K.U.Leuven

Department of Electrical Engineering (ESAT) **SISTA**

Technical report 96-03

Applications of the extended linear complementarity problem in the max-plus algebra*

B. De Schutter and B. De Moor

If you want to cite this report, please use the following reference instead:
B. De Schutter and B. De Moor, "Applications of the extended linear complementarity problem in the max-plus algebra," *Proceedings of the International Workshop on Discrete Event Systems (WODES'96)*, Edinburgh, UK, pp. 69–74, Aug. 1996.

ESAT-SISTA
K.U.Leuven
Leuven, Belgium
Current URL: https://www.esat.kuleuven.be/stadius

APPLICATIONS OF THE EXTENDED LINEAR COMPLEMENTARITY PROBLEM IN THE MAX-PLUS ALGEBRA

Bart De Schutter, Bart De Moor

ESAT-SISTA, K.U.Leuven, Kardinaal Mercierlaan 94, B-3001 Heverlee (Leuven), Belgium,
bart.deschutter@esat.kuleuven.ac.be, bart.demoor@esat.kuleuven.ac.be.

Keywords

Max-plus algebra, extended linear complementarity problem, discrete event systems, system theory

Abstract

The max-plus algebra is one of the frameworks that can be used to model discrete event systems. We show that many fundamental problems in the max-plus algebra can be reformulated as a mathematical programming problem: the Extended Linear Complementarity Problem.

1 Introduction

1.1 Overview

In this paper we present a mathematical programming problem that we have called the Extended Linear Complementarity Problem (ELCP). We briefly describe an algorithm to find all solutions of an ELCP. This algorithm yields a description of the complete solution set of the ELCP by finite points, generators for the extreme rays and a basis for the linear subspace associated with the maximal affine subspaces of the solution set. In that way it provides an insight in the geometrical structure of the solution set of the ELCP and related problems.
Next we indicate how the ELCP can be used to solve a system of multivariate max-algebraic polynomial equalities and inequalities in the max-plus algebra. This allows us to solve many other problems in the max-plus algebra such as computing max-algebraic matrix factorizations, constructing matrices with a given max-algebraic characteristic polynomial, performing state space transformations for max-linear time-invariant discrete event systems, determining partial or minimal state space realizations of the impulse response of a max-linear time-invariant discrete event system, computing singular value decompositions and QR decompositions of a matrix in the symmetrized max-plus algebra and so on.

1.2 The Max-Plus Algebra

One of the frameworks that can be used to model discrete event systems (DESs) is the max-plus algebra. The basic operations of the max-plus algebra are the maximum (represented by $\oplus$) and the addition (represented by $\otimes$):

$$x \oplus y = \max(x, y) \tag{1}$$

$$x \otimes y = x + y \tag{2}$$

with $x, y \in \mathbb{R} \cup \{-\infty\}$. The reason for choosing these symbols is that many properties from conventional linear algebra can be translated to the max-plus algebra simply by replacing $+$ by $\oplus$ and $\times$ by $\otimes$. The structure $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is called the *max-plus algebra*. The neutral element for $\oplus$ in $\mathbb{R}_{\max}$ is denoted by ε. So $\varepsilon = -\infty$. Define $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. Let $x, r \in \mathbb{R}$. The rth max-algebraic power of x is defined as follows:

$$x^{\otimes^r} = r \cdot x \ . \tag{3}$$

We have $\varepsilon^{\otimes^r} = \varepsilon$ if $r > 0$. If $r < 0$ then $\varepsilon^{\otimes^r}$ is not defined. In this paper we have $\varepsilon^{\otimes^0} = 0$ by definition. The operations $\oplus$ and $\otimes$ are extended to matrices in the usual way. So if $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ then we have $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$ for all i, j. If $A \in \mathbb{R}_\varepsilon^{m\times p}$ and $B \in \mathbb{R}_\varepsilon^{p\times n}$ then $(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj}$ for all i, j. If $k \in \mathbb{N}$ then we have $A^{\otimes^k} = \underbrace{A \otimes A \otimes \ldots \otimes A}_{k \text{ times}}$.
The matrix E_n is the n by n max-algebraic identity matrix: $(E_n)_{ii} = 0$ for all i and $(E_n)_{ij} = \varepsilon$ for all i, j with $i \neq j$. The m by n zero matrix in the max-plus algebra is denoted by $\mathcal{E}_{m\times n}$: $(\mathcal{E}_{m\times n})_{ij} = \varepsilon$ for all i, j.

In general the behavior of DESs is highly nonlinear. However, some of these systems can be described by model that is *linear* in the max-plus algebra [1, 2]: if we limit ourselves to time-invariant deterministic DESs in which the sequence of the events and the duration of the activities are fixed or can be determined in advance (such as repetitive production processes), then the behavior of a system with m inputs and l outputs can be described by equations of the form

$$x(k+1) = A \otimes x(k) \ \oplus \ B \otimes u(k) \tag{4}$$

$$y(k) = C \otimes x(k) \tag{5}$$

where $A \in \mathbb{R}_\varepsilon^{n\times n}$, $B \in \mathbb{R}_\varepsilon^{n\times m}$ and $C \in \mathbb{R}_\varepsilon^{l\times n}$ with an initial condition $x(0) = x_0$. The vector x represents the state, u is the input vector and y is the output vector of the system. We call DESs that can be described by a state space model of the form (4) – (5) *max-linear time-invariant* DESs.
The input-output behavior of DES that can be described by a model of the form (4) – (5) is given by

$$y(k) \;=\; C \otimes A^{\otimes^k} \otimes x(0) \oplus \bigoplus_{i=0}^{k-1} C \otimes A^{\otimes^{k-i-1}} \otimes B \otimes u(i)$$

for $k = 0, 1, 2, \ldots$.

For more information on the max-plus algebra and on the use of the max-plus algebra to model DESs the interested reader is referred to [1, 2, 3].

2 The Extended Linear Complementarity Problem

Consider the following problem:

Given $A \in \mathbb{R}^{p\times n}$, $B \in \mathbb{R}^{q\times n}$, $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$ and m subsets $\phi_1, \phi_2, \ldots, \phi_m$ of $\{1, 2, \ldots, p\}$, find $x \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{m} \prod_{i\in\phi_j} (Ax - c)_i \;=\; 0 \qquad (6)$$

subject to $Ax \geq c$ and $Bx = d$.

We call this problem the Extended Linear Complementarity Problem (ELCP) since it is an extension of the Linear Complementarity Problem — which is one of the fundamental problems in mathematical programming [4]. Equation (6) is called the *complementarity condition* of the ELCP. One possible interpretation of this condition is the following. Since $Ax \geq c$, all the terms in (6) are nonnegative. Hence, condition (6) is equivalent to

$$\prod_{i\in\phi_j} (Ax - c)_i \;=\; 0 \qquad \text{for } j = 1, 2, \ldots, m\,.$$

So we could say that each set ϕ_j corresponds to a group of inequalities of $Ax \geq c$ and that in each group at least one inequality should hold with equality, i.e. the corresponding residue should be equal to 0:

$$\forall j \in \{1, \ldots, m\} : \exists i \in \phi_j \text{ such that } (Ax - c)_i = 0\,.$$

In [5, 6] we have developed on algorithm to solve the ELCP. In this algorithm we take a new (in)equality into account in each step and we determine the intersection of the hyperplane or the half-space defined by the new (in)equality and the polyhedron defined by the previous (in)equalities. After each step we also immediately discard points of the resulting polyhedron that do not satisfy the complementarity condition. This finally results in three sets of vectors $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{ext}}$ and $\mathcal{X}^{\text{fin}}$ and a set Λ of ordered pairs of subsets of $\mathcal{X}^{\text{ext}}$ and $\mathcal{X}^{\text{fin}}$ such that a vector x is a solution of the ELCP if and only if there exists an ordered pair $\left(\mathcal{X}_s^{\text{ext}}, \mathcal{X}_s^{\text{fin}}\right) \in \Lambda$ such that

$$x \;=\; \sum_{x_k^{\text{c}} \in \mathcal{X}^{\text{cen}}} \lambda_k x_k^{\text{c}} + \sum_{x_k^{\text{e}} \in \mathcal{X}_s^{\text{ext}}} \kappa_k x_k^{\text{e}} + \sum_{x_k^{\text{f}} \in \mathcal{X}_s^{\text{fin}}} \mu_k x_k^{\text{f}}$$

with $\lambda_k \in \mathbb{R}$, $\kappa_k \geq 0$, $\mu_k \geq 0$ for all k and $\displaystyle\sum_k \mu_k = 1$.

If $\mathcal{P}$ is a polyhedron defined by $\mathcal{P} = \{x \,|\, Ax \geq b\}$, then the lineality space of $\mathcal{P}$ is defined by $\mathcal{L}(\mathcal{P}) = \{x \,|\, Ax = 0\}$.
We can give the following geometrical characterization to $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{ext}}$ and $\mathcal{X}^{\text{fin}}$ and Λ. Let $\mathcal{P}$ be the polyhedron defined by the system of linear equalities and inequalities of the ELCP.

- $\mathcal{X}^{\text{cen}}$ is a basis of the lineality space of $\mathcal{P}$.
- There exists a pointed polyhedron — i.e. a polyhedron with an empty lineality space — $\mathcal{P}_{\text{red}}$ such that $\mathcal{X}^{\text{ext}}$ is a set of generators of the extreme rays of $\mathcal{P}_{\text{red}}$ that satisfy the complementarity condition, and such that
- $\mathcal{X}^{\text{fin}}$ is a set of finite vertices of $\mathcal{P}_{\text{red}}$ that satisfy the complementarity condition.
- Each pair $\left(\mathcal{X}_s^{\text{ext}}, \mathcal{X}_s^{\text{fin}}\right) \in \Lambda$ determines a face $\mathcal{F}_s$ of $\mathcal{P}_{\text{red}}$ that belongs to the solution set of the ELCP: the elements of $\mathcal{X}_s^{\text{ext}}$ generate extreme rays of $\mathcal{F}_s$ and $\mathcal{X}_s^{\text{fin}}$ is the set of the finite vertices of $\mathcal{F}_s$.

This implies that in general the solution set of an ELCP consists of the union of faces of a polyhedron.

Our ELCP algorithm yields the entire solution set of the ELCP. This leads to large execution times and high storage space requirements especially if the number of variables and equations is large. Therefore, it might be interesting to develop algorithms that only find one solution of an ELCP. However, the ELCP is intrinsically a computationally hard problem since in [5, 6] we have shown that in general the ELCP with rational data is an NP-hard problem. So the ELCP can probably not be solved in polynomial time — unless the class P would coincide with the class NP [7].

For a more detailed and precise description of the ELCP algorithm the interested reader is referred to [5] and to [6], where also a worked example and some computational results can be found.

3 A link between the max-plus algebra and the ELCP

3.1 Systems of multivariate max-algebraic polynomial equalities and inequalities

Consider the following problem:

Given $p_1 + p_2$ positive integers $m_1, m_2, \ldots, m_{p_1+p_2}$ and real numbers a_{ki}, b_k and c_{kij} for $k = 1, 2, \ldots, p_1 + p_2$; $i = 1, 2, \ldots, m_k$ and $j = 1, 2, \ldots, n$, find $x \in \mathbb{R}^n$ such that

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kij}}} = b_k \tag{7}$$

for $k = 1, 2, \ldots, p_1$, and

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kij}}} \leq b_k \tag{8}$$

for $k = p_1 + 1, p_1 + 2, \ldots, p_1 + p_2$.

We call (7) – (8) a *system of multivariate max-algebraic polynomial equalities and inequalities.* Note that the exponents may be negative or real. In the next section we shall see that many important max-algebraic problems can be reformulated as a system of multivariate max-algebraic polynomial equalities and inequalities.
In [8] we have proved the following theorem:

Theorem 1 *A system of multivariate max-algebraic polynomial equalities and inequalities is equivalent to an Extended Linear Complementarity Problem.*

Note that this implies that the problem of solving a system of multivariate max-algebraic polynomial equalities and inequalities is also NP-hard.
We shall illustrate Theorem 1 by an example:

Example 1.
Consider the following system of multivariate polynomial equalities and inequalities:

$$6 \otimes x_1 \otimes x_2{}^{\otimes^{-2}} \otimes x_3{}^{\otimes^{4}} \otimes x_4{}^{\otimes^{-1}} \oplus 7 \otimes x_2{}^{\otimes^{2}} \otimes x_3 = 7 \tag{9}$$

$$3 \otimes x_1 \otimes x_2{}^{\otimes^{-1}} \otimes x_4{}^{\otimes^{-1}} \oplus 2 \otimes x_1{}^{\otimes^{-2}} \otimes x_2{}^{\otimes^{3}} \otimes x_3 \otimes x_4{}^{\otimes^{2}} = 2 \tag{10}$$

with $x \in \mathbb{R}^4$.

Consider the first term of (9). Using definitions (2) and (3) we find that this term is equivalent to

$$6 + x_1 - 2x_2 + 4x_3 - x_4 \ .$$

	$\mathcal{X}^{\text{cen}}$	$\mathcal{X}^{\text{ext}}$				$\mathcal{X}^{\text{fin}}$	
	x_1^{c}	x_1^{e}	x_2^{e}	x_3^{e}	x_4^{e}	x_1^{f}	x_2^{f}
x_1	1	0	0	0	0	0	0
x_2	0	−4	−9	1	2	2	−2
x_3	0	−1	−1	−2	−4	−4	0
x_4	1	4	14	−1	−1	−1	3

Table 1: The sets $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{ext}}$ and $\mathcal{X}^{\text{fin}}$ of the ELCP of Example 1.

s	$\mathcal{X}_s^{\text{ext}}$	$\mathcal{X}_s^{\text{fin}}$
1	$\{x_1^{\text{e}}\}$	$\{x_2^{\text{f}}\}$
2	$\{x_2^{\text{e}}\}$	$\{x_2^{\text{f}}\}$

s	$\mathcal{X}_s^{\text{ext}}$	$\mathcal{X}_s^{\text{fin}}$
3	$\{x_3^{\text{e}}\}$	$\{x_1^{\text{f}}\}$
4	$\{x_4^{\text{e}}\}$	$\{x_1^{\text{f}}\}$

Table 2: The pairs of subsets $(\mathcal{X}_s^{\text{ext}}, \mathcal{X}_s^{\text{fin}})$ that belong to the set Λ of the ELCP of Example 1.

The other term of (9) can be rewritten in a similar way. Each term has to be smaller than 7 and at least one of them has to be equal to 7. So we get a group of two inequalities in which at least one inequality should hold with equality. If we also take (10) into account, we get the following ELCP:

Given

$$A = \begin{bmatrix} -1 & 2 & -4 & 1 \\ 0 & -2 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 2 & -3 & -1 & -2 \end{bmatrix} \text{ and } c = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

find $x \in \mathbb{R}^4$ such that

$$(Ax - c)_1 \, (Ax - c)_2 + (Ax - c)_3 \, (Ax - c)_4 = 0$$

subject to $Ax \geq c$.

Table 1 lists the elements of the sets $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{ext}}$ and $\mathcal{X}^{\text{fin}}$ that are returned by the ELCP algorithm, and Table 2 contains the elements of the set Λ. Any solution of the system of multivariate max-algebraic polynomial equalities (9) – (10) can now be expressed as

$$x = \lambda x_1^{\text{c}} + \kappa x_k^{\text{e}} + x_k^{\text{f}}$$

for some $s \in \{1, 2, 3, 4\}$ with $\lambda \in \mathbb{R}$, $\kappa \geq 0$, $x_k^{\text{e}} \in \mathcal{X}_s^{\text{ext}}$ and $x_k^{\text{f}} \in \mathcal{X}_s^{\text{fin}}$. ◇

More information on how to retrieve solutions of (7) – (8) with components that are equal to ε and how to apply the ELCP approach if some of the coefficients b_k are equal to ε can be found in [6].

4 Applications

In this section we discuss some important max-algebraic problems that can be reformulated as a system of multivariate max-algebraic polynomial equalities and inequalities. These problems can also be reformulated as an ELCP and they can thus be solved using the ELCP algorithm. In general their solution set consists of the union of faces of a polyhedron. Note that for most of these problems the ELCP approach is at present the only method available to solve the problem.

4.1 Max-algebraic matrix factorizations

Consider the following problem:

> Given a matrix $A \in \mathbb{R}_\varepsilon^{m \times n}$ and $l \in \mathbb{N}_0$, find $P \in \mathbb{R}_\varepsilon^{m \times l}$ and $Q \in \mathbb{R}_\varepsilon^{l \times n}$ such that $A = P \otimes Q$.

So we have to find the entries of P and Q such that

$$\bigoplus_{k=1}^{l} p_{ik} \otimes q_{kj} = a_{ij} \qquad \text{for all } i, j ,$$

and this can clearly be considered as a system of multivariate max-algebraic polynomial equations in p_{ik} and q_{kj}.

This technique can easily be extended to the factorization of A as a product of three or more matrices of specified sizes. It is also possible to impose a certain structure on the composing matrices (e.g. triangular, diagonal, Hessenberg, ...).

4.2 Transformation of state space models

Consider a DES that can be described by a state space model of the form

$$x(k+1) = A \otimes x(k) \oplus B \otimes u(k) \qquad (11)$$
$$y(k) = C \otimes x(k) \qquad (12)$$

with initial condition $x(0) = x_0$.
Suppose that we want to find another state space model with system matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and with initial condition $\tilde{x}(0) = \tilde{x}_0$ that describes the same input-output behavior as the original state space model. This can be done as follows.
If we can find a common factor L of A and C such that $A = \hat{A} \otimes L$ and $C = \hat{C} \otimes L$ and if we define $\tilde{A} = L \otimes \hat{A}$, $\tilde{B} = L \otimes B$, $\tilde{C} = \hat{C}$ and $\tilde{x}_0 = L \otimes x_0$, then the state space model

$$\tilde{x}(k+1) = \tilde{A} \otimes \tilde{x}(k) \oplus \tilde{B} \otimes u(k)$$
$$y(k) = \tilde{C} \otimes \tilde{x}(k)$$

with initial condition $\tilde{x}(0) = \tilde{x}_0$ describes the same input-output behavior as the model (11) – (12) with initial condition $x(0) = x_0$ [8, 9].
If M is a common factor of A, B and x_0 such that $A = M \otimes \hat{A}$, $B = M \otimes \hat{B}$ and $x_0 = M \otimes \hat{x}_0$, then the state space model with $\tilde{A} = \hat{A} \otimes M$, $\tilde{B} = \hat{B}$, $\tilde{C} = C \otimes M$ and $\tilde{x}_0 = \hat{x}_0$ also describes the same input-output behavior as the model (11) – (12) with initial condition $x(0) = x_0$ [9].

So to obtain another state space realization of the given system, we try to find a factorization

$$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} \hat{A} \\ \hat{C} \end{bmatrix} \otimes L$$

or

$$\begin{bmatrix} A & B & x_0 \end{bmatrix} = M \otimes \begin{bmatrix} \hat{A} & \hat{B} & \hat{x}_0 \end{bmatrix} .$$

As has been shown in Section 4.1 these matrix factorizations can be considered as systems of multivariate max-algebraic polynomial equalities. So we can also use the ELCP approach to perform state space transformations for max-linear time-invariant DESs.

4.3 State space realization

Consider a DES that can be described an nth order state space model of the form (11) – (12) with $A \in \mathbb{R}_\varepsilon^{n \times n}$, $B \in \mathbb{R}_\varepsilon^{n \times m}$ and $C \in \mathbb{R}_\varepsilon^{l \times n}$.
If we apply a max-algebraic unit impulse: $e(k) = 0$ if $k = 0$, and $e(k) = \varepsilon$ if $k \neq 0$, to the ith input of the system and if $x(0) = \mathcal{E}_{n \times 1}$, we get

$$y(k) = C \otimes A^{\otimes^{k-1}} \otimes B_{.,i}$$

for $k = 1, 2, \ldots$ as the output of the system. Note that this output corresponds to the ith column of the matrix $G_{k-1} = C \otimes A^{\otimes^{k-1}} \otimes B$ for $k = 1, 2, \ldots$. Therefore, the sequence $\{G_k\}_{k=0}^{\infty}$ is called the *impulse response* of the DES. The G_k's are called the impulse response matrices or *Markov parameters*.
Suppose that A, B and C are unknown, and that we only know the Markov parameters, e.g. from experiments. How can we construct A, B and C from the G_k's? This process is called state space realization. If we make the dimension of A minimal, we have a minimal state space realization.

We assume that the DES can be described by an rth order state space model (see e.g. [10, 11] for methods to determine lower and upper bounds for the minimal system order). For sake of simplicity we shall only consider the partial realization problem: i.e. we look for a realization that only fits the first, say N, Markov parameters. State space realizations of the entire impulse response can be found by using a limit procedure in which we determine how the set of the partial state space realizations evolves as N goes to ∞ (See [6]). So we try to find $A \in \mathbb{R}_\varepsilon^{r \times r}$, $B \in \mathbb{R}_\varepsilon^{r \times m}$ and $C \in \mathbb{R}_\varepsilon^{l \times r}$ such that

$$C \otimes A^{\otimes^k} \otimes B = G_k \qquad \text{for } k = 0, 1, \ldots, N-1 .$$

If we write out these equations, we get

$$\bigoplus_{p=1}^{r} c_{ip} \otimes b_{pj} \;=\; (G_0)_{ij} \tag{13}$$

for all i, j, and

$$\bigoplus_{p=1}^{r} \bigoplus_{q=1}^{r} c_{ip} \otimes (A^{\otimes^k})_{pq} \otimes b_{qj} \;=\; (G_k)_{ij}$$

for all i, j and $k = 1, 2, \ldots, N-1$. Since

$$(A^{\otimes^k})_{pq} \\ = \bigoplus_{i_1=1}^{r} \bigoplus_{i_2=1}^{r} \cdots \bigoplus_{i_{k-1}=1}^{r} a_{pi_1} \otimes a_{i_1 i_2} \otimes \ldots \otimes a_{i_{k-1} q} \ ,$$

this can be rewritten as

$$\bigoplus_{p=1}^{r} \bigoplus_{q=1}^{r} \bigoplus_{s=1}^{r^{k-1}} c_{ip} \otimes \left(\bigotimes_{u=1}^{r} \bigotimes_{v=1}^{r} {a_{uv}}^{\otimes^{\gamma_{kpqsuv}}} \right) \otimes b_{qj} \\ = (G_k)_{ij} \tag{14}$$

for all i, j and $k = 1, 2, \ldots, N-1$, where γ_{kpqsuv} is the number of times that a_{uv} appears in the sth term of $(A^{\otimes^k})_{pq}$. Clearly, (13) − (14) is a system of multivariate max-algebraic polynomial equations with the entries of A, B and C as unknowns.

For more information on this method to construct minimal state space realizations and for some worked examples the interested reader is referred to [6, 9, 12, 13].

4.4 Max-max problems

Now we consider systems of max-algebraic equations that also have multivariate max-algebraic polynomials (instead of constants) on the right-hand side. Since in general there do not exist inverse elements w.r.t. $\oplus$ in $\mathbb{R}_{\max}$, we cannot simply transfer terms from the right-hand side to the left-hand side as we would do in conventional algebra. However, these problems can also be solved using the ELCP approach.

Consider the following problem:

> Given integers $m_k, p_k \in \mathbb{N}_0$ for $k = 1, 2, \ldots, q$ and real numbers a_{ki}, b_{kij}, c_{kl} and d_{klj} for $k = 1, 2, \ldots, q$; $i = 1, 2, \ldots, m_k$; $j = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, p_k$, find $x \in \mathbb{R}^n$ such that
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{b_{kij}}} \;=\; \bigoplus_{l=1}^{p_k} c_{kl} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{d_{klj}}} \tag{15}$$
>
> for $k = 1, 2, \ldots, q$.

If we define dummy variables t_1, t_2, …, t_q such that

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{b_{kij}}} = t_k$$

for $k = 1, 2, \ldots, q$, then the given problem is equivalent to

$$\begin{aligned} \bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{b_{kij}}} \otimes {t_k}^{\otimes^{-1}} &= 0 \\ \bigoplus_{l=1}^{p_k} c_{kl} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{d_{klj}}} \otimes {t_k}^{\otimes^{-1}} &= 0 \end{aligned}$$

for $k = 1, 2, \ldots, q$. This is a system of multivariate max-algebraic polynomial equalities that can be transformed into an ELCP.

4.5 Mixed max-min problems

We can also use the ELCP to solve mixed max-min problems. First we introduce the $\oplus'$ operation: $x \oplus' y = \min(x, y)$ with $x, y \in \mathbb{R}_\varepsilon$.
Now we consider the following problem:

> Given integers $m_k, m_{kl_1} \in \mathbb{N}_0$ for $k = 1, 2, \ldots, m$ and $l_1 = 1, 2, \ldots, m_k$ and real numbers $a_{kl_1 l_2}$, b_k and $c_{kl_1 l_2 j}$ for $k = 1, 2, \ldots, m$; $l_1 = 1, 2, \ldots, m_k$; $l_2 = 1, 2, \ldots, m_{kl_1}$ and $j = 1, 2, \ldots, n$, find $x \in \mathbb{R}^n$ such that
>
> $$\bigoplus_{l_1=1}^{m_k}{}' \bigoplus_{l_2=1}^{m_{kl_1}} a_{kl_1 l_2} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{c_{kl_1 l_2 j}}} = b_k$$
>
> for $k = 1, 2, \ldots, m$.

Using a technique that is similar to the one that has been used in Section 4.4 this problem can also be transformed into an ELCP.
We can also use this technique to transform systems of combined max-min equations of the following form into an ELCP:

$$\bigoplus_{l_1}{}' \bigoplus_{l_2} \bigoplus_{l_3}{}' \cdots \bigoplus_{l_q} a_{kl_1 l_2 \ldots l_q} \otimes \bigotimes_{j=1}^{n} {x_i}^{\otimes^{c_{kl_1 l_2 \ldots l_q j}}} = b_k$$

for $k = 1, 2, \ldots, m$, or analogous equations but with $\bigoplus$ replaced by $\bigoplus'$ and vice versa or when some of the equalities are replaced by inequalities.
Furthermore, we can also use this technique to solve systems of form (15) but with some of the max-algebraic summations $\bigoplus$ replaced by min-algebraic summations $\bigoplus'$ and/or with some of the equalities replaced by inequalities.

4.6 Problems in the symmetrized max-plus algebra

Although there exist no inverse elements with respect to $\oplus$ in $\mathbb{R}_{\max}$, it is possible to construct a kind of symmetrization of the max-plus algebra [1, 10]. We can also use the ELCP to solve problems in this symmetrized max-plus algebra such as constructing matrices with a given max-algebraic characteristic polynomial, determining max-algebraic singular value decompositions and max-algebraic QR decompositions of a matrix, and so on. However, it would lead us too far to go further into this matter.

For more information on this subject (and on the other problems that were discussed in this section) the reader is referred to [6, 8].

5 Conclusions and Future Research

In this paper we have introduced the Extended Linear Complementarity Problem (ELCP). The algorithm we use to solve the ELCP yields all solutions and in that way it provides a geometrical insight in the solution set of the ELCP and other problems that can be reduced to an ELCP. However, we are not always interested in obtaining all solutions of an ELCP. Therefore, our further research efforts will concentrate on algorithms that yield only one solution.
We have demonstrated that the ELCP can be used to solve a system of multivariate max-algebraic polynomial equalities and inequalities. This allows us to solve many problems in the max-plus algebra. Furthermore, we can also use the ELCP to solve problems in the max-min-plus algebra and the symmetrized max-plus algebra. Therefore, the ELCP is a powerful mathematical tool for solving max-algebraic problems. It would be interesting to make a more thorough study of the class of problems that can be reduced to solving a system of multivariate max-algebraic polynomial equalities and inequalities. Every instance of this class can then be reformulated as an ELCP.
Although in general solving an ELCP and solving a system of multivariate max-algebraic polynomial equalities and inequalities are NP-hard problems, it may be interesting to determine which subclasses of these problems can be solved with a polynomial time algorithm.

Acknowledgment

This paper presents research results of the Belgian programme on interuniversity attraction poles (IUAP-50) initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility is assumed by its authors.
Bart De Schutter is a research assistant with the N.F.W.O. (Belgian National Fund for Scientific Research) and Bart De Moor is a senior research associate with the N.F.W.O.

References

[1] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat, *Synchronization and Linearity*, John Wiley & Sons, New York, 1992.

[2] G. Cohen, D. Dubois, J.P. Quadrat, and M. Viot, "A linear-system-theoretic view of discrete-event processes and its use for performance evaluation in manufacturing", *IEEE Transactions on Automatic Control*, vol. 30, no. 3, pp. 210–220, Mar. 1985.

[3] R.A. Cuninghame-Green, *Minimax Algebra*, vol. 166 of *Lecture Notes in Economics and Mathematical Systems*, Springer-Verlag, Berlin, Germany, 1979.

[4] R.W. Cottle, J.S. Pang, and R.E. Stone, *The Linear Complementarity Problem*, Academic Press, Boston, 1992.

[5] B. De Schutter and B. De Moor, "The extended linear complementarity problem", *Mathematical Programming*, vol. 71, no. 3, pp. 289–325, Dec. 1995.

[6] B. De Schutter, *Max-Algebraic System Theory for Discrete Event Systems*, PhD thesis, Department of Electrical Engineering, K.U.Leuven, Leuven, Belgium, 1996.

[7] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W.H. Freeman and Company, San Francisco, 1979.

[8] B. De Schutter and B. De Moor, "A method to find all solutions of a system of multivariate polynomial equalities and inequalities in the max algebra", *Discrete Event Dynamic Systems: Theory and Applications*, vol. 6, no. 2, pp. 115–138, Mar. 1996.

[9] B. De Schutter and B. De Moor, "State space transformations and state space realization in the max algebra", in *Proceedings of the 34th IEEE Conference on Decision and Control*, New Orleans, Louisiana, Dec. 1995, pp. 891–896.

[10] S. Gaubert, *Théorie des Systèmes Linéaires dans les Dioïdes*, PhD thesis, Ecole Nationale Supérieure des Mines de Paris, France, July 1992.

[11] S. Gaubert, "On rational series in one variable over certain dioids", Tech. Rep. 2162, INRIA, Le Chesnay, France, Jan. 1994.

[12] B. De Schutter and B. De Moor, "Minimal realization in the max algebra is an extended linear complementarity problem", *Systems & Control Letters*, vol. 25, no. 2, pp. 103–111, May 1995.

[13] B. De Schutter and B. De Moor, "Minimal state space realization of MIMO systems in the max algebra", in *Proceedings of the 3rd European Control Conference*, Rome, Italy, Sept. 1995, pp. 411–416.