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Abstract

We derive upper bounds for the index of cyclicity of a matrix as a function of the size of the matrix. This result can be used in the characterization of the ultimate behavior of the sequence of consecutive powers of a matrix in the max-plus algebra, which has maximum and addition as its basic operations. If the matrix is irreducible then it is well known that the ultimate behavior is cyclic. For reducible matrices the behavior is more complex, but it is also cyclic in nature. The length of the cycles corresponds to the index of cyclicity of the given matrix.

1 Introduction

In this paper consider the sequence of consecutive powers of a matrix in the max-plus algebra, which has maximum and addition as basic operations. For a general matrix the ultimate behavior is cyclic. The length of the cycles corresponds to the cyclicity of the given matrix. We derive upper bounds for the index of cyclicity of a given matrix.

Our main motivation for studying this problem lies in the max-plus-algebraic system theory for discrete event systems. Typical examples of discrete event systems are flexible manufacturing systems, telecommunication networks, parallel processing systems, traffic control systems and logistic systems. The class of discrete event systems essentially consists of man-made systems that contain a finite number of resources (e.g., machines, communications channels or processors) that are shared by several users (e.g., product types, information packets or jobs) all of which contribute to the achievement of some common goal (e.g., the assembly of products, the end-to-end transmission of a set of information packets, or a parallel computation).

There are many modeling and analysis techniques for discrete event systems, such as queuing theory, (extended) state machines, max-plus algebra, formal languages, automata, temporal logic, generalized semi-Markov processes, Petri nets, perturbation analysis, computer simulation and so on (see [1, 4, 13] and the references cited therein). In general models that describe the behavior of a discrete event system are nonlinear in conventional algebra. However, there is a class of discrete event systems – the max-plus-linear discrete event systems – that can be described by a model that is "linear" in the max-plus algebra [1, 5, 6]. The

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model of a max-plus-linear discrete event system can be characterized by a triple of matrices (A, B, C), which are called the system matrices of the model.

The index of cyclicity of the system matrix A determines the length of the cycles of the ultimate cyclic behavior of the max-plus-linear system. We derive an upper bound for the index of cyclicity of a matrix as a function of the size of the matrix. This corresponds to an upper bound for the length of the cycles of the ultimate cyclic behavior of a max-plus-linear discrete event system as a function of the minimal system order.

One of the open problems in the max-plus-algebraic system theory is the minimal realization problem, which consists in determining the system matrices of the model of a max-plus-linear discrete event system starting from its impulse response[1] such that the dimensions of the system matrices are as small as possible. In order to tackle the general minimal realization problem it is useful to first study a simplified version: the Boolean minimal realization problem, in which only models with Boolean system matrices are considered. In combination with the results of [9] the results on the index of cyclicity of this paper can be used to prove that the Boolean minimal realization problem in the max-plus algebra is decidable and that it can be solved in a time that is bounded from above by a function that is exponential in the minimal system order (see [8]).

This paper is organized as follows. In Section 2 we introduce some of the notations used in the paper. We also give a short introduction to the max-plus algebra and to graph theory, and we discuss the connection between max-plus-algebraic matrix operations and graph theory. We also characterize the ultimate behavior of the sequence of consecutive powers of a general max-plus-algebraic matrix. In Section 3 we derive a new upper bound for the index of cyclicity of a matrix. Finally we present some conclusions in Section 4.

2 Notation and definitions

If A is a matrix, then a_{ij} or $(A)_{ij}$ is the entry on the ith row and the jth column. If A is an m by n matrix and if $\alpha \subseteq \{1, 2, \ldots, m\}$, $\beta \subseteq \{1, 2, \ldots, n\}$ then $A_{\alpha\beta}$ is the submatrix of A obtained by removing all rows that are not indexed by α and all columns that are not indexed by β.

The set of the real numbers is denoted by $\mathbb{R}$, the set of the nonnegative integers by $\mathbb{N}$, and the set of the positive integers by $\mathbb{N}_0$.

If S is a set, then the number of elements of S is denoted by $\#S$. If γ is a set of positive integers then the least common multiple of the elements of γ is denoted by lcm γ and the greatest common divisor of the elements of γ is denoted by gcd γ.

2.1 Max-plus algebra

The basic operations of the max-plus algebra are the maximum (represented by $\oplus$) and the addition (represented by $\otimes$):

$$\begin{aligned} x \oplus y &= \max(x, y) \\ x \otimes y &= x + y \end{aligned}$$

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[1]The impulse response is the output of the system when a certain standardized input sequence is applied to the system (see [1] for more information).

with $x, y \in \mathbb{R} \cup \{-\infty\}$. Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The operations $\oplus$ and $\otimes$ are extended to matrices as follows. If $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ then we have

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$$

for all i, j. If $A \in \mathbb{R}_\varepsilon^{m \times p}$ and $B \in \mathbb{R}_\varepsilon^{p \times n}$ then

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj}$$

for all i, j. Note that these definitions resemble the definitions of the sum and the product of matrices in linear algebra but with $\oplus$ instead of $+$ and $\otimes$ instead of $\cdot$. This analogy is one of the reasons why we call $\oplus$ the max-plus-algebraic addition and $\otimes$ the max-plus-algebraic multiplication.

The rth max-plus-algebraic power of $x \in \mathbb{R}$ is denoted by $x^{\otimes^r}$ and corresponds to rx in conventional algebra. If $r > 0$ then $\varepsilon^{\otimes^r} = \varepsilon$. If $r < 0$ then $\varepsilon^{\otimes^r}$ is not defined. In this paper we have $\varepsilon^{\otimes^0} = 0$ by definition.

The matrix E_n is the n by n max-plus-algebraic identity matrix: we have $(E_n)_{ii} = 0$ for all i and $(E_n)_{ij} = \varepsilon$ for all i, j with $i \neq j$. The matrix $\mathcal{E}_{m \times n}$ is the m by n max-plus-algebraic zero matrix: $(\mathcal{E}_{m \times n})_{ij} = \varepsilon$ for all i, j. If the dimensions of the max-plus-algebraic identity matrix or zero matrix are not indicated, then they should be clear from the context. The max-plus-algebraic matrix power of the matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ is defined as follows:

$$A^{\otimes^0} = E_n$$
$$A^{\otimes^k} = A \otimes A^{\otimes^{k-1}} \qquad \text{for } k = 1, 2, \ldots$$

If we permute the rows or the columns of the max-plus-algebraic identity matrix, we obtain a max-plus-algebraic permutation matrix. If $P \in \mathbb{R}_\varepsilon^{n \times n}$ is a max-plus-algebraic permutation matrix, then we have $P \otimes P^T = P^T \otimes P = E_n$.

2.2 Max-plus algebra and graph theory

We assume that the reader is familiar with basic concepts of graph theory such as directed graph, loop, circuit, elementary circuit and so on (see, e.g., [1]).

A directed graph $\mathcal{G}$ is called strongly connected if for any two *different*[2] vertices v_i, v_j of $\mathcal{G}$ there exists a path from v_i to v_j. A maximal strongly connected subgraph (m.s.c.s.) $\mathcal{G}_{\text{sub}}$ of a directed graph $\mathcal{G}$ is a strongly connected subgraph that is maximal, i.e., if we add an extra vertex (and some extra arcs) of $\mathcal{G}$ to $\mathcal{G}_{\text{sub}}$ then $\mathcal{G}_{\text{sub}}$ is no longer strongly connected.

If we have a directed graph $\mathcal{G}$ with set of vertices $\mathcal{V} = \{1, 2, \ldots, n\}$ and if we associate a real number w_{ij} with each arc (j, i) of $\mathcal{G}$, then we say that $\mathcal{G}$ is a weighted directed graph. We call w_{ij} the weight of the arc (j, i). Note that the first subscript of w_{ij} corresponds to the final (and not the initial) vertex of the arc (j, i).

With every weighted graph $\mathcal{G}$ with set of vertices $\mathcal{V} = \{1, 2, \ldots, n\}$ there corresponds a matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ such that $a_{ij} = w_{ij}$ if there is an arc (j, i) in $\mathcal{G}$ with weight w_{ij} and $a_{ij} = \varepsilon$

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[2]Most authors do not add the extra condition that the vertices should be different. However, this definition which was taken from [1] makes some of the subsequent definitions, theorems and proofs easier to formulate.

if there is no arc (j, i) in $\mathcal{G}$. We say that $\mathcal{G}$ is the precedence graph of A. The precedence graph of a given matrix $A \in \mathbb{R}_{\varepsilon}^{n \times n}$ will be denoted by $\mathcal{G}(A)$.

Consider a matrix $A \in \mathbb{R}_{\varepsilon}^{n \times n}$ and its precedence graph $\mathcal{G}(A)$. The average weight of a circuit $i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_l \rightarrow i_1$ in $\mathcal{G}(A)$ is defined as the sum of the weights of the arcs that compose the circuit divided by the length of the circuit: $\frac{1}{l}(a_{i_2 i_1} + a_{i_3 i_2} + \cdots + a_{i_l i_{l-1}} + a_{i_1 i_l})$. A circuit of $\mathcal{G}(A)$ is called critical if it has maximum average weight. The critical graph $\mathcal{G}^{\mathrm{c}}(A)$ consists of those vertices and arcs of $\mathcal{G}(A)$ that belong to some critical circuit of $\mathcal{G}(A)$.

A matrix $A \in \mathbb{R}_{\varepsilon}^{n \times n}$ is called irreducible if its precedence graph is strongly connected. Note that the 1 by 1 max-plus-algebraic zero matrix $[\varepsilon]$ is the only max-plus-algebraic zero matrix that is irreducible.

The index of cyclicity [2] or cyclicity [1] or index of *imprimitivity*[3] [3, 10] of an m.s.c.s. is the greatest common divisor of the lengths of all the elementary circuits of the given m.s.c.s. If an m.s.c.s. or a graph contains no circuits then its index of cyclicity is equal to 1 by definition. The index of cyclicity $c(\mathcal{G})$ of a graph $\mathcal{G}$ is the least common multiple of the indices of cyclicity of its m.s.c.s.'s. Consider a matrix $A \in \mathbb{R}_{\varepsilon}^{n \times n}$. The cyclicity of a matrix $A \in \mathbb{R}_{\varepsilon}^{n \times n}$ is denoted by $c(A)$ and is equal to the cyclicity of the critical graph of the precedence graph of A. So $c(A) = c(\mathcal{G}^{\mathrm{c}}(A))$.

The following theorem gives a relation between the index of cyclicity of an irreducible max-plus algebraic matrix A and the ultimate behavior of the sequence $\{A^{\otimes^k}\}_{k=0}^{\infty}$.

Theorem 2.1 *If $A \in \mathbb{R}_{\varepsilon}^{n \times n}$ is irreducible, then*

$$\exists \lambda \in \mathbb{R}_{\varepsilon}, \exists k_0 \in \mathbb{N} \text{ such that } \forall k \geqslant k_0 : A^{\otimes^{k+c}} = \lambda^{\otimes^c} \otimes A^{\otimes^k}$$

where c is the cyclicity of A.

Proof: See, e.g., [1, 5, 11]. ∎

The following theorem is the max-plus-algebraic analogue of a well-known result from matrix algebra that states that any square matrix can be transformed into a block upper diagonal matrix with irreducible blocks by simultaneously reordering the rows and columns of the matrix (see, e.g., [1, 2, 3, 10, 12] for the proof of this theorem):

Theorem 2.2 *If $A \in \mathbb{R}_{\varepsilon}^{n \times n}$ then there exists a max-plus-algebraic permutation matrix $P \in \mathbb{R}_{\varepsilon}^{n \times n}$ such that the matrix $\hat{A} = P \otimes A \otimes P^T$ is a max-plus-algebraic block upper triangular matrix of the form*

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \ldots & \hat{A}_{1l} \\ \mathcal{E} & \hat{A}_{22} & \ldots & \hat{A}_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{E} & \mathcal{E} & \ldots & \hat{A}_{ll} \end{bmatrix} \tag{1}$$

with $l \geqslant 1$ and where the matrices $\hat{A}_{11}, \hat{A}_{22}, \ldots, \hat{A}_{ll}$ are square and irreducible. The matrices $\hat{A}_{11}, \hat{A}_{22}, \ldots, \hat{A}_{ll}$ are uniquely determined to within simultaneous permutation of their rows and columns, but their ordering in (1) is not necessarily unique.

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[3]We prefer to use the word "index of cyclicity" in this paper in order to avoid confusion with the concept "index of primitivity" [2, 20] of a nonnegative matrix A, which is defined to be the least positive integer $\gamma(A)$ such that all the entries of $A^{\gamma(A)}$ are positive.

The form in (1) is called the max-plus-algebraic Frobenius normal form of the matrix A. If A is irreducible then there is only one block in (1) and then A is a max-plus-algebraic Frobenius normal form of itself. Each diagonal block of $\hat{A}$ corresponds to an m.s.c.s. of the precedence graph of $\hat{A}$. If $\hat{A} = P \otimes A \otimes P^T$ is the max-plus-algebraic Frobenius normal form of $A \in \mathbb{R}_\varepsilon^{n \times n}$ where P is a max-plus-algebraic permutation matrix, then we have $A = P^T \otimes \hat{A} \otimes P$. Hence,

$$A^{\otimes k} = (P^T \otimes \hat{A} \otimes P)^{\otimes k} = P^T \otimes \hat{A}^{\otimes k} \otimes P$$

for all $k \in \mathbb{N}$. Therefore, we may consider without loss of generality the sequence $\{\hat{A}^{\otimes k}\}_{k=0}^\infty$ instead of $\{A^{\otimes k}\}_{k=0}^\infty$. Furthermore, since the transformation from A to $\hat{A}$ corresponds to a simultaneous reordering of the rows and columns of A (or to a reordering of the vertices of $\mathcal{G}(A)$), we have $c(A) = c(\hat{A})$. For the ultimate behavior of the sequence $\{\hat{A}^{\otimes k}\}_{k=0}^\infty$ we have:

Theorem 2.3 *Let $\hat{A} \in \mathbb{R}_\varepsilon^{n \times n}$ be a matrix of the form (1) where the matrices $\hat{A}_{11}$, $\hat{A}_{22}$, ..., $\hat{A}_{ll}$ are square and irreducible. Let λ_i and c_i be respectively the max-plus-algebraic eigenvalue and the cyclicity of $\hat{A}_{ii}$ for $i = 1, 2, \ldots, l$. Define sets α_1, α_2, ..., α_l such that $\hat{A}_{\alpha_i \alpha_j} = \hat{A}_{ij}$ for all i, j with $i \leqslant j$. Define*

$$S_{ij} = \big\{ \{i_0, i_1, \ldots, i_s\} \subseteq \{1, 2, \ldots, l\} \,\big|\, i = i_0 < i_1 < \ldots < i_s = j \text{ and}$$

$$\hat{A}_{i_r i_{r+1}} \neq \mathcal{E} \text{ for } r = 0, 1, \ldots, s - 1 \big\}$$

$$\Gamma_{ij} = \big\{ t \,\big|\, \exists \gamma \in S_{ij} \text{ such that } t \in \gamma \big\}$$

$$\Lambda_{ij} = \begin{cases} \displaystyle\bigcup_{t \in \Gamma_{ij}} \{\lambda_t\} & \text{if } \Gamma_{ij} \neq \emptyset, \\ \{\varepsilon\} & \text{if } \Gamma_{ij} = \emptyset, \end{cases}$$

$$c_{ij} = \begin{cases} \operatorname{lcm}\{c_t \,|\, t \in \Gamma_{ij}\} & \text{if } \Gamma_{ij} \neq \emptyset, \\ 1 & \text{otherwise}, \end{cases}$$

for all i, j with $i < j$. We have

$$\forall i, j \in \{1, 2, \ldots, l\} \text{ with } i > j : \left(\hat{A}^{\otimes k}\right)_{\alpha_i \alpha_j} = \mathcal{E} \quad \text{for all } k \in \mathbb{N} \ .$$

Moreover, there exists an integer $K \in \mathbb{N}$ such that

$$\forall i \in \{1, 2, \ldots, l\} : \left(\hat{A}^{\otimes k + c_i}\right)_{\alpha_i \alpha_i} = \lambda_i^{\otimes c_i} \otimes \left(\hat{A}^{\otimes k}\right)_{\alpha_i \alpha_i} \quad \text{for all } k \geqslant K$$

and

$$\forall i, j \in \{1, 2, \ldots, l\} \text{ with } i < j, \forall p \in \alpha_i, \forall q \in \alpha_j, \exists \gamma_0, \gamma_1, \ldots, \gamma_{c_{ij}-1} \in \Lambda_{ij} \text{ such that}$$

$$\left(\hat{A}^{\otimes k c_{ij} + c_{ij} + s}\right)_{pq} = \gamma_s^{\otimes c_{ij}} \otimes \left(\hat{A}^{\otimes k c_{ij} + s}\right)_{pq} \tag{2}$$

for all $k \geqslant K$ and for $s = 0, 1, \ldots, c_{ij} - 1$.

For each combination i, j, p, q with $i < j$, $p \in \alpha_i$ and $q \in \alpha_j$, there exists at least one index $s \in \{0, 1, \ldots, c_{ij} - 1\}$ such that the smallest γ_s for which (2) holds is equal to $\max \Lambda_{ij}$. Furthermore, there exists an integer K such that

$$\forall i, j \in \{1, \ldots, l\} \text{ with } i > j, \ \forall p \in \alpha_i, \forall q \in \alpha_j, \ \exists s \in \{0, \ldots, c_{ij} - 1\} \ \text{ such that}$$

$$\left(\hat{A}^{\otimes kc_{ij}+s+c_{ij}} \ \oplus \ \hat{A}^{\otimes kc_{ij}+s+c_{ij}+1} \ \oplus \ \ldots \ \oplus \ \hat{A}^{\otimes kc_{ij}+s+2c_{ij}-1} \right)_{pq} =$$

$$\lambda_{ij}{}^{\otimes c_{ij}} \otimes \left(\hat{A}^{\otimes kc_{ij}+s} \ \oplus \ \hat{A}^{\otimes kc_{ij}+s+1} \ \oplus \ \ldots \ \oplus \ \hat{A}^{\otimes kc_{ij}+s+c_{ij}-1} \right)_{pq} \quad \text{for all } k \geqslant K, \quad (3)$$

where $\lambda_{ij} = \max \Lambda_{ij}$.

Proof: See [7]. ∎

Note that the largest possible value for c_{ij} in this theorem is equal to $c(A)$. Furthermore, (2) and (3) also hold if we replace c_{ij} by c. Therefore, we will study c in more detail in the next section.

Remark 2.4 In this section we have treated the connection between max-plus-algebraic matrices and graphs. For nonnegative matrices we can introduce similar definitions (see, e.g., [3]). The precedence graph of a real matrix $A \in \mathbb{R}^{n \times n}$ is a graph with set of vertices $\{1, 2, \ldots, n\}$ and an arc (j, i) with weight a_{ij} for every nonzero entry a_{ij}. So here the absence of the arc (j, i) corresponds to a weight $a_{ij} = 0$, whereas for a max-plus-algebraic the absence of the arc (j, i) corresponds to a weight $a_{ij} = \varepsilon$ (Note that 0 is the zero element in conventional algebra and that ε is the zero element in the max-plus algebra). In [2, 3, 20] the index of cyclicity or index of imprimitivity is defined for an *irreducible* nonnegative real matrix $A \in \mathbb{R}^{n \times n}$. It corresponds to the index of cyclicity of the precedence graph of A and is also equal to the number of eigenvalues of maximum modulus of A. Note however that in the max-plus algebra the index of cyclicity is also defined for *reducible* matrices. ◇

3 Upper bounds for the index of cyclicity of a matrix

3.1 Tight upper bounds

In this section we consider tight upper bounds for the index of cyclicity of a matrix.

Lemma 3.1 *If a graph $\mathcal{G}$ with n vertices is strongly connected then we have $c(\mathcal{G}) \leqslant n$.*

Proof: If $\mathcal{G}$ contains only one vertex and no loop then we have $c(\mathcal{G}) = 1 \leqslant 1 = n$.
From now on we assume that there is at least one arc in $\mathcal{G}$. Since $\mathcal{G}$ is strongly connected, it contains only one m.s.c.s. Hence, $c(\mathcal{G})$ is the greatest common divisor of the lengths of the elementary circuits in $\mathcal{G}$. Since the maximal possible length of an elementary circuit of a graph with n vertices is n, $c(\mathcal{G})$ is maximal if there is only one circuit in $\mathcal{G}$ and if this circuit has length n. In that case we have $c(\mathcal{G}) = n$. In the other cases, $c(\mathcal{G})$ will be less than n. ∎

Example 3.2
Let $n \in \mathbb{N}_0$ and consider the circuit graph $\mathcal{G}_{\text{circ},n}$ with set of vertices $\{1, 2, \ldots, n\}$ and arcs $1 \to 2, 2 \to 3, \ldots, n-1 \to n$ and $n \to 1$ where all the arcs have weight 0 (see Figure 1). We
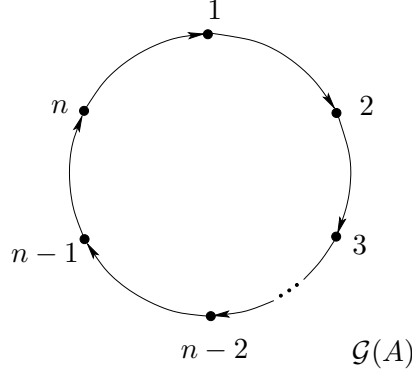
Figure 1: The circuit graph $\mathcal{G}_{\mathrm{circ},n}$ is a graph with set of vertices $\{1, 2, \ldots, n\}$ and arcs $1 \to 2$, $2 \to 3$, $\ldots$, $n-1 \to n$ and $n \to 1$. All the arcs have weight 0.

have $c(\mathcal{G}_{\mathrm{circ},n}) = n$. Furthermore, it is easy to verify that $\mathcal{G}_{\mathrm{circ},n}$ is the precedence graph of the max-plus-algebraic matrix

$$A = \begin{bmatrix} \varepsilon & \varepsilon & \ldots & \varepsilon & 0 \\ 0 & \varepsilon & \ldots & \varepsilon & \varepsilon \\ \varepsilon & 0 & \ldots & \varepsilon & \varepsilon \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varepsilon & \varepsilon & \ldots & 0 & \varepsilon \end{bmatrix} . \tag{4}$$

Since $\mathcal{G}^{\mathrm{c}}(A)$ coincides with $\mathcal{G}(A)$, we have $c(A) = c(\mathcal{G}^{\mathrm{c}}(A)) = c(\mathcal{G}(A)) = c(\mathcal{G}_{\mathrm{circ},n}) = n$. $\qquad \square$

Let $n, s \in \mathbb{N}_0$ with $s \leqslant n$. Define

$$F_{\mathrm{aux}}(n, s) = \max_{\substack{l_i \in \{1, 2, \ldots, n\} \\ l_1 + \cdots + l_s \leqslant n}} \mathrm{lcm}(l_1, l_2, \ldots, l_s) \tag{5}$$

and

$$F(n) = \max_{1 \leqslant s \leqslant n} F_{\mathrm{aux}}(n, s) \ . \tag{6}$$

The values of $F_{\mathrm{aux}}(n, s)$ and $F(n)$ for $n, s = 1, 2, \ldots, 10$ are listed in Table 1.

Proposition 3.3 *For any $n \in \mathbb{N}_0$ $F(n)$ is an upper bound for the index of cyclicity of any n by n max-plus-algebraic matrix. Furthermore, this upper bound is tight, i.e., there exists a matrix $A \in \mathbb{R}_{\varepsilon}^{n \times n}$ such that $c(A) = F(n)$.*

Proof: Let us first prove that $F(n)$ is an upper bound for the index of cyclicity of an arbitrary n by n max-plus-algebraic matrix (see also [19, Theorem 2.2]). Consider a matrix $A \in \mathbb{R}_{\varepsilon}^{n \times n}$. Let $C_1, C_2, \ldots, C_s$ be the m.s.c.s.'s of $\mathcal{G}^{\mathrm{c}}(A)$ that have more than one vertex or that contain at least one circuit. Note that the other m.s.c.s.'s have cyclicity 1. So they do not influence the value of $c(\mathcal{G}^{\mathrm{c}}(A))$. If l_i is the number of vertices of C_i, then we have $c(C_i) \leqslant l_i$ by Lemma 3.1. Since $c(A) = \mathrm{lcm}(c(C_1), c(C_2), \ldots, c(C_s))$ and since $c(C_1) + c(C_2) + \cdots + c(C_s) \leqslant l_1 + l_2 + \cdots + l_s \leqslant n$, we have $c(A) \leqslant F_{\mathrm{aux}}(n, s) \leqslant F(n)$.

Now we prove that there exists a matrix $A \in \mathbb{R}_{\varepsilon}^{n \times n}$ such that $c(A) = F(n)$. Let $l_1, l_2, \ldots, l_s$ be a combination for which $F(n)$ and $F_{\mathrm{aux}}(n, s)$ reach their maximum. Define $m_1 = 0$ and

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F_{aux}	s										$F(n)$
n	1	2	3	4	5	6	7	8	9	10	
1	1	–	–	–	–	–	–	–	–	–	1
2	2	1	–	–	–	–	–	–	–	–	2
3	3	2	1	–	–	–	–	–	–	–	3
4	4	3	2	1	–	–	–	–	–	–	4
5	5	6	3	2	1	–	–	–	–	–	6
6	6	6	6	3	2	1	–	–	–	–	6
7	7	12	6	6	3	2	1	–	–	–	12
8	8	15	12	6	6	3	2	1	–	–	15
9	9	20	15	12	6	6	3	2	1	–	20
10	10	21	30	15	12	6	6	3	2	1	30

Table 1: The values of $F_{\text{aux}}(n, s)$ and $F(n)$ for $n = 1, 2, \ldots, 10$ and $s = 1, 2, \ldots, n$.

$m_i = l_1 + l_2 + \cdots + l_{i-1}$ for all $i > 1$. Now consider a graph $\mathcal{G}$ consisting of separate (i.e., mutually not connected) subgraphs $C_1, C_2, \ldots, C_s, D_1, D_2, \ldots, D_{n-m_{s+1}}$, where C_i is a circuit graph with set of vertices $\{v_{m_i+1}, v_{m_i+2}, \ldots, v_{m_i+l_i}\}$ and arcs $v_{m_i+1} \to v_{m_i+2}$, $v_{m_i+2} \to v_{m_i+3}$, $\ldots$, $v_{m_i+l_i-1} \to v_{m_i+l_i}$, $v_{m_i+l_i} \to v_{m_i+1}$, and where D_i is a graph with only one vertex $v_{m_{s+1}+i}$ and no arc. The index of cyclicity of $\mathcal{G}$ is then equal to $\text{lcm}(l_1, l_2, \ldots, l_s) = F(n)$.

If all the arcs in $\mathcal{G}$ have weight 0, then $\mathcal{G}$ is the precedence graph of the max-plus-algebraic matrix

$$
A = \begin{bmatrix}
A_{11} & \varepsilon & \ldots & \varepsilon & \varepsilon \\
\varepsilon & A_{22} & \ldots & \varepsilon & \varepsilon \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\varepsilon & \varepsilon & \ldots & A_{ss} & \varepsilon \\
\varepsilon & \varepsilon & \ldots & \varepsilon & E_{n-m_{s+1}}
\end{bmatrix},
$$

where A_{ii} is an l_i by l_i matrix of the form (4) for $i = 1, 2, \ldots, s$. Hence, $c(A) = c(\mathcal{G}) = F(n)$. ∎

Example 3.4 We have $F(10) = 30$. For $n = 30$ the maximum in (6) is reached for $s = 3$, $l_1 = 2$, $l_2 = 3$ and $l_3 = 5$. Now consider the matrix

$$
A = \left[\begin{array}{cc|ccc|ccccc}
\varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\hline
\varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\hline
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon
\end{array}\right].
$$

This matrix is in max-plus-algebraic Frobenius normal form and its block structure is indicated by the vertical and horizontal lines. The precedence graph of A is represented in Figure 2. We have $c(A) = c(\mathcal{G}(A)) = \text{lcm}(2, 3, 5) = 30$. □
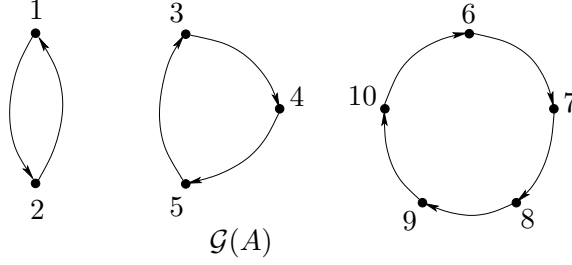
Figure 2: The precedence graph $\mathcal{G}(A)$ of the matrix A of Example 3.4. All the arcs have weight 0.

In [17, 18] an efficient method is given to compute F without making use of F_{aux} (see also Appendix A). In general there does not exist a closed form expression for F.

Although $F(n)$ is a tight upper bound for the index of cyclicity of an n by n matrix, there does not exist a closed-form expression for $F(n)$. Therefore, we closed-form upper bounds for the index of cyclicity of a matrix as a function of the size of the matrix in the next section.

3.2 Closed-form upper bounds for the index of cyclicity

For large-sized matrices the following proposition yields a rather tight upper bound for the index of cyclicity (see Figure 3):

Proposition 3.5 *For all $n \geqslant 4$ we have*

$$F(n) \leqslant G_{\text{m}}(n) \stackrel{\text{def}}{=} \exp\left(\sqrt{n \log n} \left(1 + \frac{\log \log n - 0.975}{2 \log n} \right) \right) \ .$$

Proof: Let $n \in \mathbb{N}$. Let us denote the maximum order[4] of a permutation of n elements by $g(n)$. If we consider all distinct representations of n as a sum of positive integers and if for each representation we consider the least common multiple of the integers in the representation, then $g(n)$ is equal to the maximum of these least common multiples (see, e.g., [16] or the references therein). Hence, $F(n) = g(n)$. Furthermore, in [15] it has been shown that $g(n) \leqslant G_{\text{m}}(n)$ if $n \geqslant 4$. ∎

Remark 3.6 In fact Theorem 2 of [15] erroneously states that $g(n) \leqslant G_{\text{m}}(n)$ if $n \geqslant 3$, but $g(n) \leqslant G_{\text{m}}(n)$ only holds if $n > 3$ since $g(3) = 3 > 2.967 \approx G_{\text{m}}(3)$ (see also Figure 4). Other (less tight) upper bounds for $g(n)$ can be found in [14, 15]. ◇

Now we shall derive another upper bound for $F(n)$ that is more tight than the upper bound of Proposition 3.5 if n is small.

Lemma 3.7 *If $A \in \mathbb{R}_{\varepsilon}^{n \times n}$ then we have $c(A) \leqslant \exp\left(\dfrac{n}{e}\right)$.*

Proof: Let $\mathcal{G}^{\text{c}}$ be the critical graph of $\mathcal{G}(A)$. So $c(A) = c(\mathcal{G}^{\text{c}})$. Recall that if we have an m.s.c.s. with m vertices in $\mathcal{G}^{\text{c}}$ then the index of cyclicity of this m.s.c.s. is less than or equal

---

[4]The order of a permutation is the least common multiple of the lengths of the disjoint cycles that compose the permutation.
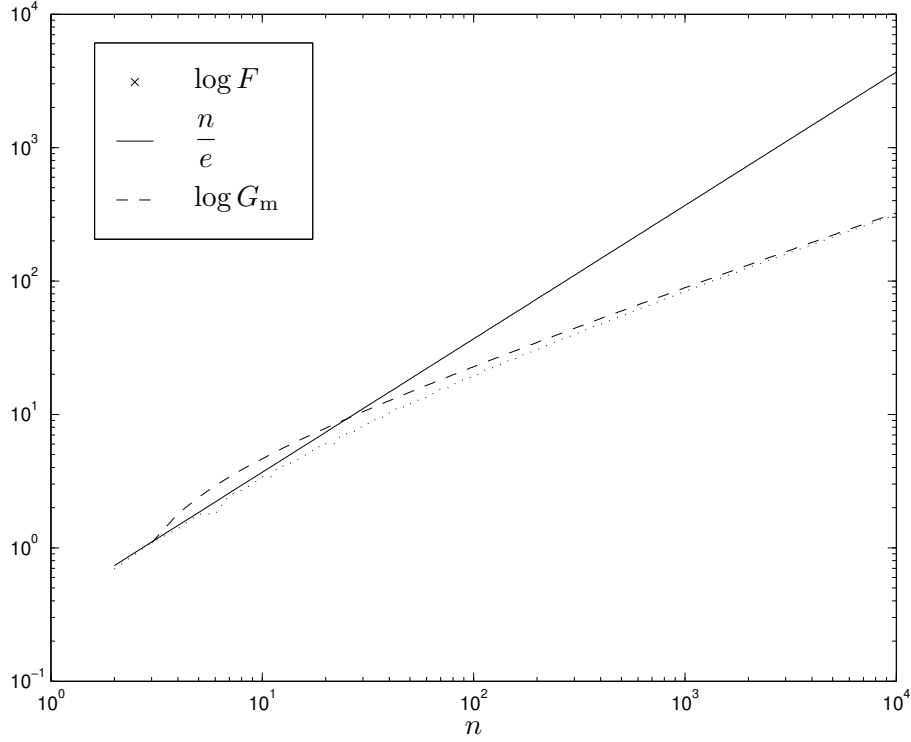
Figure 3: The plots of n versus the logarithm of the functions F, $\exp\left(\dfrac{n}{e}\right)$ and G_{m}.

to m by Lemma 3.1.

The index of cyclicity c of $\mathcal{G}^{\mathrm{c}}$ is equal to the least common multiple of the indices of cyclicity of its m.s.c.s.'s. So if $\mathcal{G}^{\mathrm{c}}$ has r m.s.c.s.'s and if m_i is the number of vertices of m.s.c.s. i, then we have

$$c \leqslant m_1 \, m_2 \, \ldots \, m_r \; . \tag{7}$$

Note that $1 \leqslant r \leqslant n$. Let us now compute the maximal value of the right-hand side of (7) subject to $m_i \geqslant 0$ for $i = 1, 2, \ldots, r$ and $m_1 + m_2 + \cdots + m_r = n$ for a fixed value of r. Since $m_r = n - m_1 - m_2 - \cdots - m_{r-1}$, this leads to the following optimization problem:

$$\max_{m_1, \ldots, m_{r-1} \geqslant 0} f(m_1, m_2, \ldots, m_{r-1})$$

with

$$f(m_1, m_2, \ldots, m_{r-1}) = m_1 \, m_2 \, \ldots \, m_{r-1} \left(n - m_1 - m_2 - \cdots - m_{r-1} \right) \; .$$

If we consider a point on the border of the feasible region, i.e., a point with $m_j = 0$ for some j, then the value of the objective function is equal to 0, which is clearly not the largest possible value of f over the feasible region. Therefore, we now look for unconstrained maxima of f that lie in the interior of the feasible region. A necessary condition for an unconstrained maximum is that the gradient of f is equal to the zero vector, or equivalently, $\dfrac{\partial f}{\partial m_j} = 0$ for all j. We have

$$\frac{\partial f}{\partial m_j} = m_1 \, m_2 \, \ldots \, m_{j-1} \, m_{j+1} \, \ldots \, m_{r-1} \left(n - m_1 - m_2 - \cdots - m_{r-1} \right) + m_1 \, m_2 \, \ldots \, m_{r-1} \, (-1)$$
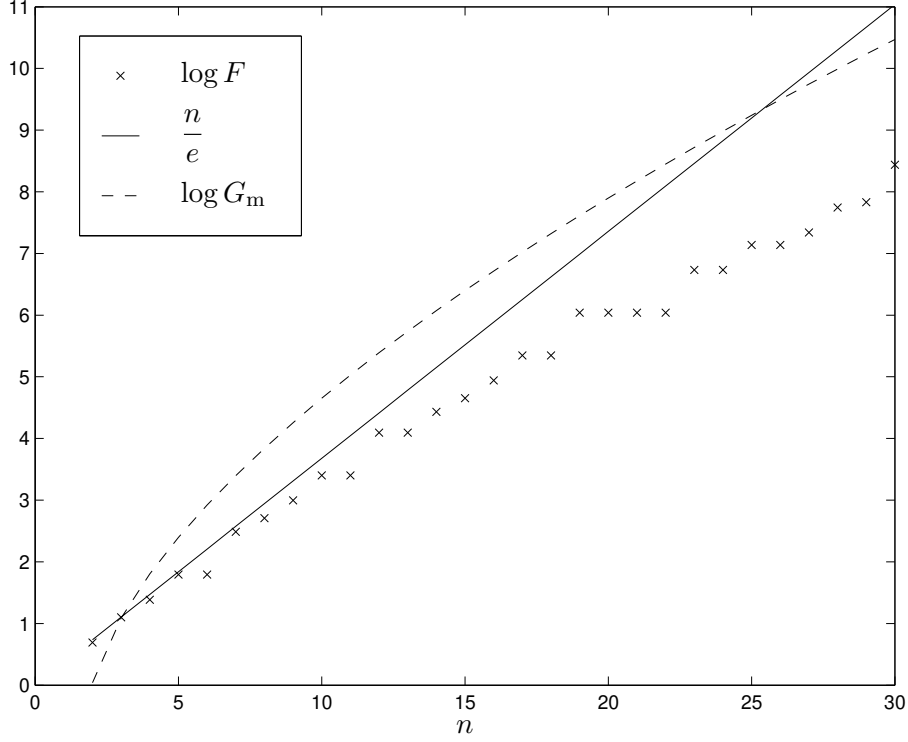
10

Figure 4: The plots of n versus the logarithm of the functions F, $\exp\left(\dfrac{n}{e}\right)$ and G_{m} for small values of n. The two latter functions have been plotted as continuous functions of n. The functions $\exp\left(\dfrac{n}{e}\right)$ and G_{m} intersect for $n \approx 25.391$. Note that $G_{\mathrm{m}}(n) \geqslant g(n)$ does not hold for $n = 2$ and $n = 3$ (cf. Proposition 3.5 and Remark 3.6).

$$
\begin{aligned}
&= m_1\, m_2\, \ldots\, m_{j-1}\, m_{j+1}\, \ldots\, m_{r-1}\, (n - m_1 - m_2 - \cdots - m_{r-1} - m_j) \\
&= m_1\, m_2\, \ldots\, m_{j-1}\, m_{j+1}\, \ldots\, m_{r-1}\, (n - s - m_j)\ ,
\end{aligned}
$$

with

$$
s = m_1 + m_2 + \cdots + m_{r-1}\ . \tag{8}
$$

If we consider a point for which $m_k = 0$ for some $k \in \{1, 2, \ldots, r-1\}$ then the value of the objective function f will be equal to 0, which is clearly not the maximal value. The only other point for which the gradient of f is equal to the zero vector is defined by

$$
n - s - m_j = 0 \qquad \text{for all } j\ .
$$

Hence, $m_j = n - s$ for all j. From (8) it follows that $s = (r-1)(n-s)$ or $s = n - \dfrac{n}{r}$. Hence, f will reach its maximal value over the feasible region in the point with $m_j = n - s = \dfrac{n}{r}$ for all j, and the value of f in this point is $\left(\dfrac{n}{r}\right)^r$. As a consequence, we have

$$
c \leqslant \max_{r \in \{1, 2, \ldots, n\}} \left(\frac{n}{r}\right)^r \leqslant \max_{r \in [1, n]} \left(\frac{n}{r}\right)^r\ . \tag{9}
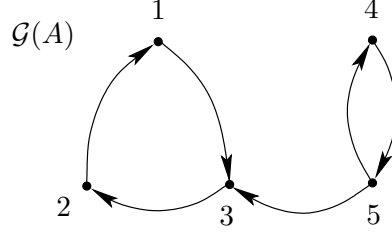$$

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Figure 5: The precedence graph of the matrix A of Example 3.8. All the arcs have weight 0.

Now we determine the maximal value of $g(r) = \left(\dfrac{n}{r}\right)^r = \exp\left(r\log\dfrac{n}{r}\right) = \exp\left(-r\log\dfrac{r}{n}\right)$ for $r \in [1, n]$. We have

$$
\begin{aligned}
\frac{dg}{dr} &= \exp\left(-r\log\frac{r}{n}\right)\left(-\log\frac{r}{n} - r\frac{n}{r}\frac{1}{n}\right) \\
&= \left(\frac{n}{r}\right)^r\left(-\log\frac{r}{n} - 1\right) \quad .
\end{aligned}
$$

We have $\dfrac{dg(r)}{dr} = 0$ if $r = \dfrac{n}{e}$. Furthermore, $\dfrac{dg(r)}{dr} \geqslant 0$ if $r \leqslant \dfrac{n}{e}$, and $\dfrac{dg(r)}{dr} \leqslant 0$ if $r \geqslant \dfrac{n}{e}$. So g reaches its maximum on the interval $[1, n]$ for $r = \dfrac{n}{e}$. Since $g\left(\dfrac{n}{e}\right) = \exp\left(\dfrac{n}{e}\right)$, it follows from (9) that $c \leqslant \exp\left(\dfrac{n}{e}\right)$. ∎

Example 3.8 Consider the matrix

$$
A = \begin{bmatrix}
\varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\
0 & \varepsilon & \varepsilon & \varepsilon & 0 \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\
\varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon
\end{bmatrix} \quad .
$$

The precedence graph of A is represented in Figure 5. This graph has two m.s.c.s.'s: $\mathcal{G}_1$ with vertices 1, 2 and 3 and $\mathcal{G}_2$ with vertices 4 and 5. We have $c(\mathcal{G}_1) = 3$ and $c(\mathcal{G}_2) = 2$. Hence, $c(\mathcal{G}) = 6$. Note that $c(\mathcal{G}) \leqslant \exp\left(\dfrac{5}{e}\right) \approx 6.29$. □

In Figures 4 and 3 we have plotted the functions F, $\exp\left(\dfrac{n}{e}\right)$ and G_{m}. Clearly, for large values of n the function G_{m} yields an upper bound for F that is tighter than $\exp\left(\dfrac{n}{e}\right)$. On the other hand for $n \leqslant 25$ the upper bound $\exp\left(\dfrac{n}{e}\right)$ is tighter than $G_{\mathrm{m}}(n)$.

4   Conclusions

In this paper we have derived upper bounds for the index of cyclicity of a general matrix in the max-plus algebra as a function of the size of the matrix. These results can be used in the max-plus-algebraic system theory for discrete event systems: they can be used to prove that

the Boolean minimal realization problem in the max-plus algebra can be solved in a time that is bounded from above by a function that is exponential in the minimal system order [8].

Topics for future research include the derivation of tighter upper bounds for the index of cyclicity of an n by n max-plus-algebraic matrix for small values of n.

References

[1] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat, *Synchronization and Linearity*. New York: John Wiley & Sons, 1992.

[2] A. Berman and R.J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*. New York: Academic Press, 1979.

[3] R.A. Brualdi and H.J. Ryser, *Combinatorial Matrix Theory*, vol. 39 of *Encyclopedia of Mathematics and Its Applications*. Cambridge, UK: Cambridge University Press, 1991.

[4] C.G. Cassandras, S. Lafortune, and G.J. Olsder, "Introduction to the modelling, control and optimization of discrete event systems," in *Trends in Control: A European Perspective* (A. Isidori, ed.), pp. 217–291, Berlin, Germany: Springer-Verlag, 1995.

[5] G. Cohen, D. Dubois, J.P. Quadrat, and M. Viot, "A linear-system-theoretic view of discrete-event processes and its use for performance evaluation in manufacturing," *IEEE Transactions on Automatic Control*, vol. 30, no. 3, pp. 210–220, Mar. 1985.

[6] R.A. Cuninghame-Green, *Minimax Algebra*, vol. 166 of *Lecture Notes in Economics and Mathematical Systems*. Berlin, Germany: Springer-Verlag, 1979.

[7] B. De Schutter, "On the ultimate behavior of the sequence of consecutive powers of a matrix in the max-plus algebra," *Linear Algebra and Its Applications*, vol. 307, no. 1–3, pp. 103–117, Mar. 2000.

[8] B. De Schutter, V. Blondel, R. de Vries, and B. De Moor, "On the boolean minimal realization problem in the max-plus algebra," *Systems & Control Letters*, vol. 35, no. 2, pp. 69–78, Sept. 1998.

[9] B. De Schutter and B. De Moor, "On the sequence of consecutive powers of a matrix in a Boolean algebra," *SIAM Journal on Matrix Analysis and Applications*, vol. 21, no. 1, pp. 328–354, 1999.

[10] F.R. Gantmacher, *The Theory of Matrices*, vol. 2. New York: Chelsea Publishing Company, 1959.

[11] S. Gaubert, "On rational series in one variable over certain dioids," Tech. rep. 2162, INRIA, Le Chesnay, France, Jan. 1994.

[12] F. Harary, "A graph theoretic approach to matrix inversion by partitioning," *Numerische Mathematik*, vol. 4, pp. 128–135, 1962.

[13] Y.C. Ho, ed., *Discrete Event Dynamic Systems: Analyzing Complexity and Performance in the Modern World*. Piscataway, New Jersey: IEEE Press, 1992.

[14] J.P. Massias, "Majoration explicite de l'ordre maximum d'un élément du groupe symétrique," *Annales de la Faculté des Sciences de Toulouse, Mathématiques*, vol. VI, pp. 269–281, 1984.

[15] J.P. Massias, J.L. Nicolas, and G. Robin, "Effective bounds for the maximal order of an element in the symmetric group," *Mathematics of Computation*, vol. 53, no. 188, pp. 665–678, Oct. 1989.

[16] W. Miller, "The maximum order of an element of a finite symmetric group," *American Mathematics Monthly*, vol. 94, pp. 497–506, June–July 1987.

[17] J.L. Nicolas, "Calcul de l'ordre maximum d'un élément du groupe symétrique S_n," *Revue Française d'Informatique et de Recherche Opérationelle*, vol. 3, no. R-2, pp. 43–50, 1969.

[18] J.L. Nicolas, "Ordre maximal d'un élément du groupe S_n des permutations et "highly composite numbers"," *Bulletin de la Société Mathématique de France*, vol. 97, pp. 129–191, 1969.

[19] R.D. Nussbaum, "Convergence of iterates of a nonlinear operator arising in statistical mechanics," *Nonlinearity*, vol. 4, pp. 1223–1240, 1991.

[20] R.S. Varga, *Matrix Iterative Analysis*. Englewood Cliffs, New Jersey: Prentice-Hall, 1962.

Appendix A: A recursion-based method for computing F

Let p_i be the ith prime number. So $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ...

Note that computing F using formulas (5) and (6) is computationally quite demanding. Therefore, and for sake of completeness, we now repeat the recursion-based method for computing F that has been derived in [17, 18] and which is based on the following lemma.

Lemma A.1 *Let $N \in \mathbb{N}$ and let p_{q_n} be the largest prime that is less than or equal to n for $n = 1, 2, \ldots, N$. Consider the set of functions H_1, H_2, ..., H_{q_N} that are defined by the following recursion formula:*

$$H_k(n) = \max\left(H_{k-1}(n), \, \max\left\{ p_k^i H_{k-1}(n - p_k^i) \,\big|\, i \in \mathbb{N}_0, \, p_k^i \leqslant n \right\} \right) \qquad (10)$$

for $k = 2, 3, \ldots, q_N$ and $n = 0, 1, \ldots, N$ where $\max \emptyset = 1$ by definition, and by the following boundary condition:

$$H_1(n) = \max\left\{ 2^i \,\big|\, i \in \mathbb{N}_0, 2^i \leqslant n \right\} \qquad (11)$$

for $n = 0, 1, \ldots, N$ where $\max \emptyset = 1$ by definition. If we define the function H by $H(n) = H^{(q_n)}(n)$ for $n = 1, 2, \ldots, N$, then we have

$$H(n) = \max\left\{ \prod_{r=1}^{s} p_{i_r}^{\alpha_r} \,\middle|\, s \in \mathbb{N}_0, \, i_1, i_2, \ldots, i_s, \alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{N}_0, \, \sum_{r=1}^{s} p_{i_r}^{\alpha_r} \leqslant n \right\} \qquad (12)$$

for $n = 1, 2, \ldots, N$ where $\max \emptyset = 1$ by definition.

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Proof: Define $\mathcal{H}_k(0) = \mathcal{H}_k(1) = \{1\}$ and

$$\mathcal{H}_k(n) = \left\{ \prod_{r=1}^{s} p_{i_r}^{\alpha_r} \, \middle| \, s \in \mathbb{N}_0, \ i_1, i_2, \ldots, i_s \in \{1, 2, \ldots, k\}, \ \alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{N}_0, \ \sum_{r=1}^{s} p_{i_r}^{\alpha_r} \leqslant n \right\}$$

for $k = 1, 2, \ldots, q_N$ and $n = 2, 3, \ldots, N$. Let us show by induction that

$$H_k(n) = \max \mathcal{H}_k(n) \tag{13}$$

for $k = 1, 2, \ldots, q_N$ and $n = 1, 2, \ldots, N$. Note that we have $H_k(0) = H_k(1) = 1$ for all k since $\max \emptyset = 1$ by definition.

Since $p_1 = 2$ and since $\max \emptyset = 1$ by definition, it follows from (11) that (13) holds for $k = 1$ and $n = 1, 2, \ldots, N$.

Let $K \in \{1, 2, \ldots, q_N - 1\}$. Now we assume that (13) holds for $k = 1, 2, \ldots, K$ and $n = 1, 2, \ldots, N$, and we show that it also holds for $k = K + 1$ and $n = 1, 2, \ldots, N$.

We have

$$\max \mathcal{H}_{K+1}(n)$$

$$= \max \left(\max \mathcal{H}_K(n), \ \max \left\{ p_{K+1}^i \, \middle| \, i \in \mathbb{N}_0, \ p_{K+1}^i \leqslant n \right\}, \right.$$

$$\max \left\{ p_{K+1}^i \prod_{r=1}^{s-1} p_{i_r}^i \, \middle| \, s \in \mathbb{N} \setminus \{0, 1\}, \ i_1, i_2, \ldots, i_{s-1} \in \{1, 2, \ldots, K\}, \right.$$

$$\left. \left. \alpha_1, \alpha_2, \ldots, \alpha_{s-1}, i \in \mathbb{N}_0, \ p_{K+1}^i + \sum_{r=1}^{s} p_{i_r}^{\alpha_r} \leqslant n \right\} \right) \tag{14}$$

$$= \max \left(H_K(n), \ \max \left\{ p_{K+1}^i \max \mathcal{H}_K(n - p_{K+1}^i) \, \middle| \, i \in \mathbb{N}_0, \ p_{K+1}^i \leqslant n \right\} \right) \tag{15}$$

$$= \max \left(H_K(n), \ \max \left\{ p_{K+1}^i H_K(n - p_{K+1}^i) \, \middle| \, i \in \mathbb{N}_0, \ p_{K+1}^i \leqslant n \right\} \right)$$

$$= H_{K+1}(n, s) \qquad \text{(by (10))}.$$

In (15) we have used the fact that $\mathcal{H}_k(0) = \mathcal{H}_k(1) = \{1\}$ to merge the second and the third set that appear on the right-hand of (14). So (13) holds for $k = K + 1$ and $n = 1, 2, \ldots, N$. As a consequence, (13) holds for $k = 2, 3, \ldots, q_N$ and $n = 1, 2, \ldots, N$.

Since $p_k > n$ if $k > q_n$, indices i_r with $i_r > q_n$ do not contribute to the set that appears on the right-hand side of (12). Hence, (12) holds for $n = 1, 2, \ldots, N$. ∎

Remark A.2 Note that we have $p_{q_n} \leqslant n$. However, it can be shown [17, 18] that (12) already holds if we define the function H by $H(n) = H^{(r_n)}(n)$ for $n = 1, 2, \ldots, N$, where p_{r_n} is the largest prime that divides $H(n)$. We have $\lim\limits_{n \to \infty} \dfrac{p_{r_n}}{\sqrt{n \log n}} = 1$. Furthermore, there exists a constant $c > 1$ such that $p_{r_n} \leqslant c\sqrt{n \log n}$ for all n [17, 18][5]. ◇

Proposition A.3 *We have* $F(n) = H(n)$ *for* $n = 1, 2, \ldots$

---

[5]It can be verified experimentally that if we take $c = 1.2$ then we have $p_{r_n} \leqslant c\sqrt{n \log n}$ for all $n \geqslant 3$.

Proof: We have $F(1) = H(1)$. Now we show that we also have $F(n) = H(n)$ for $n \geqslant 2$.

Let $n \in \mathbb{N} \setminus \{0, 1\}$ and $l_1, l_2, \ldots, l_s$ be a combination for which $F(n)$ reaches its maximum. Note that if we have $l_1 = 1$ for some i, or $l_i = l_j$ for some i, j with $i \neq j$, then there are redundant terms in the combination $l_1, l_2, \ldots, l_s$ that can be removed without changing the value of $\mathrm{lcm}(l_1, l_2, \ldots, l_s)$. Therefore, we may assume without loss of generality that $l_i \neq 1$ for all i and that $l_i \neq l_j$ for all i, j with $i \neq j$.

Let us now show that we may also assume without loss of generality that $\gcd(l_i, l_j) = 1$ for all i, j with $i \neq j$. Indeed, assume that $\gcd(l_1, l_2) = w > 1$. Then there exist integers $\hat{l}_1$ and $\hat{l}_2$ such that $l_1 = w\hat{l}_1$, $l_2 = w\hat{l}_2$ and $\gcd(\hat{l}_1, \hat{l}_2) = 1$. Hence, $\mathrm{lcm}(l_1, l_2) = w\hat{l}_1\hat{l}_2 = \mathrm{lcm}(\hat{l}_1, l_2)$. Furthermore, $\hat{l}_1 + l_2 + \cdots + l_s < l_1 + l_2 + \cdots + l_s \leqslant n$ and $\mathrm{lcm}(\hat{l}_1, l_2, \ldots, l_s) = \mathrm{lcm}(l_1, l_2, \ldots, l_s)$. As a consequence, we also have $F(n) = \mathrm{lcm}(\hat{l}_1, l_2, \ldots, l_s)$.

So from now on we assume that $\gcd(l_i, l_j) = 1$ for all i, j with $i \neq j$. This implies that the l_i's can be written as products of powers of mutually different prime numbers and that $F(n) = \mathrm{lcm}(l_1, l_2, \ldots, l_s) = l_1 l_2 \ldots l_s$. Assume that $l_i = \displaystyle\prod_{j=u_i}^{v_i} p_{k_j}^{\alpha_j}$ for $i = 1, 2, \ldots, s$ with $k_i \neq k_j$ for all i, j with $i \neq j$, with $\alpha_j \in \mathbb{N}_0$ for all j, and with $u_1 = 1$, $u_{i+1} = v_i + 1$ for $i = 1, 2, \ldots, s-1$.

Now we have $F(n) = \displaystyle\prod_{i=1}^{s} l_i = \displaystyle\prod_{i=1}^{v_s} p_{k_i}^{\alpha_i}$. Furthermore, since $p_{k_i}^{\alpha_i} + p_{k_j}^{\alpha_j} \leqslant p_{k_i}^{\alpha_i} p_{k_j}^{\alpha_j}$ for all i, j, we have $\displaystyle\sum_{i=1}^{\hat{s}} p_{k_i}^{\alpha_i} \leqslant \displaystyle\sum_{i=1}^{s} l_i \leqslant n$. Hence, it follows from Lemma A.1 that $F(n) \leqslant H(n)$. Furthermore, from the definition of F and H it follows that $F(n) \geqslant H(n)$. Hence, $F(n) = H(n)$. ∎