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A Note on the Characteristic Equation in the Max-Plus Algebra

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ABSTRACT

In this paper we discuss the characteristic equation of a matrix in the max-plus algebra. In their *Linear Algebra and Its Applications* paper (vol. 101, pp. 87–108, 1988) Olsder and Roos have used a transformation between the max-plus algebra and linear algebra to show that the Cayley-Hamilton theorem also holds in the max-plus algebra. We show that the derivation of Olsder and Roos is not entirely correct and we give the correct formulas for the coefficients of this alternative version of the max-algebraic characteristic equation. We also give a counterexample for a conjecture of Olsder in which he states necessary and sufficient conditions for the coefficients of the max-algebraic characteristic equation.

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1. INTRODUCTION

There are many ways to model and to analyze discrete event systems (examples of which are flexible manufacturing systems, subway traffic networks, parallel processing systems, telecommunication networks, logistic systems, etc.). In general these systems lead to a non-linear description in linear algebra. However, for timed event graphs — a subclass of discrete event systems — we obtain a 'linear' state space model when we use

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the max-plus algebra to describe the behavior of the system [1, 2]. The basic operations of the max-plus algebra are the maximum and the addition. There exists a remarkable similarity between these operations on the one hand and the basic operations of conventional algebra (addition and multiplication) on the other hand: many properties and concepts of linear algebra, such as Cramer's rule, the Cayley-Hamilton theorem, eigenvalues, eigenvectors, ... also have a max-algebraic equivalent (See e.g. [1, 11]).

An important concept in the study of the steady state behavior of timed event graphs is the max-algebraic eigenvalue [1, 2]. In [8, 9, 10, 11] Olsder and Roos have used a kind of (entrywise) exponential matrix mapping to define the max-algebraic equation, to show that every matrix has at least one max-algebraic eigenvalue and to prove a max-algebraic equivalent of the Cayley-Hamilton theorem. In this paper we correct an error in their derivation. We also give a counterexample for a conjecture of Olsder in which he states necessary and sufficient conditions for the coefficients of the max-algebraic characteristic equation.

This paper is organized as follows. In Section 2 we introduce some notations and we give a short introduction to the max-plus algebra. In Section 3 we recapitulate the reasoning Olsder and Roos have used in [11] to derive the characteristic equation of a matrix in the max-plus algebra. We show where their reasoning goes wrong and how it can be corrected. We also include an example in which we compute the max-algebraic characteristic equation of a given matrix. In Section 4 we give a counterexample for a conjecture of Olsder in which he states necessary and sufficient conditions for the coefficients of the max-algebraic characteristic equation.

2. PRELIMINARIES

2.1. Notations and Definitions

If $\mathcal{A}$ is a set then $\#\mathcal{A}$ denotes the number of elements of $\mathcal{A}$. We use $\mathcal{P}_n$ to represent the set of all permutations of n numbers. The set of the even permutations of n numbers is denoted by $\mathcal{P}_{n,\text{even}}$ and the set of the odd permutations of n numbers is denoted by $\mathcal{P}_{n,\text{odd}}$. An element σ of $\mathcal{P}_n$ will be considered as a mapping from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$. The signature of the permutation σ is denoted by $\operatorname{sgn}(\sigma)$. We use $\mathcal{C}_n^k$ to represent the set of all subsets with k elements of the set $\{1, 2, \ldots, n\}$.

Let A be an n by n matrix and let $\varphi \subseteq \{1, 2, \ldots, n\}$. The submatrix obtained by removing all rows and columns of A except for those indexed by φ is denoted by $A_{\varphi\varphi}$. The matrix $A_{\varphi\varphi}$ is called a principal submatrix of the matrix A.

DEFINITION 1. Let f and g be real functions. The function f is asymptotically equivalent to g in the neighborhood of ∞, denoted by $f(x) \sim g(x)$, $x \to \infty$, if $\lim_{x\to\infty} f(x)/g(x) = 1$.
We say that $f(x) \sim 0$, $x \to \infty$ if there exists a real number K such that $f(x) = 0$ for all $x \geq K$.
If F and G are real m by n matrix-valued functions then we have $F(x) \sim G(x)$, $x \to \infty$ if $f_{ij}(x) \sim g_{ij}(x)$, $x \to \infty$ for all i, j.

Note that the main difference with the conventional definition of asymptotic equivalence is that Definition 1 also allows us to say that a function is asymptotically equivalent to 0.

2.2. *The Max-Plus Algebra*

In this section we give a short introduction to the max-plus algebra. We shall only state the definitions and properties that we need in the remainder of this paper. A more complete overview of the max-plus algebra can be found in [1, 3]. The basic max-algebraic operations are defined as follows:

$$\begin{aligned} a \oplus b &= \max(a, b) \ , \\ a \otimes b &= a + b \ , \end{aligned}$$

where $a, b \in \mathbb{R} \cup \{-\infty\}$. The reason for using these symbols is that there is a remarkable analogy between $\oplus$ and addition, and between $\otimes$ and multiplication: many concepts and properties from linear algebra (such as the Cayley-Hamilton theorem, eigenvectors and eigenvalues, Cramer's rule, ...) can be translated to the max-plus algebra by replacing $+$ by $\oplus$ and $\times$ by $\otimes$. The structure $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is called the max-plus algebra. The zero element for $\oplus$ is $\varepsilon \stackrel{\text{def}}{=} -\infty$: we have $a \oplus \varepsilon = a = \varepsilon \oplus a$ for all $a \in \mathbb{R} \cup \{\varepsilon\}$. Define $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$.

Let $k \in \mathbb{N}$. The kth max-algebraic power of $a \in \mathbb{R}$ is defined as follows: $a^{\otimes^k} = ka$. If $k > 0$ then $\varepsilon^{\otimes^k} = \varepsilon$; $\varepsilon^{\otimes^0}$ is not defined.

The rules for the order of evaluation of the max-algebraic operators are similar to those of conventional algebra. So max-algebraic power has the highest priority, and $\otimes$ has a higher priority than $\oplus$.

The matrix E_n is the n by n identity matrix in the max-plus algebra: $(E_n)_{ii} = 0$ for all i and $(E_n)_{ij} = \varepsilon$ for all i, j with $i \neq j$. The max-algebraic operations are extended to matrices as follows. If $A \in \mathbb{R}_\varepsilon^{m\times n}$ and $\alpha \in \mathbb{R}_\varepsilon$ then we have

$$(\alpha \otimes A)_{ij} = \alpha \otimes a_{ij} \qquad \text{for all } i, j \ .$$

If $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ then

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} \qquad \text{for all } i, j \ .$$

If $A \in \mathbb{R}_\varepsilon^{m \times p}$ and $B \in \mathbb{R}_\varepsilon^{p \times n}$ then

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj} \qquad \text{for all } i, j\,.$$

Note that these definitions are analogous to the definitions for the multiplication of a matrix by a scalar, the addition of two matrices and the multiplication of two matrices in linear algebra, but with $+$ replaced by $\oplus$ and $\times$ by $\otimes$.

If $A \in \mathbb{R}_\varepsilon^{n \times n}$ and if $k \in \mathbb{N}$, then the kth max-algebraic power of A is defined recursively as follows: $A^{\otimes^k} = A^{\otimes^{k-1}} \otimes A$ if $k > 0$ and $A^{\otimes^0} = E_n$.

If $a_0, a_1, \ldots, a_n \in \mathbb{R}_\varepsilon$ then $a_0 \otimes \lambda^{\otimes^n} \oplus a_1 \otimes \lambda^{\otimes^{n-1}} \oplus \ldots \oplus a_{n-1} \otimes \lambda \oplus a_n$ is called a max-algebraic polynomial in the variable λ. If the coefficient a_0 is equal to 0 (the identity element for $\otimes$), we say that the max-algebraic polynomial is a *monic* max-algebraic polynomial.

3. THE CHARACTERISTIC EQUATION IN THE MAX-PLUS ALGEBRA

In this section we recapitulate the reasoning Olsder and Roos have used in [11] to derive the max-algebraic characteristic equation of a matrix (See also [1]). We show where their reasoning goes wrong and how it can be corrected.

If $a, b, c \in \mathbb{R}_\varepsilon$ then we have

$$\begin{aligned} a \oplus b = c \qquad &\Leftrightarrow \qquad z^a + z^b \sim \alpha z^c,\ z \to \infty\ , \\ a \otimes b = c \qquad &\Leftrightarrow \qquad z^a z^b = z^c \quad \text{for all } z \in \mathbb{R}_0^+\ , \end{aligned}$$

where $\alpha = 1$ if $a \neq b$ and $\alpha = 2$ if $a = b$ and where $z^\varepsilon = 0$ for all $z \in \mathbb{R}_0^+$ by definition.

If $A \in \mathbb{R}_\varepsilon^{n \times n}$ then z^A is a real n by n matrix-valued function with domain of definition $\mathbb{R}_0^+$ that is defined by $(z^A)_{ij} = z^{a_{ij}}$ for all i, j. The *dominant* of A is defined as follows:

$$\operatorname{dom}_\oplus A = \begin{cases} \text{the highest exponent in } \det z^A & \text{if } \det z^A \neq 0\,, \\ \varepsilon & \text{otherwise.} \end{cases}$$

The characteristic polynomial of the matrix-valued function z^A (in linear algebra) is given by

$$\det\left(\lambda(z) I_n - z^A\right) = \lambda^n(z) + \gamma_1(z)\lambda^{n-1}(z) + \ldots + \gamma_{n-1}(z)\lambda(z) + \gamma_n(z)$$

with

$$\gamma_k(z) = (-1)^k \sum_{\varphi \in \mathcal{C}_n^k} \det z^{A_{\varphi\varphi}} \quad . \tag{1}$$

The Cayley-Hamilton theorem applied to the matrix-valued function z^A yields

$$(z^A)^n + \gamma_1(z)(z^A)^{n-1} + \ldots + \gamma_{n-1}(z)(z^A) + \gamma_n(z) I_n = 0 \tag{2}$$

for all $z \in \mathbb{R}_0^+$. In [11] Olsder and Roos claim that the highest degree in (1) is equal to $\max\left\{ \mathrm{dom}_\oplus A_{\varphi\varphi} \,\middle|\, \varphi \in \mathcal{C}_n^k \right\}$ and that

$$\gamma_k(z) \;\sim\; (-1)^k \, \bar{\gamma}_k \, z^{\max\left\{ \mathrm{dom}_\oplus A_{\varphi\varphi} \,\middle|\, \varphi \in \mathcal{C}_n^k \right\}} \ , \ z \to \infty \ ,$$

where $\bar{\gamma}_k$ is equal to the number of even permutations that contribute to the highest degree in (1) minus the number of odd permutations that contribute to the highest degree in (1). However, the highest degree in (1) is not necessarily equal to $\max\left\{ \mathrm{dom}_\oplus A_{\varphi\varphi} \,\middle|\, \varphi \in \mathcal{C}_n^k \right\}$, since if the number of even permutations that contribute to $z^{\max\left\{ \mathrm{dom}_\oplus A_{\varphi\varphi} \,\middle|\, \varphi \in \mathcal{C}_n^k \right\}}$ is equal to the number of odd permutations that contribute to $z^{\max\left\{ \mathrm{dom}_\oplus A_{\varphi\varphi} \,\middle|\, \varphi \in \mathcal{C}_n^k \right\}}$, the term that contains $z^{\max\left\{ \mathrm{dom}_\oplus A_{\varphi\varphi} \,\middle|\, \varphi \in \mathcal{C}_n^k \right\}}$ disappears.‡ We shall illustrate this with an example:

Example 2. Consider the matrix

$$A = \begin{bmatrix} -2 & 1 & \varepsilon \\ 1 & 0 & 1 \\ \varepsilon & 0 & 2 \end{bmatrix} \quad .$$

The matrix-valued function z^A is given by

$$z^A = \begin{bmatrix} z^{-2} & z & 0 \\ z & 1 & z \\ 0 & 1 & z^2 \end{bmatrix} \quad .$$

We have

$$\begin{aligned}
\det z^{A_{\{1,2\},\{1,2\}}} &= \begin{bmatrix} z^{-2} & z \\ z & 1 \end{bmatrix} = z^{-2} - z^2 \ , \\
\det z^{A_{\{1,3\},\{1,3\}}} &= \begin{bmatrix} z^{-2} & 0 \\ 0 & z^2 \end{bmatrix} = 1 \ , \\
\det z^{A_{\{2,3\},\{2,3\}}} &= \begin{bmatrix} 1 & z \\ 1 & z^2 \end{bmatrix} = z^2 - z \ .
\end{aligned}$$

‡This phenomenon does not occur for the matrices of the examples given in [8, 9, 10, 11] and in which the calculation and the properties of the max-algebraic characteristic equation are illustrated. These examples are thus correct.

So $\mathrm{dom}_\oplus A_{\{1,2\},\{1,2\}} = 2$, $\mathrm{dom}_\oplus A_{\{1,3\},\{1,3\}} = 0$ and $\mathrm{dom}_\oplus A_{\{2,3\},\{2,3\}} = 2$. However, since

$$\begin{aligned}
\gamma_2(z) &= (-1)^2 \left(\det z^{A_{\{1,2\},\{1,2\}}} + \det z^{A_{\{1,3\},\{1,3\}}} + \det z^{A_{\{2,3\},\{2,3\}}}\right) \\
&= z^{-2} - z^2 + 1 + z^2 - z \\
&= -z + 1 + z^{-2} \ ,
\end{aligned}$$

the highest degree in $\gamma_2(z)$ is equal to 1 whereas $\max\left\{ \mathrm{dom}_\oplus A_{\varphi\varphi} \,\middle|\, \varphi \in \mathcal{C}_3^2 \right\} = 2 \neq 1$.

The highest degree term in (1) can be determined as follows. Define

$$\Gamma_k = \left\{ \xi \,\middle|\, \exists \{i_1, i_2, \ldots, i_k\} \in \mathcal{C}_n^k,\ \exists \sigma \in \mathcal{P}_k \text{ such that } \xi = \sum_{r=1}^{k} a_{i_r i_{\sigma(r)}} \right\}$$

for $k = 1, 2, \ldots, n$. For every $k \in \{1, 2, \ldots, n\}$ and for every $\xi \in \Gamma_k$ we define

$$\begin{aligned}
I_k^{\mathrm{e}}(\xi) &= \#\left\{ \sigma \in \mathcal{P}_{k,\mathrm{even}} \,\middle|\, \exists \{i_1, i_2, \ldots, i_k\} \in \mathcal{C}_n^k \right. \\
&\qquad\qquad \left. \text{such that } \sum_{r=1}^{k} a_{i_r i_{\sigma(r)}} = \xi \right\} \\
I_k^{\mathrm{o}}(\xi) &= \#\left\{ \sigma \in \mathcal{P}_{k,\mathrm{odd}} \,\middle|\, \exists \{i_1, i_2, \ldots, i_k\} \in \mathcal{C}_n^k \right. \\
&\qquad\qquad \left. \text{such that } \sum_{r=1}^{k} a_{i_r i_{\sigma(r)}} = \xi \right\} \\
I_k(\xi) &= I_k^{\mathrm{e}}(\xi) - I_k^{\mathrm{o}}(\xi) \ .
\end{aligned}$$

Since (1) can be rewritten as

$$\begin{aligned}
\gamma_k(z) &= (-1)^k \sum_{\{i_1,\ldots,i_k\} \in \mathcal{C}_n^k} \sum_{\sigma \in \mathcal{P}_k} \mathrm{sgn}\,(\sigma) \prod_{r=1}^{k} (z^A)_{i_r i_{\sigma(r)}} \\
&= (-1)^k \sum_{\{i_1,\ldots,i_k\} \in \mathcal{C}_n^k} \sum_{\sigma \in \mathcal{P}_k} \mathrm{sgn}\,(\sigma)\, z^{\left(\sum_{r=1}^{k} a_{i_r i_{\sigma(r)}} \right)} \ ,
\end{aligned}$$

the highest degree that appears in $\gamma_k(z)$ is given by

$$c_k \stackrel{\mathrm{def}}{=} \max\{\, \xi \in \Gamma_k \mid I_k(\xi) \neq 0 \,\} \ ,$$

and the coefficients of the characteristic equation of z^A satisfy:

$$\gamma_k(z) \;\sim\; (-1)^k \, I_k(c_k) \, z^{c_k} \;,\; z \to \infty \;.$$

Define $\hat{\gamma}_k = (-1)^k I_k(c_k)$ for $k = 1, 2, \ldots, n$. Let $\mathcal{I} = \{\, k \,|\, \hat{\gamma}_k > 0 \,\}$ and $\mathcal{J} = \{\, k \,|\, \hat{\gamma}_k < 0 \,\}$. Note that we have $I_1^{\mathrm{o}}(\xi) = 0$ and $I_1^{\mathrm{e}}(\xi) > 0$ for every $\xi \in \Gamma_1 = \{\, a_{ii} \,|\, i = 1, 2, \ldots, n \,\}$. This implies that $I_1(c_1) > 0$ and that $\hat{\gamma}_1 < 0$. Hence, we always have $1 \in \mathcal{J}$.

It is easy to verify that if A is a square matrix with entries in $\mathbb{R}_\varepsilon$ then we have

$$\left(z^A\right)^k \;\sim\; z^{\left(A^{\otimes^k}\right)} \;,\; z \to \infty \;. \tag{3}$$

As a consequence, (2) results in

$$z^{\left(A^{\otimes^n}\right)} + \sum_{k \in \mathcal{I}} \hat{\gamma}_k \, z^{c_k} z^{\left(A^{\otimes^{n-k}}\right)} \;\sim\; \sum_{k \in \mathcal{J}} \hat{\gamma}_k \, z^{c_k} z^{\left(A^{\otimes^{n-k}}\right)} \;,\; z \to \infty \;.$$

Since all the terms of this expression have positive coefficients, comparison of the highest degree terms of corresponding entries on the left-hand and the right-hand side of this expression leads to the following identity in $\mathbb{R}_{\max}$:

$$A^{\otimes^n} \;\oplus\; \bigoplus_{k \in \mathcal{I}} c_k \otimes A^{\otimes^{n-k}} = \bigoplus_{k \in \mathcal{J}} c_k \otimes A^{\otimes^{n-k}} \;.$$

This equation can be considered as a max-algebraic version of the Cayley-Hamilton theorem if we define the *max-algebraic characteristic equation* of A as

$$\lambda^{\otimes^n} \;\oplus\; \bigoplus_{k \in \mathcal{I}} c_k \otimes \lambda^{\otimes^{n-k}} = \bigoplus_{k \in \mathcal{J}} c_k \otimes \lambda^{\otimes^{n-k}} \;. \tag{4}$$

Let us now calculate the max-algebraic characteristic equation of the matrix A of Example 2.

Example 3. For the matrix

$$A = \begin{bmatrix} -2 & 1 & \varepsilon \\ 1 & 0 & 1 \\ \varepsilon & 0 & 2 \end{bmatrix}$$

we have $\Gamma_1 = \{2, 0, -2\}$, $\Gamma_2 = \{2, 1, 0, -2, \varepsilon\}$, $\Gamma_3 = \{4, 0, -1, \varepsilon\}$ and

$$\begin{array}{lllll} I_1(2) = 1, & I_1(0) = 1, & I_1(-2) = 1, & & \\ I_2(2) = 0, & I_2(1) = -1, & I_2(0) = 1, & I_2(-2) = 1, & I_2(\varepsilon) = -1, \\ I_3(4) = -1, & I_3(0) = 1, & I_3(-1) = -1, & I_3(\varepsilon) = 1 \;. & \end{array}$$

Hence, $c_1 = 2$, $c_2 = 1$ and $c_3 = 4$. Since $\hat{\gamma}_1 = -1$, $\hat{\gamma}_2 = -1$ and $\hat{\gamma}_3 = 1$, we have $\mathcal{I} = \{3\}$ and $\mathcal{J} = \{1,2\}$. So the max-algebraic characteristic equation of A is:

$$\lambda^{\otimes^3} \oplus 4 = 2 \otimes \lambda^{\otimes^2} \oplus 1 \otimes \lambda \ .$$

Furthermore,

$$A^{\otimes^3} \oplus 4 \otimes E_3 = \begin{bmatrix} 4 & 3 & 4 \\ 3 & 4 & 5 \\ 3 & 4 & 6 \end{bmatrix} = 2 \otimes A^{\otimes^2} \oplus 1 \otimes A \ .$$

So A satisfies its max-algebraic characteristic equation.

Remarks

1. For the matrix A of Example 2 we have $c_2 = 1$ whereas the dominants of the principal 2 by 2 submatrices of A: $A_{\{1,2\},\{1,2\}}$, $A_{\{1,3\},\{1,3\}}$ and $A_{\{2,3\},\{2,3\}}$ are 2, 0 and 2 respectively. This shows that in general the dominant cannot be used to define the coefficients of the max-algebraic characteristic equation.

2. Define $d_k = \max\{\,\xi \,|\, \xi \in \Gamma_k\,\}$ and $\delta_k = I_k(d_k)$ for $k = 1, 2, \ldots, n$. Note that d_k is equal to $\max\left\{\mathrm{dom}_\oplus A_{\varphi\varphi} \,|\, \varphi \in \mathcal{C}_n^k\right\}$ and that δ_k is equal to $\bar{\gamma}_k$. If we follow the formulas of [1, 11] literally, we should set c_k equal to d_k if $\delta_k = 0$ and put k in $\mathcal{J}$.

   In [8, 9, 10] a derivation that is similar to that of [11] has been presented, but there c_k was set equal to ε if $\delta_k = 0$.

   However, the following example shows that in general neither of these definitions leads to a valid max-algebraic analogue of the Cayley-Hamilton theorem: when we take these definitions for the coefficients of the max-algebraic characteristic equation then there exist matrices that do not satisfy their max-algebraic characteristic equation.

   Example 4. For the matrix

$$A = \begin{bmatrix} \varepsilon & \varepsilon & 0 & \varepsilon \\ 0 & 0 & \varepsilon & \varepsilon \\ 1 & 1 & 1 & \varepsilon \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

   we have $d_1 = 1$, $\delta_1 = 1$, $d_2 = 1$, $\delta_2 = 0$, $d_3 = 1$, $\delta_3 = 0$, $d_4 = 0$ and $\delta_4 = 0$.

   The definition of [11] would result in

$$\lambda^{\otimes^4} = 1 \otimes \lambda^{\otimes^3} \oplus 1 \otimes \lambda^{\otimes^2} \oplus 1 \otimes \lambda \oplus 0 \ . \qquad (5)$$

However, since

$$A^{\otimes^4} = \begin{bmatrix} 3 & 3 & 3 & \varepsilon \\ 2 & 2 & 2 & \varepsilon \\ 4 & 4 & 4 & \varepsilon \\ 4 & 4 & 4 & -4 \end{bmatrix}$$

and

$$1 \otimes A^{\otimes^3} \ \oplus \ 1 \otimes A^{\otimes^2} \ \oplus \ 1 \otimes A \ \oplus \ 0 \otimes E_4 = \begin{bmatrix} 3 & 3 & 3 & \varepsilon \\ 2 & 2 & 2 & \varepsilon \\ 4 & 4 & 4 & \varepsilon \\ 4 & 4 & 4 & 0 \end{bmatrix} ,$$

the matrix A does not satisfy (5).

The definition of [8, 9, 10] would result in the following max-algebraic characteristic equation for A:

$$\lambda^{\otimes^4} = 1 \otimes \lambda^{\otimes^3} \ . \tag{6}$$

However, since

$$1 \otimes A^{\otimes^3} = \begin{bmatrix} 3 & 3 & 3 & \varepsilon \\ 2 & 2 & 2 & \varepsilon \\ 4 & 4 & 4 & \varepsilon \\ 4 & 4 & 4 & -2 \end{bmatrix} ,$$

the matrix A does not satisfy (6) either.

The max-algebraic characteristic equation of A is given by

$$\lambda^{\otimes^4} \ \oplus \ 0 \otimes \lambda^{\otimes^2} = 1 \otimes \lambda^{\otimes^3} \ .$$

Note that A satisfies this equation since

$$A^{\otimes^4} \ \oplus \ 0 \otimes A^{\otimes^2} = \begin{bmatrix} 3 & 3 & 3 & \varepsilon \\ 2 & 2 & 2 & \varepsilon \\ 4 & 4 & 4 & \varepsilon \\ 4 & 4 & 4 & -2 \end{bmatrix} = 1 \otimes A^{\otimes^3} \ .$$

4. A COUNTEREXAMPLE FOR A CONJECTURE OF OLSDER

In [8] Olsder states that if an equation of the form

$$\lambda^{\otimes^3} \ \oplus \ c_1 \otimes \lambda = c_2 \otimes \lambda^{\otimes^2} \ \oplus \ c_0 \tag{7}$$

has less than three (possibly coinciding) solutions then it cannot be the max-algebraic characteristic equation of a 3 by 3 matrix. Next he proposes the following conjecture:

CONJECTURE 5. A monic [max-algebraic polynomial][§] equation of degree n with the highest and one but highest order terms at different sides of the equality sign is a [max-algebraic] characteristic equation of an $n \times n$ matrix if and only if this equation has the maximum number of possibly coinciding real solutions. With "possibly coinciding" is meant that an arbitrary small perturbation of the coefficients exists, in the usual $\|\cdot\|_2$ norm, such that the perturbed equation has the maximum number of solutions which are all different.

Now we show by a counterexample that the statement made in [8] about (7) does not hold. Hence, Conjecture 5 does not hold either. Note that for this example the various (correct and wrong) definitions for the coefficients of the max-algebraic characteristic equation all yield the same result.

EXAMPLE 6. Consider

$$A = \begin{bmatrix} -1 & \varepsilon & 3 \\ 0 & -2 & \varepsilon \\ \varepsilon & 0 & \varepsilon \end{bmatrix} .$$

The different definitions for the coefficients of the max-algebraic characteristic equation discussed in Section 3 all result in the following max-algebraic characteristic equation for A:

$$\lambda^{\otimes^3} \ \oplus \ (-3) \otimes \lambda = (-1) \otimes \lambda^{\otimes^2} \ \oplus \ 3 \ . \tag{8}$$

In Figure 1 we have plotted the graphs of the max-algebraic polynomials on the left-hand and the right-hand side of (8). Clearly, (8) has only one simple root, viz. $\lambda^* = 1$. Hence, Olsder's statement that (7) cannot be the max-algebraic characteristic equation of a 3 by 3 matrix if it has less than three (possibly coinciding) solutions, does not hold. As a consequence, Conjecture 5 does not hold either.

Remark: In contrast to linear algebra, there exist no inverse elements w.r.t. $\oplus$ in $\mathbb{R}_\varepsilon$. To overcome this problem we can use the symmetrized max-plus algebra $\mathbb{S}_{\max}$ [6, 7], which is a kind of symmetrization of the max-plus algebra. This symmetrization leads to the introduction of the $\ominus$ operator,

[§]The words between square brackets have been added to make the formulation consistent with the one that has been used in the previous sections.

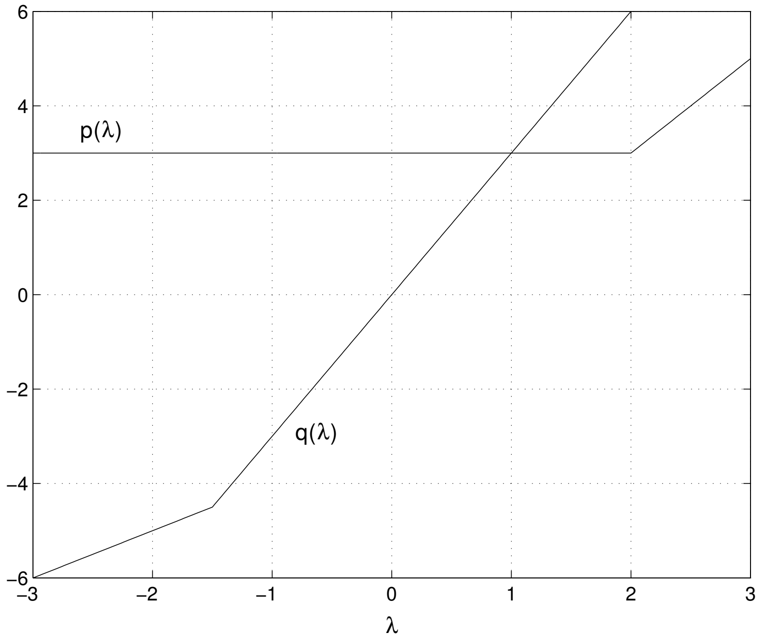

Figure 1: The graphs of the max-algebraic polynomials on the left-hand and the right-hand side of (8): $p(\lambda) = (-1) \otimes \lambda^{\otimes^2} \oplus 3$ and $q(\lambda) = \lambda^{\otimes^3} \oplus (-3) \otimes \lambda$.

which could be compared with the $-$ operator from conventional algebra. Once this $\ominus$ operator has been introduced we can define a max-algebraic equivalent of e.g. the determinant (where we replace $+$ by $\oplus$, $\times$ by $\otimes$ and $-$ by $\ominus$ in the conventional definition of the determinant). Once we have defined the max-algebraic determinant we can define the max-algebraic characteristic polynomial of a matrix. It is worth mentioning here that this leads to another version of the max-algebraic characteristic equation that differs from the characteristic equation introduced above, but for which the Cayley-Hamilton theorem also holds. For more information on the symmetrized max-plus algebra $\mathbb{S}_{\max}$ and the max-algebraic characteristic equation in $\mathbb{S}_{\max}$ the interested reader is referred to [1, 6, 7]. In [4, 5] we have given necessary conditions for the coefficients of the max-algebraic characteristic polynomial in $\mathbb{S}_{\max}$ of a matrix with entries in $\mathbb{R}_\varepsilon$. If the size of the matrix is less than or equal to 4 these conditions are also sufficient.

5. CONCLUSION

In this paper we have presented a corrected version of the derivation of the characteristic equation of a matrix in the max-plus algebra. Furthermore, we have also given a counterexample for a conjecture of Olsder in which he states necessary and sufficient conditions for the coefficients of the max-algebraic characteristic equation.

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