Psi^{\mathrm{c}}, \\ M & \text{if } j \in \Psi, \end{cases}$$

where M is a real number the exact value of which will be determined later on: we shall select the value of M such that $\tilde{x}$ will be a solution of $\mathcal{S}$.

Let us now determine under which conditions $\tilde{x}$ will be a solution of $\mathcal{S}$. Since $c_{kij} \geqslant 0$ for all k, i and all $j \in \Psi$, the left-hand sides of the system $(1)-(2)$ can only increase if we replace x by $\tilde{x}$. Now we determine conditions on M such that the left-hand sides do not increase if we replace x by $\tilde{x}$. If we select M such that

$$a_{ki} \otimes \bigoplus_{j=1}^{n} \tilde{x}_j^{\otimes^{c_{kij}}} \leqslant b_k \tag{4}$$

for all k, i, then the left-hand sides of $(1)-(2)$ will not change if we replace x by $\tilde{x}$, and then $\tilde{x}$ will be a solution of $\mathcal{S}$.

Consider arbitrary indices k and i. Since all the components of $\tilde{x}$ are finite, (4) can be rewritten as

$$a_{ki} + \sum_{j \in \Psi^{c}} c_{kij}\tilde{x}_j + \sum_{j \in \Psi} c_{kij}\tilde{x}_j \leqslant b_k \ ,$$

which is in its turn equivalent to

$$a_{ki} + \sum_{j \in \Psi^{c}} c_{kij}x_j + \sum_{j \in \Psi} c_{kij}M \leqslant b_k \ . \tag{5}$$

If $\sum_{j \in \Psi} c_{kij}$ is equal to 0, then $c_{kij} = 0$ for all $j \in \Psi$ since $c_{kij} \geqslant 0$ for all $j \in \Psi$. So in that case we have $\bigoplus_{j \in \Psi} \tilde{x}_j^{\otimes^{c_{kij}}} = 0 = \bigoplus_{j \in \Psi} x_j^{\otimes^{c_{kij}}}$ and since x is a solution of $\mathcal{S}$, this means that condition (4) is satisfied.

From now on we assume that $\sum_{j \in \Psi} c_{kij} \neq 0$. Hence, $\sum_{j \in \Psi} c_{kij} > 0$. As a consequence, condition (5) can be rewritten as

$$M \leqslant \frac{b_k - a_{ki} - \sum_{j \in \Psi^{c}} c_{kij}x_j}{\sum_{j \in \Psi} c_{kij}} \tag{6}$$

The right-hand side of this expression is defined since b_k, a_{ki} and $\sum_{j \in \Psi^{c}} c_{kij}x_j$ are finite and since $\sum_{j \in \Psi} c_{kij} \neq 0$.

A sufficient condition for (4) to hold is that (6) is satisfied for all k, i for which $\sum_{j \in \Psi} c_{kij} \neq 0$.

If $\sum_{j \in \Psi} c_{kij} = 0$ for all k, i, then we may choose an arbitrary value for M, e.g., $M = 0$. So if

we select M such that

$$M = \min\left(\left\{\left.\frac{b_k - a_{ki} - \sum_{j \in \Psi^c} c_{kij} x_j}{\sum_{j \in \Psi} c_{kij}}\right| \sum_{j \in \Psi} c_{kij} \neq 0\right\} \cup \{0\}\right), \qquad (7)$$

then $\tilde{x}$ is a solution of $\mathcal{S}$. Since the right-hand side of (7) is finite, M is finite. As a consequence, the components of $\tilde{x}$ are also finite. $\qquad \square$

In order to be able to apply the ELCP technique to solve $(1)-(2)$, we have to assume that x contains only finite components. However, solutions with components that are equal to ε can be retrieved by applying a limit argument in which we allow some of the λ_k's or κ_k's in (3) to become infinite, but in a controlled way, since we only allow infinite components that are equal to ε and since negative powers of ε are not defined. Solutions obtained in this way will correspond to points at infinity of the polyhedron $\mathcal{P}$ defined by the system of linear equalities and inequalities of the ELCP that corresponds to the system $(1)-(2)$. A formal justification and an example of this limit technique can be found in [6].

Remark 2.4 Let $\mathcal{S}$ be a system of multivariate max-algebraic polynomial equalities and inequalities of the form $(1)-(2)$. Let $\Upsilon = \{\, j \,|\, c_{kij} \geqslant 0 \text{ for all } k, i \,\}$.
If some of the b_k's are equal to ε, then $\mathcal{S}$ cannot have finite solutions. However, we can still use the ELCP approach to solve $\mathcal{S}$ if we use the following procedure. We introduce a positive real number ξ and we transform every equation of the form $\bigoplus_i t_i = \varepsilon$ or $\bigoplus_i t_i \leqslant \varepsilon$ into

$\bigoplus_i t_i \leqslant -\xi$. Now we have a system $\mathcal{S}(\xi)$ of multivariate max-algebraic polynomial equalities and inequalities with finite right-hand sides that can be solved using the ELCP approach. If we let ξ go to ∞ and if we see how the solution set of the intermediate ELCPs evolves, we obtain the solutions of $\mathcal{S}$.
It can be shown [6] that if ξ is large enough the components of the finite points of the solution set of the intermediate ELCPs will depend affinely on ξ, i.e., if ξ is large enough then the ith component of any finite point $x(\xi)$ of $\mathcal{S}(\xi)$ can be written as $x_i(\xi) = a_i\xi + b_i$ for some $a_i, b_i \in \mathbb{R}$. Furthermore, if ξ is large enough then the solution set of all the intermediate ELCPs can be described by the same sets of central and extreme generators (see [6]). So in order to determine how the solution set of the intermediate ELCPs evolves as ξ tends to ∞, we only have to solve a *finite* number of intermediate ELCPs: we solve intermediate ELCPs for some values $\xi_1, \xi_2, \ldots, \xi_r$ of ξ until we notice that from a certain value of ξ on the minimal complete sets of central and extreme generators do not change any more and the components of the finite points depend affinely on ξ.
Note that we have to take care that in this way we do not create solutions with components that are equal to ∞ or solutions with components that are equal to ε but that are not indexed by Υ. Sometimes it is useful to normalize the representation of the solution set of the intermediate ELCPs in order to be able to see how the solution set evolves as ξ increases (see also Section 3.3).
Since the $\oplus$ operation hides small numbers from larger numbers, we could also use the following threshold procedure. First we select a positive real number ξ that is several orders of

magnitude larger than $\dfrac{\alpha + \beta + \gamma}{\delta}$ where

$$\alpha = \max\{\, |a_{ki}| \,\big|\, a_{ki} \text{ is finite} \,\}$$
$$\beta = \max\{\, |b_k| \,\big|\, b_k \text{ is finite} \,\}$$
$$\gamma = \max\{\, |c_{kij}| \,\big|\, c_{kij} \text{ is finite} \,\}$$
$$\delta = \min\{\, c_{kij} \,|\, c_{kij} > 0 \,\} \ .$$

This heuristic rule for selecting ξ is based on expression (7).

Once we have found a solution x of $\mathcal{S}(\xi)$, we replace every negative component of x that has the same order of magnitude as ξ and that is not bounded from below by ε provided that in this way only components indexed by Υ are replaced by ε and provided that x has no positive components of the same order of magnitude as ξ. Note that positive components of the same order of magnitude as ξ would have to be replaced by ∞, but ∞ does not belong to $\mathbb{R}_\varepsilon$. If one uses this threshold technique, it is advisable to check whether the resulting solutions are truly solutions of $\mathcal{S}$ since it is possible that wrong results are obtained if ξ is not large enough. An example of the application of both the limit and the threshold technique that have been discussed in this remark can be found in [6]. $\diamond$

3 State space realization of impulse responses of max-linear discrete event systems

3.1 Some extra definitions

Consider a discrete event system that can be described by the following nth order state space model:

$$x(k + 1) = A \otimes x(k) \ \oplus \ B \otimes u(k) \tag{8}$$
$$y(k) = C \otimes x(k) \tag{9}$$

with $A \in \mathbb{R}_\varepsilon^{n\times n}$, $B \in \mathbb{R}_\varepsilon^{n\times m}$ and $C \in \mathbb{R}_\varepsilon^{l\times n}$. The vector x represents the state, u the input vector and y the output vector of the system. A discrete event system the behavior of which can be modeled by a description of the form $(8) - (9)$ will be called a *max-linear time-invariant discrete event system*.

A max-algebraic unit impulse is a sequence that is defined as follows:

$$e(k) = \begin{cases} 0 & \text{if } k = 0\,, \\ \varepsilon & \text{if } k \neq 0\,, \end{cases}$$

for $k = 0, 1, 2, \ldots$ Let $i \in \{1, 2, \ldots, m\}$. If we apply a max-algebraic unit impulse to the ith input of the system and if we assume that $x(0) = \mathcal{E}_{n\times 1}$, then we get

$$y(k) = C \otimes A^{\otimes^{k-1}} \otimes B_{.,i} \qquad \text{for } k = 1, 2, 3, \ldots$$

as the output of the discrete event system. This output is called the impulse response due to a max-algebraic impulse at the ith input. Note that $y(k)$ corresponds to the ith column of the matrix $G_{k-1} \overset{\text{def}}{=} C \otimes A^{\otimes^{k-1}} \otimes B$ for $k = 1, 2, 3, \ldots$ Therefore, the sequence $\{G_k\}_{k=0}^\infty$

is called the *impulse response* of the discrete event system. The G_k's are called the *impulse response matrices*.

Suppose that the system matrices A, B and C of the system are unknown, and that we only know the impulse response $\{G_k\}_{k=0}^{\infty}$ of the system. The problem of constructing the system matrices A, B and C from the impulse response is $\{G_k\}_{k=0}^{\infty}$ is called the *state space realization* problem. The smallest possible dimension of the system matrix A over all possible state space realizations of the given impulse response is called the *minimal system order*, and the corresponding triple (A, B, C) is a called a *minimal state space realization*.

The minimal state space realization problem for max-linear time-invariant discrete event systems has been studied by many authors and for some specific cases the problem has been solved [4, 5, 13, 14, 17, 18, 19, 20, 21, 22, 23]. Related results can be found in [24, 25].

In this paper we shall present a method that will always result in a minimal state space realization. If we use the ELCP algorithm of [8] to solve the resulting system of multivariate max-algebraic polynomial equalities and inequalities, we can — at least theoretically — compute *all* the minimal state space realizations of a given impulse response. Moreover, our method works for both single-input single-output (SISO) and multiple-input multiple-output (MIMO) systems. However, the major drawback of our method is that at this moment there are no efficient, polynomial time algorithms available to solve some of the subproblems encountered in this approach.

Definition 3.1 (Ultimately geometric impulse response) *Let $\{G_k\}_{k=0}^{\infty}$ be the impulse response of a max-linear time-invariant discrete event system. If*

$$\exists k_0 \in \mathbb{N},\ \exists c \in \mathbb{N}_0,\ \exists \lambda \in \mathbb{R}_\varepsilon \ \text{ such that } \forall k \geqslant k_0\ :\ G_{k+c} = \lambda^{\otimes^c} \otimes G_k\ , \qquad (10)$$

then we say that the impulse response $\{G_k\}_{k=0}^{\infty}$ is ultimately geometric.

The term "ultimately geometric" has been introduced by Gaubert in [14, 15]. It can be shown (see, e.g., [1, 2, 14]) that a sufficient — but not necessary — condition for the impulse response of a discrete event system described by $(8)-(9)$ to be ultimately geometric is that the system matrix A is irreducible, i.e., $(A \oplus A^{\otimes^2} \oplus \ldots \oplus A^{\otimes^n})_{ij} \neq \varepsilon$ for all i, j. This will be the case for a discrete event system without separate independent subsystems and with a cyclic behavior or with feedback from the output to the input (such as, e.g., a flexible production system in which the parts are carried around on a limited number of pallets that circulate in the system [2]).

In general, the impulse response of a max-linear time-invariant discrete event system can be characterized by the following theorem:

Theorem 3.2 *If $\{G_k\}_{k=0}^{\infty}$ is the impulse response of a max-linear time-invariant discrete event system with m inputs and l outputs then we have*

$$\forall i \in \{1, 2, \ldots, l\},\ \forall j \in \{1, 2, \ldots, m\},\ \exists c \in \mathbb{N}_0,$$

$$\exists \lambda_1, \lambda_2, \ldots, \lambda_c \in \mathbb{R}_\varepsilon,\ \exists k_0 \in \mathbb{N} \ \text{ such that } \forall k \geqslant k_0\ :$$

$$(G_{kc+c+s-1})_{ij} = \lambda_s^{\otimes^c} \otimes (G_{kc+s-1})_{ij}\ \text{ for } s = 1, 2, \ldots, c. \qquad (11)$$

Proof: This is a direct consequence of, e.g., Corollary 1.1.9 of [13, p. 166] or of Proposition 1.2.2 of [14]. □

If a sequence $\{G_k\}_{k=0}^{\infty}$ exhibits a behavior of the form (11) then we say that the sequence is *ultimately periodic*.

Proposition 3.3 *A sequence $\{G_k\}_{k=0}^{\infty}$ with $G_k \in \mathbb{R}_{\varepsilon}^{l \times m}$ for all k is the impulse response of a max-linear time-invariant discrete event system if and only if it is an ultimately periodic sequence.*

Proof: A proof of this proposition for SISO systems can be found in, e.g., [1, 13, 13]. The extension to MIMO systems is straightforward (see also [6]). $\qquad\square$

For linear time-invariant systems we can use similarity transformations to obtain equivalent state space realizations (see, e.g., [16]). The following proposition shows that we can use max-algebraic similarity transformations to obtain equivalent state space realizations for max-linear time-invariant discrete event systems.

Proposition 3.4 (Max-algebraic similarity transformation) *Let $T \in \mathbb{R}_{\varepsilon}^{n \times n}$ be max-invertible. If (A, B, C) is an nth order state space realization of the impulse response of a max-linear time-invariant discrete event system then $(T \otimes A \otimes T^{\otimes^{-1}}, T \otimes B, C \otimes T^{\otimes^{-1}})$ is an equivalent realization.*

Let $G = \{G_k\}_{k=0}^{\infty}$ be an ultimately periodic sequence with $G_k \in \mathbb{R}_{\varepsilon}^{l \times m}$ for all k. So by Proposition 3.3 G is the impulse response of a max-linear time-invariant discrete event system. We shall construct state space realizations of G in two steps. First we determine all the minimal state space realizations of a finite subsequence of G (i.e., we solve the *partial state space realization* problem) and then we construct all the minimal state space realizations of the full sequence.

We assume that the max-linear system can be described by an rth order state space system (see, e.g., [13, 14] for methods to determine lower and upper bounds for the minimal system order).

3.2 Partial state space realization

First we consider the partial state space realization problem: we try to determine a state space realization that fits the first N terms of the sequence $G = \{G_k\}_{k=0}^{\infty}$ for some $N \in \mathbb{N}_0$. So we have to find $A \in \mathbb{R}_{\varepsilon}^{r \times r}$, $B \in \mathbb{R}_{\varepsilon}^{r \times m}$ and $C \in \mathbb{R}_{\varepsilon}^{l \times r}$ such that

$$C \otimes A^{\otimes^k} \otimes B = G_k \qquad \text{for } k = 0, 1, \dots, N-1 \,. \tag{12}$$

If we write out the equations of the form (12), we get

$$\bigoplus_{p=1}^{r} c_{ip} \otimes b_{pj} \;=\; (G_0)_{ij}$$

for $i = 1, 2, \dots, l$ and $j = 1, 2, \dots, m$, and

$$\bigoplus_{p=1}^{r} \bigoplus_{q=1}^{r} c_{ip} \otimes (A^{\otimes^k})_{pq} \otimes b_{qj} \;=\; (G_k)_{ij} \tag{13}$$

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for $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, N - 1$.

Since

$$(A^{\otimes^k})_{pq} = \bigoplus_{i_1=1}^{r} \bigoplus_{i_2=1}^{r} \cdots \bigoplus_{i_{k-1}=1}^{r} a_{pi_1} \otimes a_{i_1 i_2} \otimes \ldots \otimes a_{i_{k-1} q} \;,$$

equation (13) can be rewritten as

$$\bigoplus_{p=1}^{r} \bigoplus_{q=1}^{r} \bigoplus_{s=1}^{r^{k-1}} c_{ip} \otimes \bigotimes_{u=1}^{r} \bigotimes_{v=1}^{r} a_{uv}^{\otimes^{\gamma_{kpqsuv}}} \otimes b_{qj} = (G_k)_{ij} \qquad (14)$$

where γ_{kpqsuv} is the number of times that a_{uv} appears in the sth term of $(A^{\otimes^k})_{pq}$. Note that if a_{uv} does not appear in that term we have $\gamma_{kpqsuv} = 0$ since $a^{\otimes^0} = 0 \cdot a = 0$, which is the identity element for $\otimes$. If we use the fact that $x \oplus x = x$ and $x \otimes y \leqslant x \otimes x \oplus y \otimes y$ for all $x, y \in \mathbb{R}_\varepsilon$, we can remove many redundant terms. Suppose that after removing the redundant terms, there are w_{kij} terms left in (14). Note that $w_{kij} \leqslant r^{k+1}$.

If we put the entries of A, B and C in one large vector x of length $r(r + m + l)$, we have to solve a system of multivariate max-algebraic polynomial equations of the following form:

$$\bigoplus_{p=1}^{r} \bigotimes_{q=1}^{r(r+m+l)} x_q^{\otimes^{\delta_{0ijpq}}} = (G_0)_{ij} \qquad (15)$$

$$\bigoplus_{p=1}^{w_{kij}} \bigotimes_{q=1}^{r(r+m+l)} x_q^{\otimes^{\delta_{kijpq}}} = (G_k)_{ij} \qquad (16)$$

for $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, N - 1$. If all the impulse response matrices have finite entries then it follows from Proposition 2.3 that $(15) - (16)$ always has a finite solution. This solution can be found using the ELCP approach. If some impulse response matrices have entries that are equal to ε, we can also use the ELCP approach to solve the system of max-algebraic polynomial equalities $(15) - (16)$ if we apply the threshold or the limit procedure described in Remark 2.4. Note that all the exponents in $(15) - (16)$ are nonnegative. Once we have found a solution of $(15) - (16)$, we extract the entries of the system matrices A, B and C from x. This results in a partial realization of the given impulse response.

The set of all the rth order state space realizations of the first N terms of the impulse response $G = \{G_k\}_{k=0}^{\infty}$ will be denoted by $\mathcal{R}_r(G, N)$. So

$$\mathcal{R}_r(G, N) = \Big\{ (A, B, C) \, \Big| \, A \in \mathbb{R}_\varepsilon^{r \times r}, B \in \mathbb{R}_\varepsilon^{r \times m}, C \in \mathbb{R}_\varepsilon^{l \times r} \text{ and}$$

$$C \otimes A^{\otimes^k} \otimes B = G_k \text{ for } k = 0, 1, \ldots, N - 1 \Big\} \;.$$

3.3 Realizations of the Entire Impulse Response

Suppose that $\mathcal{R}_r(G, N)$ is nonempty for all $N \in \mathbb{N}_0$. If we want to find all rth order state space realizations of G, we have to determine $\lim_{N \to \infty} \mathcal{R}_r(G, N)$. When N becomes larger and larger, there are two possible situations that can occur:

(1) there exists an index $N_0 \in \mathbb{N}_0$ such that $\mathcal{R}_r(G, N) = \mathcal{R}_r(G, N_0)$ for all $N \geqslant N_0$;

(2) the sequence $\{\mathcal{R}_r(G,N)\}_{N=1}^\infty$ does not become stationary after a finite number of terms.

The first situation typically occurs if G is ultimately geometric. Unfortunately, it is not obvious how N_0 can be determined without explicitly computing the terms of the sequence $\{\mathcal{R}_r(G,N)\}_{N=1}^\infty$. Therefore, we start with an arbitrary integer $\tilde{N} \in \mathbb{N}$ and we construct the sequence $\mathcal{R}_r(G,\tilde{N})$, $\mathcal{R}_r(G,\tilde{N}+1)$, $\mathcal{R}_r(G,\tilde{N}+2)$, ... and we check whether this sequence becomes stationary from a certain index $\tilde{N}_0$ on. It is obvious that we have to take our estimate of $\tilde{N}$ large enough. In practice it appears that we should at least include the transient behavior and the first cycles of the geometric behavior.

Case (2) occurs if G is not ultimately geometric since then G cannot be realized by a triple (A,B,C) with an irreducible A matrix. So A has to contain entries that are equal to ε. Since $\mathcal{R}_r(G,N)$ always contains finite elements for any $N \in \mathbb{N}$ (under the assumption that all the entries of the Markov parameters are finite), the sequence $\{\mathcal{R}_r(G,N)\}_{N=1}^\infty$ cannot reach its limit after a finite number of terms in this case. However, we can still use the ELCP approach by applying a limit procedure and by observing how $\mathcal{R}_r(G,N)$ evolves as N goes to ∞. In the limit some of the entries of the system matrix A will become equal to ε. In order to be able to determine to evolution of $\mathcal{R}_r(G,N)$ as N goes to ∞ it is advisable to perform certain normalizations and to sort the extreme generators and the finite points lexicographically[4] before listing them. The following proposition shows how an arbitrary triple of system matrices (A,B,C) can be normalized.

Proposition 3.5 *Let $G = \{G_k\}_{k=0}^\infty$ be the impulse response of a max-linear time-invariant discrete event system, and let $L^*(G)$ be the set of the smallest possible values for the λ_s's in (11). Let (A,B,C) be a minimal state space realization of G and let n be the minimal system order. If $\lambda = \max L^*(G)$ and if $\lambda \neq \varepsilon$, then there exists a max-algebraic similarity transformation that transforms (A,B,C) in an equivalent state space realization $(\tilde{A},\tilde{B},\tilde{C})$ of G with $\|\tilde{A}\|_\oplus = \lambda$, $\|\tilde{B}\|_\oplus = 0$ and $\tilde{a}_{11} \geqslant \tilde{a}_{22} \geqslant \ldots \geqslant \tilde{a}_{nn}$.*

Proof: See [6]. □

So by applying max-algebraic similarity transformations we can always bring a minimal state space realization into a normalized form. Therefore, we may always add the following extra constraints to the system of multivariate max-algebraic polynomial equalities $(15)-(16)$ if we are computing minimal state space realizations:

$$\bigoplus_{i=1}^r \bigoplus_{j=1}^r a_{ij} = \lambda \tag{17}$$

$$\bigoplus_{i=1}^r \bigoplus_{j=1}^m b_{ij} = 0 \tag{18}$$

$$a_{ii} \geqslant a_{i+1,i+1} \qquad \text{for } i = 1,2,\ldots,r-1, \tag{19}$$

where $\lambda = \max L^*(G) \neq \varepsilon$ (Note that λ can only be equal to ε if $G_k = \mathcal{E}_{l \times m}$ for all k). So instead of determining the evolution of $\mathcal{R}_r(G,N)$, we determine the evolution of

$$\mathcal{R}_r^{\text{nor}}(G,N) \stackrel{\text{def}}{=} \{(A,B,C) \in \mathcal{R}_r(G,N) \,|\, \|A\|_\oplus = \lambda, \; \|B\|_\oplus = 0 \text{ and}$$

$$a_{11} \geqslant a_{22} \geqslant \ldots \geqslant a_{rr}\} \;.$$

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[4]A vector $x \in \mathbb{R}^n$ is lexicographically greater than or equal to a vector $y \in \mathbb{R}^n$ if and only if the first non-zero component of $x - y$ is greater than or equal to 0.

Once we have determined $\mathcal{R}_r^{\mathrm{nor}}(G) \stackrel{\text{def}}{=} \lim_{N \to \infty} \mathcal{R}_r^{\mathrm{nor}}(G, N)$, we can reconstruct the elements of the set $\mathcal{R}_r(G) \stackrel{\text{def}}{=} \lim_{N \to \infty} \mathcal{R}_r(G, N)$ by applying max-algebraic similarity transformations to the elements of $\mathcal{R}_r^{\mathrm{nor}}(G)$:

$$\mathcal{R}_r(G) = \left\{ \left(T \otimes A \otimes T^{\otimes^{-1}}, T \otimes B, C \otimes T^{\otimes^{-1}} \right) \, \middle| \, (A, B, C) \in \mathcal{R}_r^{\mathrm{nor}}(G) \text{ and } \right.$$

$$\left. T \in \mathbb{R}_\varepsilon^{r \times r} \text{ is max-invertible} \right\} .$$

This procedure will be illustrated in the next section.

If the above procedure does not yield a realization of the complete impulse response, we have to augment r and repeat the procedure of Sections 3.2 and 3.3 until we finally get a solution.

4 A worked example

Example 4.1 We consider the sequence

$$g = \{g_k\}_{k=0}^\infty = 0, 1, 0, 3, 0, 5, 0, 7, 0, 9, 0, 11, 0, 13, 0, 15, \ldots$$

of an example of [13, 14]. This sequence is not ultimately geometric, but it does exhibit an ultimately periodic behavior since we have

$$g_{2k+2} = 0^{\otimes^2} \otimes g_{2k} \quad \text{and} \quad g_{2k+3} = 1^{\otimes^2} \otimes g_{2k+1} \qquad \text{for all } k \in \mathbb{N} .$$

So it follows from Proposition 3.3 that there exists a max-linear time-invariant SISO discrete event system that has g as its impulse response. Using the methods of [13, 14] to determine upper and lower bounds for the minimal system order, we find that the minimal system order is equal to 3 (see also [6]).

Let us now use the ELCP approach to construct the set of all 3rd order state space realizations of g. Since g is not ultimately geometric, it cannot be realized by a triple (A, B, C) for which all the entries of A are finite. Therefore, we shall determine how the set $\mathcal{R}_3^{\mathrm{nor}}(g, N)$ evolves as N goes to ∞. The set $L^*(g)$ of the smallest possible values for the λ_s's in (11) is given by $L^*(g) = \{0, 1\}$. Since $\max L^*(g) = 1$, we have

$$\mathcal{R}_3^{\mathrm{nor}}(g, N) = \left\{ (A, B, C) \in \mathcal{R}_3(g, N) \, \middle| \, \|A\|_\oplus = 1, \ \|B\|_\oplus = 0 \text{ and } a_{11} \geqslant a_{22} \geqslant a_{33} \right\} .$$

If we use the ELCP algorithm of [8] to solve the ELCP that corresponds to $\mathcal{R}_3^{\mathrm{nor}}(g, N)$, we get the extreme generators and the finite points of Table 1 and the pairs of maximal cross-complementary subsets of Table 2 for any $N \geqslant 5$. There are no central generators. We have $\mathcal{R}_3^{\mathrm{nor}}(g, 2l + 1) = \mathcal{R}_3^{\mathrm{nor}}(g, 2l + 2)$ for all $l \geqslant 2$. Note that the components of the finite points of $\mathcal{R}_3^{\mathrm{nor}}(g, N)$ depend affinely on l where $l = \left\lfloor \dfrac{N - 1}{2} \right\rfloor$.

If l goes to ∞ then only $x_5^{\mathrm{f}}(l)$ and $x_6^{\mathrm{f}}(l)$ have components that are bounded from above. Define $\tilde{x}_1^{\mathrm{f}} = \lim_{l \to \infty} x_5^{\mathrm{f}}(l)$ and $\tilde{x}_2^{\mathrm{f}} = \lim_{l \to \infty} x_6^{\mathrm{f}}(l)$. Note that all the extreme generators except for $\tilde{x}_1^{\mathrm{e}} \stackrel{\text{def}}{=} x_1^{\mathrm{e}}$ and $\tilde{x}_2^{\mathrm{e}} \stackrel{\text{def}}{=} x_4^{\mathrm{e}}$ become redundant when l goes to ∞. As a consequence, the set

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	$\mathcal{X}^{\mathrm{ext}}(l)$											
	x_1^{e}	x_2^{e}	x_3^{e}	x_4^{e}	x_5^{e}	x_6^{e}	x_7^{e}	x_8^{e}	x_9^{e}	x_{10}^{e}	x_{11}^{e}	x_{12}^{e}
a_{11}	0	0	0	0	0	0	0	0	0	0	0	0
a_{12}	-1	-1	0	0	0	0	0	0	0	0	0	0
a_{13}	-1	0	-1	0	0	0	0	0	0	0	0	0
a_{21}	0	0	0	-1	-1	0	0	0	0	0	0	0
a_{22}	0	0	0	0	0	-1	0	0	0	0	0	0
a_{23}	0	0	0	0	0	0	0	0	0	0	0	0
a_{31}	0	0	0	-1	0	0	-1	0	0	0	0	0
a_{32}	0	0	0	0	0	0	0	0	0	0	0	0
a_{33}	0	0	0	0	0	-1	0	-1	0	0	0	0
b_1	-1	0	0	0	0	0	0	0	0	0	0	0
b_2	0	0	0	-1	0	0	0	0	-1	0	0	0
b_3	0	0	0	-1	0	0	0	0	0	-1	0	0
c_1	1	0	0	0	0	0	0	0	0	0	0	0
c_2	0	0	0	1	0	0	0	0	0	0	-1	0
c_3	0	0	0	1	0	0	0	0	0	0	0	-1

	$\mathcal{X}^{\mathrm{fin}}(l)$									
	$x_1^{\mathrm{f}}(l)$	$x_2^{\mathrm{f}}(l)$	$x_3^{\mathrm{f}}(l)$	$x_4^{\mathrm{f}}(l)$	$x_5^{\mathrm{f}}(l)$	$x_6^{\mathrm{f}}(l)$	$x_7^{\mathrm{f}}(l)$	$x_8^{\mathrm{f}}(l)$	$x_9^{\mathrm{f}}(l)$	$x_{10}^{\mathrm{f}}(l)$
a_{11}	0	0	0	0	0	0	0	0	0	0
a_{12}	1	1	1	0	$2-2l$	$1-2l$	$3-4l$	$2-4l$	$2-4l$	$1-4l$
a_{13}	1	1	0	1	$1-2l$	$2-2l$	$2-4l$	$3-4l$	$1-4l$	$2-4l$
a_{21}	$2-4l$	$1-4l$	$2-4l$	$3-4l$	$1-2l$	$2-2l$	0	1	1	1
a_{22}	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$
a_{23}	1	1	1	1	1	1	1	1	1	1
a_{31}	$1-4l$	$2-4l$	$3-4l$	$2-4l$	$2-2l$	$1-2l$	1	0	1	1
a_{32}	1	1	1	1	1	1	1	1	1	1
a_{33}	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$
b_1	0	0	0	0	0	0	$1-2l$	$1-2l$	$-2l$	$-2l$
b_2	$-4l$	$-2l$	$1-2l$	$1-4l$	0	$-2l$	0	$-2l$	0	$-2l$
b_3	$-2l$	$-4l$	$1-4l$	$1-2l$	$-2l$	0	$-2l$	0	$-2l$	0
c_1	0	0	0	0	0	0	$2l-1$	$2l-1$	$2l$	$2l$
c_2	$2l$	0	-1	$2l-1$	$-2l$	0	$-2l$	0	$-2l$	0
c_3	0	$2l$	$2l-1$	-1	0	$-2l$	0	$-2l$	0	$-2l$

Table 1: The generators and the finite points of the sets $\mathcal{R}_3^{\mathrm{nor}}(g, 2l+1)$ and $\mathcal{R}_3^{\mathrm{nor}}(g, 2l+2)$ of Example 4.1 for $l \geqslant 2$.

s	$\mathcal{X}_s^{\mathrm{ext}}(l)$	$\mathcal{X}_s^{\mathrm{fin}}(l)$
1	$\{x_1^{\mathrm{e}}, x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_9^{\mathrm{e}}, x_{12}^{\mathrm{e}}\}$	$\{x_6^{\mathrm{f}}(l), x_8^{\mathrm{f}}(l), x_{10}^{\mathrm{f}}(l)\}$
2	$\{x_1^{\mathrm{e}}, x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{10}^{\mathrm{e}}, x_{11}^{\mathrm{e}}\}$	$\{x_5^{\mathrm{f}}(l), x_7^{\mathrm{f}}(l), x_9^{\mathrm{f}}(l)\}$
3	$\{x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_4^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_9^{\mathrm{e}}, x_{12}^{\mathrm{e}}\}$	$\{x_1^{\mathrm{f}}(l), x_4^{\mathrm{f}}(l), x_6^{\mathrm{f}}(l)\}$
4	$\{x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_4^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{10}^{\mathrm{e}}, x_{11}^{\mathrm{e}}\}$	$\{x_2^{\mathrm{f}}(l), x_3^{\mathrm{f}}(l), x_5^{\mathrm{f}}(l)\}$

Table 2: The pairs of maximal cross-complementary subsets of the sets $\mathcal{X}^{\mathrm{ext}}(l)$ and $\mathcal{X}^{\mathrm{fin}}(l)$ of Example 4.1 for $l \geqslant 2$.

$\mathcal{R}_3^{\mathrm{nor}}(g) = \lim_{N \to \infty} \mathcal{R}_3^{\mathrm{nor}}(g, N)$ corresponds to the extreme generators and the "finite" points of Table 3 and the set

$$\tilde{\Lambda} = \left\{ \left(\{\tilde{x}_1^{\mathrm{e}}\}, \{\tilde{x}_1^{\mathrm{f}}\}\right), \left(\{\tilde{x}_1^{\mathrm{e}}\}, \{\tilde{x}_2^{\mathrm{f}}\}\right), \left(\{\tilde{x}_2^{\mathrm{e}}\}, \{\tilde{x}_1^{\mathrm{f}}\}\right), \left(\{\tilde{x}_2^{\mathrm{e}}\}, \{\tilde{x}_2^{\mathrm{f}}\}\right) \right\}$$

of ordered pairs of maximal cross-complementary subsets of $\tilde{\mathcal{X}}^{\mathrm{ext}}$ and $\tilde{\mathcal{X}}^{\mathrm{fin}}$.
The elements of $\mathcal{R}_3(g)$ can now be reconstructed from $\mathcal{R}_3^{\mathrm{nor}}(g)$ by applying max-algebraic similarity transformations to the elements of $\mathcal{R}_3^{\mathrm{nor}}(g)$.

In [6] we have used the technique of [13, 14] to construct a 3rd order state space realization (A_1, B_1, C_1) of the given impulse response[5]. This led to

$$A_1 = \begin{bmatrix} 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 2 \\ \varepsilon & 0 & \varepsilon \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ \varepsilon \end{bmatrix} \text{ and } C_1 = \begin{bmatrix} 0 & \varepsilon & 1 \end{bmatrix} .$$

The triple (A_1, B_1, C_1) can be brought into a normalized form by performing a max-algebraic similarity transformation with $T = \begin{bmatrix} 0 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & 1 \end{bmatrix}$ (cf. Proposition 3.5). This results in the triple $(\tilde{A}_1, \tilde{B}_1, \tilde{C}_1) \in \mathcal{R}_3^{\mathrm{nor}}(g)$ with

$$\tilde{A}_1 = \begin{bmatrix} 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 \\ \varepsilon & 1 & \varepsilon \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} 0 \\ 0 \\ \varepsilon \end{bmatrix} \text{ and } \tilde{C}_1 = \begin{bmatrix} 0 & \varepsilon & 0 \end{bmatrix} .$$

Note that the triple $(\tilde{A}_1, \tilde{B}_1, \tilde{C}_1)$ corresponds to the "finite" point $\tilde{x}_1^{\mathrm{f}}$. □

For other examples of state space realization for SISO systems the interested reader is referred to [6, 7, 9, 10]. In [11] we have given an example with a MIMO system.

5 Conclusions and future research

We have used the fact that a system of multivariate polynomial equalities and inequalities in the max-plus algebra can be transformed into an Extended Linear Complementarity Problem

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[5]Note that this technique will only result in a *minimal* state space realization of the given impulse response if the upper bound given by this theorem is equal to the minimal system order, which is not always the case (see [6]).

	$\tilde{\mathcal{X}}^{\mathrm{ext}}$		$\tilde{\mathcal{X}}^{\mathrm{fin}}$	
	$\tilde{x}_1^{\mathrm{e}}$	$\tilde{x}_2^{\mathrm{e}}$	$\tilde{x}_1^{\mathrm{f}}$	$\tilde{x}_2^{\mathrm{f}}$
a_{11}	0	0	0	0
a_{12}	-1	0	ε	ε
a_{13}	-1	0	ε	ε
a_{21}	0	-1	ε	ε
a_{22}	0	0	ε	ε
a_{23}	0	0	1	1
a_{31}	0	-1	ε	ε
a_{32}	0	0	1	1
a_{33}	0	0	ε	ε
b_1	-1	0	0	0
b_2	0	-1	0	ε
b_3	0	-1	ε	0
c_1	1	0	0	0
c_2	0	1	ε	0
c_3	0	1	0	ε

Table 3: The generators and the "finite" points of the set $\mathcal{R}_3^{\mathrm{nor}}(g)$ of Example 4.1.

(ELCP) to find all fixed order partial state space realizations of a max-linear multiple input multiple output discrete event system given its impulse response. The ELCP algorithm of [8] that was used to solve a system of multivariate max-algebraic polynomial equalities and inequalities finds all finite solutions. We have also discussed how solutions with components that are infinite can be reconstructed, and how the set of all the minimal state space realizations of the entire impulse response can be obtained by solving several partial state space realization problems.

For the state space realization problem the ELCP approach yields *all* solutions. However, this approach leads to large computation times and storage space requirements if the minimal system order and the number of terms of the impulse response that should be realized are large. Therefore, it might be interesting to develop (heuristic) algorithms that only find one solution. Furthermore, it is still an open question how to determine the minimal number N such that all partial state space realizations of the first N terms of the impulse response are also realizations of the entire impulse response.

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