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B. De Schutter and B. De Moor

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ESAT-SISTA  
K.U.Leuven  
Leuven, Belgium  
phone: +32-16-32.17.09 (secretary)  
fax: +32-16-32.19.70  
URL: <http://www.esat.kuleuven.ac.be/sista-cosic-docarch>

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# Optimal traffic light control for a single intersection

Bart De Schutter\* and Bart De Moor†

*ESAT-SISTA, K.U. Leuven, Kardinaal Mercierlaan 94, B-3001 Leuven (Heverlee), Belgium.  
bart.deschutter@esat.kuleuven.ac.be, bart.demoor@esat.kuleuven.ac.be*

**Abstract**— We consider a single intersection of two two-way streets with controllable traffic lights on each corner. We construct a model that describes the evolution of the queue lengths in each lane as a function of time, and we discuss how (sub)optimal traffic light switching schemes for this system can be determined.

## I. Introduction

In this paper we study the optimal traffic light control problem for a single intersection. Given the arrival rates and the maximal departure rates of vehicles at the intersection we compute traffic light switching schemes that minimize criteria such as average queue length, worst case queue length, average waiting time, . . . , thereby augmenting the flow of traffic and diminishing the effects of traffic congestion. We show that for a special class of objective functions (i.e., objective functions that depend strictly monotonously on the queue lengths at the traffic light switching time instants), the optimal traffic light switching scheme can be computed very efficiently. Moreover, if the objective function is linear, the problem reduces to a linear programming problem.

## II. The set-up and the model of the system

We consider an intersection of two two-way streets (see Figure 1). There are four lanes  $L_1$ ,  $L_2$ ,  $L_3$  and  $L_4$ , and on each corner of the intersection there is a traffic light ( $T_1$ ,  $T_2$ ,  $T_3$  and  $T_4$ ). For sake of simplicity we assume that the traffic lights can be either green or red. The average arrival rate of cars in lane  $L_i$  is  $\lambda_i$ . When the traffic light is green, the average departure rate in lane  $L_i$  is  $\mu_i$ . Let  $t_0, t_1, t_2, \dots$  be the time instants at which the traffic lights switch from green to red or vice versa. The traffic light switching scheme is shown in Table 1. Let  $l_i(t)$  be the queue length (i.e., the number of cars waiting) in lane  $L_i$  at time instant  $t$ . Normally,  $l_i(t)$  can only take on integer values, but we use continuous approximations for the queue lengths. Note that although in general this will not lead to an optimal traffic light switching scheme, the traffic light

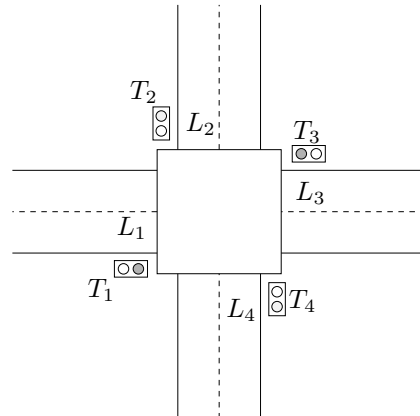


Figure 1: A traffic light controlled intersection of two two-way streets.

Period	$T_1$	$T_2$	$T_3$	$T_4$
$t_0 - t_1$	red	green	red	green
$t_1 - t_2$	green	red	green	red
$t_2 - t_3$	red	green	red	green
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: The traffic light switching scheme.

switching scheme that we shall obtain will be a good approximation to the optimal scheme. Furthermore, in practice there will also be uncertainty due to inaccurate measurements and the time-varying nature of the arrival and departure rates. Define  $\delta_k = t_{k+1} - t_k$  for  $k \in \mathbb{N}$ .

Let us now write down the equations that describe the relation between the switching time instants and the queue lengths.

When the traffic light  $T_1$  is red, there are arrivals at lane  $L_1$  and no departures. Hence,

$$\frac{dl_1(t)}{dt} = \lambda_1$$

for  $t \in (t_{2k}, t_{2k+1})$  with  $k \in \mathbb{N}$ , and

$$l_1(t_{2k+1}) = l_1(t_{2k}) + \lambda_1 \delta_{2k}$$

\* Senior Research Assistant with the F.W.O. (Fund for Scientific Research – Flanders).

† Senior Research Associate with the F.W.O.

for  $k = 0, 1, 2, \dots$ . When the traffic light  $T_1$  is green, there are arrivals and departures at lane  $L_1$ . Since the net arrival rate is  $\lambda_1 - \mu_1$  and since the queue length  $l_1(t)$  cannot be negative, we have:

$$\frac{dl_1(t)}{dt} = \begin{cases} \lambda_1 - \mu_1 & \text{if } l_1(t) > 0 \\ 0 & \text{if } l_1(t) = 0 \end{cases}$$

for  $t \in (t_{2k+1}, t_{2k+2})$  with  $k \in \mathbb{N}$ . So

$$l_1(t_{2k+2}) = \max(l_1(t_{2k+1}) + (\lambda_1 - \mu_1)\delta_{2k+1}, 0)$$

for  $k = 0, 1, 2, \dots$ . Note that we also have

$$l_1(t_{2k+1}) = \max(l_1(t_{2k}) + \lambda_1\delta_{2k}, 0)$$

for  $k = 0, 1, 2, \dots$ , since  $l_1(t) \geq 0$  for all  $t$ .

We can write down similar equations for  $l_2(t_k)$ ,  $l_3(t_k)$  and  $l_4(t_k)$ . So if we define

$$\begin{aligned} x_k &= [l_1(t_k) \quad l_2(t_k) \quad l_3(t_k) \quad l_4(t_k)]^T \\ b_1 &= [\lambda_1 \quad \lambda_2 - \mu_2 \quad \lambda_3 \quad \lambda_4 - \mu_4]^T \\ b_2 &= [\lambda_1 - \mu_1 \quad \lambda_2 \quad \lambda_3 - \mu_3 \quad \lambda_4]^T, \end{aligned}$$

then we have

$$x_{2k+1} = \max(x_{2k} + b_1\delta_{2k}, 0) \quad (1)$$

$$x_{2k+2} = \max(x_{2k+1} + b_2\delta_{2k+1}, 0) \quad (2)$$

for  $k = 0, 1, 2, \dots$

The model we have derived above is different from the models used by most other researchers due to the fact that we consider red-green cycle lengths that may vary from cycle to cycle. Furthermore, we consider non-saturated intersections, i.e., we allow queue lengths to become equal to 0 during the green cycle. For more information on other models that describe the evolution of the queue lengths at a traffic-light-controlled intersection and on optimal traffic light control the interested reader is referred to [4, 5, 6, 7, 9, 10] and the references given therein.

### III. Optimal traffic light control

From now on we assume that the arrival and departure rates are known. We want to compute an optimal sequence  $t_0, t_1, \dots, t_N$  of switching time instants that minimizes a given criterion  $J$ . Possible objective functions are:

- (weighted) average queue length over all queues:

$$J_1 = \sum_{i=1}^4 w_i \frac{1}{t_N} \int_0^{t_N} l_i(t) dt, \quad (3)$$

- (weighted) average queue length over the worst queue:

$$J_2 = \max_i \left( w_i \frac{1}{t_N} \int_0^{t_N} l_i(t) dt \right), \quad (4)$$

- (weighted) worst case queue length:

$$J_3 = \max_{i,t} (w_i l_i(t)), \quad (5)$$

- (weighted) average waiting time over all queues:

$$J_4 = \sum_{i=1}^4 w_i \frac{1}{\lambda_i t_N} \int_0^{t_N} l_i(t) dt, \quad (6)$$

- (weighted) average waiting time over the worst queue:

$$J_5 = \max_i \left( w_i \frac{1}{\lambda_i t_N} \int_0^{t_N} l_i(t) dt \right), \quad (7)$$

where  $w_i > 0$  for all  $i$ . Note that  $J_1$  and  $J_4$  are in fact equivalent in the sense that for any weight vector  $w$  for  $J_1$  there exists a weight vector  $\tilde{w}$  for  $J_4$  such that  $J_1$  and  $J_4$  are equal. This also holds for  $J_2$  and  $J_5$ .

Define the following sets:

$$\begin{aligned} \alpha(N) &= \left\{ 0, 1, \dots, \left\lfloor \frac{N-1}{2} \right\rfloor \right\} \\ \beta(N) &= \left\{ 0, 1, \dots, \left\lfloor \frac{N}{2} \right\rfloor - 1 \right\}. \end{aligned}$$

We can impose some extra conditions such as minimum and maximum durations for the red and green time, maximum queue lengths, and so on. This leads to the following problem:

$$\text{minimize } J \quad (8)$$

subject to

$$\delta_{\min,r} \leq \delta_{2k} \leq \delta_{\max,r} \quad \text{for } k \in \alpha(N) \quad (9)$$

$$\delta_{\min,g} \leq \delta_{2k+1} \leq \delta_{\max,g} \quad \text{for } k \in \beta(N) \quad (10)$$

$$x_k \leq x_{\max} \quad \text{for } k = 1, \dots, N \quad (11)$$

$$x_{2k+1} = \max(x_{2k} + b_1\delta_{2k}, 0) \quad \text{for } k \in \alpha(N) \quad (12)$$

$$x_{2k+2} = \max(x_{2k+1} + b_2\delta_{2k+1}, 0) \quad \text{for } k \in \beta(N). \quad (13)$$

Now we discuss some methods to solve problem (8)–(13).

Consider (12) for an arbitrary index  $k$ . It is easy to verify that this equation is equivalent to:

$$x_{2k+1} - x_{2k} - b_1\delta_{2k} \geq 0$$

$$x_{2k+1} \geq 0$$

$$\sum_{i=1}^4 (x_{2k+1} - x_{2k} - b_1\delta_{2k})_i (x_{2k+1})_i = 0$$

(See also [3]). We can repeat this reasoning for (13) and for each  $k$ .

If we define  $x^* = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$  and  $\delta^* = \begin{bmatrix} \delta_0 \\ \vdots \\ \delta_{N-1} \end{bmatrix}$ , we finally get a problem of the form

$$\text{minimize } J \quad (14)$$

subject to

$$Ax^* + B\delta^* + c \geq 0 \quad (15)$$

$$x^* \geq 0 \quad (16)$$

$$Ex^* + D\delta^* + f \geq 0 \quad (17)$$

$$(Ax^* + B\delta^* + c)^T x^* = 0. \quad (18)$$

The system (15)–(18) is a special case of an Extended Linear Complementarity Problem (ELCP) [1, 2]. In [1, 2] we have developed an algorithm to compute a parametric description of the complete solution set of an ELCP. Once this description has been obtained, we can determine the values of the parameters for which  $J$  reaches a global minimum over the solution set of the ELCP. It can be shown that  $J_3$  is a convex function of the parameters [3]. Furthermore, our computational experiments have shown that the determination of the minimum value of  $J_1$ ,  $J_2$ ,  $J_4$  or  $J_5$  over the solution set of the ELCP is a well-behaved problem in the sense that using a local minimization routine (that uses, e.g., sequential quadratic programming) starting from different initial points always yields the same numerical result (within a certain tolerance).

However, the general ELCP is an NP-hard problem [1, 2]. Furthermore, the size of the parametric description of the solution set of an ELCP with  $p$  (non-redundant) inequality constraints in an  $n$ -dimensional space is  $O\left(p^{\lfloor \frac{n}{2} \rfloor}\right)$  if  $p \gg n \gg 1$ . This implies that the approach sketched above is not feasible if the number of switching cycles  $N$  is large. In [3] we have developed a method to compute suboptimal solutions of problem (8)–(13) by considering several ELCPs of a smaller size.

Note that if there is no upper bound on  $x_k$ , then problem (8)–(13) reduces to a constrained optimization problem in  $\delta^*$ , which could be solved using a constrained minimization procedure. The major disadvantage of this approach is that in general the resulting objective function is neither convex nor concave so that the minimization routine will only return a local minimum. However, in [3] we have shown that  $J_3$  is convex as a function of  $\delta^*$ . This implies that problem (8)–(13) with  $J = J_3$  can be solved efficiently (if there is no upper bound on the queue lengths or if we introduce a convex penalty term if one or more components of  $x_{\max}$  are finite).

Now we define some objective functions that can be considered as approximations to the objective func-

tions  $J_i$  ( $i = 1, \dots, 5$ ). Although these approximations will lead to suboptimal solutions with respect to  $J_i$ , the corresponding optimization problems can be solved much more efficiently<sup>†</sup>.

Note that the objective functions  $J_i$  do not depend explicitly on  $x^*$  since for given  $x_0$ ,  $\lambda_i$ 's and  $\mu_i$ 's, the components of  $x^*$  (i.e., the  $x_k$ 's) are uniquely determined by  $\delta^*$ . The approximate objective functions that we introduce now will depend explicitly on  $x^*$  and  $\delta^*$ . For a given  $x_0$ ,  $x^*$  and  $\delta^*$  we define the function  $\tilde{l}_i$  as the piecewise-linear function that interpolates in the points  $(t_k, (x_k)_i)$  for  $k = 0, \dots, N$ . The approximate objective functions  $\tilde{J}_i$  ( $i = 1, \dots, 5$ ) are defined as in (3)–(7) but with  $l_i$  replaced by  $\tilde{l}_i$ .

We say that the vectors  $x^*$  and  $\delta^*$  are *compatible* for a given  $x_0$  if the corresponding sequences  $\{x_k\}_{k=0}^N$  and  $\{\delta_k\}_{k=0}^{N-1}$  satisfy the recurrence equations (1) and (2) for all  $k$ .

**Proposition 1** *If  $x^*$  and  $\delta^*$  are compatible for a given  $x_0$  then we have  $J_3(\delta) = \tilde{J}_3(x, \delta)$  and  $J_l(\delta) \leq \tilde{J}_l(x, \delta)$  for  $l = 1, 2, 4, 5$ .*

**Proof:** See [3]. □

Let the queue length vector  $x^*$  and the switching interval vector  $\delta^*$  be compatible for a given  $x_0$ . Note that if the queue lengths never stay 0 in a time interval then  $J_l(\delta^*)$  and  $\tilde{J}_l(x^*, \delta^*)$  are equal. In practice, an optimal traffic light switching scheme implies the absence of long periods in which no cars wait in one lane while in the other lanes the queue lengths increase, i.e., the periods during which the queue length in some lane is equal to 0 are in general short for optimal traffic light switching schemes. This implies that for traffic light switching schemes in the neighborhood of the optimal scheme  $\tilde{J}_l$  will be a good approximation of  $J_l$ .

Now we show that the use of the approximate objective function  $\tilde{J}_1$  or  $\tilde{J}_4$  leads to an optimization problem that can be solved more efficiently than the original problem in which  $J_1$  or  $J_4$  is used.

Let  $\mathcal{P}$  be the problem (8)–(13). We define the “relaxed” problem  $\tilde{\mathcal{P}}$  corresponding to the original problem  $\mathcal{P}$  as follows:

$$\text{minimize } \tilde{J} \quad (19)$$

subject to

$$\delta_{\min, r} \leq \delta_{2k} \leq \delta_{\max, r} \quad \text{for } k \in \alpha(N) \quad (20)$$

<sup>†</sup>Recall that we have already introduced an approximation by considering continuous queue lengths. Furthermore, there is also some uncertainty and variation in time of the arrival and departure rates, which makes that in general computing the exact optimal traffic light switching scheme is not feasible. Moreover, in practice we are more interested in quickly obtaining a good approximation of the optimal traffic light switching scheme than in spending a large amount of time to obtain the exact optimal switching scheme.

$$\delta_{\min, g} \leq \delta_{2k+1} \leq \delta_{\max, g} \quad \text{for } k \in \beta(N) \quad (21)$$

$$0 \leq x_k \leq x_{\max} \quad \text{for } k = 1, \dots, N, \quad (22)$$

$$x_{2k+1} \geq x_{2k} + b_1 \delta_{2k} \quad \text{for } k \in \alpha(N) \quad (23)$$

$$x_{2k+2} \geq x_{2k+1} + b_2 \delta_{2k+1} \quad \text{for } k \in \beta(N) . \quad (24)$$

Note that compared to the original problem we have replaced (12)–(13) by the relaxed equations  $0 \leq x_k$ , (23) and (24), i.e., we do not take the maximum condition into account.

**Proposition 2** *If  $J$  is a strictly monotonous function of  $x^*$  — i.e., if for any  $\delta^*$  with positive components and for all  $\tilde{x}^*, \hat{x}^*$  with  $0 \leq \tilde{x}^* \leq \hat{x}^*$  and with  $\tilde{x}_j^* < \hat{x}_j^*$  for at least one index  $j$ , we have  $J(\tilde{x}^*, \delta^*) < J(\hat{x}^*, \delta^*)$  — then any solution of the relaxed problem  $\tilde{\mathcal{P}}$  is also a solution of the original problem  $\mathcal{P}$ .*

**Proof:** See [3]. □

In [3] we have shown that  $\tilde{J}_1$  and  $\tilde{J}_4$  are strictly monotonous functions of  $x^*$ , i.e., they satisfy the conditions of Proposition 2.

Note that in general it is easier to solve the relaxed problem  $\tilde{\mathcal{P}}$  since the set of feasible solutions of  $\tilde{\mathcal{P}}$  is a convex set, whereas the set of feasible solutions of  $\mathcal{P}$  is in general not convex since it consists of the union of a subset of the set of faces of the polyhedron defined by the system of inequalities (20)–(24) (See [1, 2]).

In [3] we have also shown that if all the  $\delta_k$ 's are equal then  $\tilde{J}_1$  and  $\tilde{J}_4$  are linear, strictly monotonous functions of  $x^*$ , which implies that problem (8)–(13) then reduces to a linear programming problem, which can be solved efficiently using (variants of the) simplex method or using an interior point method (see, e.g., [8]).

For a worked-out example of the computation of (sub)optimal traffic light schemes for the set-up that has been considered in this paper the interested reader is referred to [3].

#### IV. Conclusions and further research

We have derived a model that describes the evolution of the queue lengths at an intersection of two two-way streets with controllable traffic lights on each corner. We have indicated how an optimal traffic light switching scheme for the given system can be determined. We have shown that for objective functions that depend strictly monotonously on the queue lengths at the traffic light switching time instants, the optimal traffic light switching scheme can be computed very efficiently. We have discussed some approximate objective functions for which this property holds. Moreover, if the objective function is linear, the problem reduces to a linear programming problem.

Topics for further research include: inclusion of an all-red phase and/or an amber phase, extension of the

results to traffic networks or clusters of intersections, extension to models with integer queue lengths, development of efficient algorithms to compute optimal traffic light switching schemes for the objective functions discussed in this paper and for other more general objective functions.

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