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Generalized Linear Complementarity Problems and the Analysis of Continuously Variable Systems and Discrete Event Systems

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Abstract. We present an overview of our research on the use of generalized linear complementarity problems (LCPs) for analysis of continuously variable systems and discrete event systems. We indicate how the Generalized LCP can be used to analyze piecewise-linear resistive electrical circuits. Next we discuss how the Extended LCP can be used to solve some fundamental problems that arise in max-algebraic system theory for discrete event systems. This shows that generalized LCPs appear in the analysis and modeling of certain continuously variable systems and discrete event systems. Since hybrid systems exhibit characteristics of both continuously variable systems and discrete event systems, this leads to the question as to whether generalized LCPs can also play a role in the modeling and analysis of certain classes of hybrid systems.

1 Introduction

In our research we have developed extensions of the linear complementarity problem (LCP), which is one of the basic problems in mathematical programming. We have used one extension in the analysis of electrical circuits with piecewise-linear characteristics [5, 13], which can be considered as examples of continuously variable systems (CVSs). Another extension of the LCP has been used in the analysis of a class of discrete event systems (DESs) that can be described by a state space model that is linear in the max-plus algebra [8, 9]. In this paper we present an overview of this research. Since hybrid systems exhibit characteristics of both CVSs and DESs, this suggests that extensions of the LCP will probably also be useful in the analysis of hybrid systems.

2 Generalized and Extended Linear Complementarity Problems

One of the possible formulations of the Linear Complementarity Problem (LCP) is the following [3]:

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Given $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, find w, $z \in \mathbb{R}^n$ such that $w \geq 0$, $z \geq 0$, $w = q + Mz$ and $z^T w = 0$.

The LCP has numerous applications such as quadratic programming problems, determination of the Nash equilibrium of a bimatrix game problem, the market equilibrium problem, the optimal invariant capital stock problem, the optimal stopping problem, etc. [3].

In [4, 5] De Moor introduced the following generalization of the LCP:

Given $Z \in \mathbb{R}^{p \times n}$ and m subsets $\phi_1, \ldots, \phi_m$ of $\{1, \ldots, n\}$, find a non-trivial $u \in \mathbb{R}^n$ such that $\displaystyle\sum_{j=1}^{m} \prod_{i \in \phi_j} u_i = 0$ subject to $u \geq 0$ and $Zu = 0$.

This problem is called the Generalized LCP (GLCP). In Sect. 3 we shall see that the GLCP can be used to determine operating points and transfer characteristics of piecewise-linear resistive electrical circuits.

Another extension of the LCP, the Extended LCP (ELCP), is defined as follows [6, 7]:

Given $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{q \times n}$, $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$ and m subsets $\phi_1, \ldots,$ ϕ_m of $\{1, \ldots, p\}$, find $x \in \mathbb{R}^n$ such that $\displaystyle\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i = 0$ subject to $Ax \geq c$ and $Bx = d$.

In Sect. 4 we shall see that the ELCP can be used to solve many problems that arise in the system theory for max-linear time-invariant DESs.

It can be shown that the ELCP is a generalization of the GLCP and that the homogeneous ELCP and the GLCP are equivalent [6, 7]. In [4] De Moor has developed an algorithm to compute the complete solution set of a GLCP. In [7] we have extended this algorithm in order to compute the complete solution set of an ELCP.

3 The GLCP and Piecewise-Linear Resistive Electrical Circuits

In this section we consider electrical circuits that may contain the following elements: linear resistive elements, piecewise-linear (PWL) resistors (the resistors are not required to be either voltage or current controlled), and PWL controlled sources (all four types) with one controlling variable (the characteristics may be multi-valued). The key idea behind the reformulation of the equations that describe the relations between the voltages and currents in the circuit as a (special case of a) GLCP is an intelligent parameterization of the PWL characteristics.

If x is a vector, then we define $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$, where the operations are performed componentwise. An equivalent definition is:

$$x = x^+ - x^- \,, \qquad x^+, x^- \geq 0 \,, \qquad (x^+)^T x^- = 0 \ .$$

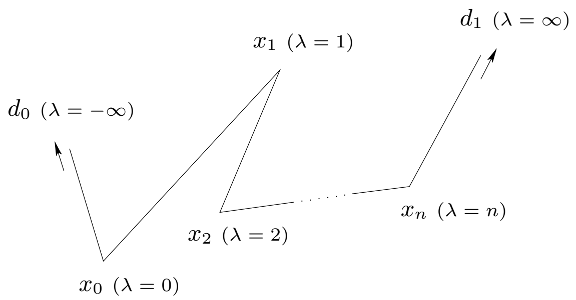

Fig. 1. A one-dimensional PWL curve.

For sake of simplicity we consider only two-terminal resistors since they can be described by a one-dimensional PWL manifold[3]. It is easy to verify that a one-dimensional PWL curve characterized by $n+1$ breakpoints $x_0, \ldots, x_n$ and two directions d_0 and d_1 (see Fig. 1) can be parameterized as follows [4, 13]:

$$
\begin{aligned}
x \;=\; & x_0 + d_0\,\lambda^- + (x_1 - x_0)\,\lambda^+ + \sum_{k=2}^{n} (x_k - 2x_{k-1} + x_{k-2})(\lambda - k + 1)^+ + \\
& (d_1 - x_n + x_{n-1})(\lambda - n)^+ \ . \qquad (1)
\end{aligned}
$$

Introducing auxiliary variables $\lambda_i = \lambda - i$ yields a description of the following form:

$$
\begin{aligned}
x \;&=\; x_0 + A\,y^- + B\,y^+ \\
C\,(y^+ - y^-) \;&=\; d \\
y^+, y^- \;&\geq\; 0 \\
(y^+)^T y^- \;&=\; 0
\end{aligned}
$$

where $y = [\,\lambda \;\; \lambda_1 \;\ldots\; \lambda_n\,]^T$.

If we extract all nonlinear resistors out of the electrical circuit, the resulting N-port contains only linear resistive elements and independent sources. As a consequence, the relation between the branch currents and voltages of this N-port is described by a system of linear equations. If we combine these equations with the PWL descriptions (1) of the nonlinear resistors, we finally get a system of the form:

$$
Mw^+ + Nw^- = q\,, \qquad w^+, w^- \geq 0\,, \qquad (w^+)^T(w^-) = 0 \ , \qquad (2)
$$

where the vector w contains the parameters λ and λ_i of the PWL descriptions of all the nonlinear resistors. It is easy to verify that after multiplying q by a nonnegative homogenization parameter α and including the extra condition

[3] If we also allow multi-terminal nonlinear resistors, which can be modeled by higher-dimensional PWL manifolds, we shall obtain the general GLCP that has been defined in Sect. 2 instead of the special GLCP of (2) (See [4]).

$\alpha \geq 0$, (2) can be considered as a special case of the GLCP. If we solve (2), we get the complete set of operating points of the electrical circuit.

In a similar way we can determine the driving-point characteristic (i_{in} versus v_{in}) and transfer characteristics of the electrical circuit [13].

The behavior of an electrical network consisting of linear resistors, capacitors, inductors, transformers, gyrators and ideal diodes can be described by a model of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

subject to the conditions

$$y(t) \geq 0, \quad u(t) \geq 0, \quad (y(t))^T u(t) = 0 \tag{3}$$

(see e.g., [12]). In order to compute the *stationary points* of such an electrical circuit, we add the condition $\dot{x}(t) = 0$, which leads to an LCP [12]. If we replace (3) by more general conditions of the form $w_i \geq 0$, $z_i \geq 0$, $w_i\, z_i = 0$, where w_i and z_i are components of u, y or x, then we get (a special case of) an ELCP.

4 The ELCP and Max-Linear Time-Invariant DESs

In general the description of DESs is nonlinear. However, there exists a class of DESs — the so-called *max-linear DESs* — for which the description becomes "linear" when we express it in the max-plus algebra [1, 2]. Loosely speaking we could say that this subclass corresponds to the class of deterministic time-invariant DESs in which only synchronization and no concurrency occurs.

The basic operations of the max-plus algebra are maximization (represented by $\oplus$) and addition (represented by $\otimes$). There exists a remarkable analogy between the basic operations of the max-plus algebra on the one hand, and the basic operations of conventional algebra (addition and multiplication) on the other hand. As a consequence many concepts and properties of conventional algebra (such as Cramer's rule, eigenvectors and eigenvalues, the Cayley-Hamilton theorem, ...) also have a max-algebraic analogue. Furthermore, this analogy also allows us to translate many concepts, properties and techniques from conventional linear system theory to system theory for max-linear time-invariant DESs. However, there are also some major differences that prevent a straightforward translation of properties, concepts and algorithms from conventional linear algebra and linear system theory to max-plus algebra and max-algebraic system theory for DESs.

If we write down a model for a max-linear DES and if we use the symbols $\oplus$ and $\otimes$ to denote maximization and addition, we obtain a description of the following form:

$$x(k+1) = A \otimes x(k) \;\oplus\; B \otimes u(k) \tag{4}$$

$$y(k) = C \otimes x(k) \;, \tag{5}$$

where x is the state vector, u the input vector and y the output vector. For a manufacturing system $u(k)$ would typically represent the time instants at which raw material is fed to the system for the $(k+1)$st time; $x(k)$ the time instants at which the machines start processing the kth batch of intermediate products; and $y(k)$ the time instants at which the kth batch of finished products leaves the system. In analogy with the state space model for linear time-invariant discrete-time systems, a model of the form (4) – (5) is called a *max-linear time-invariant state space model*.

Let $x, r \in \mathbb{R}$. The rth max-algebraic power of x is denoted by $x^{\otimes^r}$ and corresponds to rx in conventional algebra.

Now consider the following problem:

> Given $p_1 + p_2$ positive integers $m_1, \ldots, m_{p_1+p_2}$ and real numbers a_{ki}, b_k and c_{kij} for $k = 1, \ldots, p_1 + p_2$, $i = 1, \ldots, m_k$ and $j = 1, \ldots, n$, find $x \in \mathbb{R}^n$ such that
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{c_{kij}}} = b_k \qquad \text{for } k = 1, \ldots, p_1 , \tag{6}$$
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{c_{kij}}} \leq b_k \qquad \text{for } k = p_1 + 1, \ldots, p_1 + p_2 . \tag{7}$$

We call (6) – (7) a *system of multivariate max-algebraic polynomial equalities and inequalities*. Note that the exponents may be negative or real.

In [6, 10] we have shown that the problem of solving a system of multivariate max-algebraic polynomial equalities and inequalities can be recast as an ELCP. This enables us to solve many important problems that arise in the max-plus algebra and in the system theory for max-linear DESs such as: computing max-algebraic matrix factorizations, performing max-algebraic state space transformations, computing state space realizations of the impulse response of a max-linear time-invariant DES, constructing matrices with a given max-algebraic characteristic polynomial, computing max-algebraic singular value decompositions and QR decompositions, and so on [6, 7, 8, 9, 10].

Although the analogues of these problems in conventional algebra and linear system theory are easy to solve, the max-algebraic problems are not that easy to solve and for almost all of them the ELCP approach is at present the only way to solve the problem.

5 Conclusions and Future Research

We have defined two extensions of the linear complementarity problem (LCP): the Generalized Linear Complementarity Problem (GLCP) and the Extended Linear Complementarity Problem (ELCP). First we have indicated how the GLCP can be used to analyze piecewise-linear resistive electrical circuits, which are examples of continuously variable systems (CVSs). Next we have indicated how the ELCP can be used to solve some problems that arise in the max-algebraic system theory for max-linear discrete event systems (DESs). So generalized LCPs appear in the analysis and modeling of certain classes of CVSs and DESs. Since hybrid systems exhibit characteristics of both CVSs and DESs, it would be interesting to determine whether the GLCP, the ELCP or other — even more general — generalized LCPs also play a role in the modeling and analysis of certain classes of hybrid systems. The results of [11] seem to indicate that this is indeed the case.

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References

1. F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat, *Synchronization and Linearity*. New York: John Wiley & Sons, 1992.
2. G. Cohen, D. Dubois, J.P. Quadrat, and M. Viot, "A linear-system-theoretic view of discrete-event processes and its use for performance evaluation in manufacturing," *IEEE Trans. on Aut. Control*, vol. 30, no. 3, pp. 210–220, Mar. 1985.
3. R.W. Cottle, J.S. Pang, and R.E. Stone, *The Linear Complementarity Problem*. Boston: Academic Press, 1992.
4. B. De Moor, *Mathematical Concepts and Techniques for Modelling of Static and Dynamic Systems*. PhD thesis, Fac. of Applied Sc., K.U.Leuven, Belgium, 1988.
5. B. De Moor, L. Vandenberghe, and J. Vandewalle, "The generalized linear complementarity problem and an algorithm to find all its solutions," *Math. Prog.*, vol. 57, pp. 415–426, 1992.
6. B. De Schutter, *Max-Algebraic System Theory for Discrete Event Systems*. PhD thesis, Fac. of Applied Sc., K.U.Leuven, Belgium, 1996.
7. B. De Schutter and B. De Moor, "The extended linear complementarity problem," *Math. Prog.*, vol. 71, no. 3, pp. 289–325, Dec. 1995.
8. B. De Schutter and B. De Moor, "Minimal realization in the max algebra is an extended linear complementarity problem," *Syst. & Control Letters*, vol. 25, no. 2, pp. 103–111, May 1995.
9. B. De Schutter and B. De Moor, "Applications of the extended linear complementarity problem in the max-plus algebra," *Proc. of WODES'96 (Internat. Workshop on Discrete Event Syst.)*, Edinburgh, UK, pp. 69–74, Aug. 1996.
10. B. De Schutter and B. De Moor, "A method to find all solutions of a system of multivariate polynomial equalities and inequalities in the max algebra," *Discrete Event Dynamic Systems: Theory and Appl.*, vol. 6, no. 2, pp. 115–138, Mar. 1996.
11. B. De Schutter and B. De Moor, "Optimal traffic signal control for a single intersection," Tech. rep. 96-90, ESAT/SISTA, K.U.Leuven, Belgium, Dec. 1996.
12. J.M. Schumacher, "Some modeling aspects of unilaterally constrained dynamics," *Proc. of the ESA Internat. Workshop on Adv. Math. Methods in the Dynamics of Flexible Bodies*, Noordwijk, The Netherlands, June 1996.
13. L. Vandenberghe, B. De Moor, and J. Vandewalle, "The generalized linear complementarity problem applied to the complete analysis of resistive piecewise-linear circuits," *IEEE Trans. on Circ. and Syst.*, vol. 36, no. 11, pp. 1382–1391, Nov. 1989.