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Technical report 98-41

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If you want to cite this report, please use the following reference instead:
B. De Schutter, V. Blondel, and B. De Moor, "On the complexity of the boolean minimal realization problem in the max-plus algebra," *Advances in Systems, Signals, Control and Computers* (Proceedings of the International Conference on Systems, Signals, Control, Computers (SSCC'98), Durban, South Africa, Sept. 1998) (V.B. Bajić, ed.), vol. II, pp. 451–455, 1998.

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ON THE COMPLEXITY OF THE BOOLEAN MINIMAL REALIZATION PROBLEM IN THE MAX-PLUS ALGEBRA

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ABSTRACT

One of the open problems in the max-plus-algebraic system theory for discrete event systems is the minimal realization problem. We consider a simplified version of the general minimal realization problem: the boolean minimal realization problem, i.e., we consider models in which the entries of the system matrices are either equal to the max-plus-algebraic zero element or to the max-plus-algebraic identity element. We show that the corresponding decision problem (i.e., deciding whether or not a boolean realization of a given order exists) is decidable, and that the boolean minimal realization problem can be solved in a number of elementary operations that is bounded from above by an exponential of the square of (any upper bound of) the minimal system order.

INTRODUCTION

The max-plus-algebra [1, 2], which has maximization and addition as its basic operations, is one of the frameworks that can be used to model a class of discrete event systems (DESs). Typical examples of DESs are flexible manufacturing systems, telecommunication networks, parallel processing systems and logistic systems. One of the characteristic features of DESs, as opposed to continuous variable systems (i.e., systems the behavior of which can be described by difference or differential equations), is that their dynamics are *event-driven* as opposed to *time-driven*: the behavior of a DES is governed by events rather than by ticks of a clock. An event corresponds to the start or the end of an activity. If we consider a production system then possible events are: the completion of a part on a machine, a machine breakdown, or a buffer becoming empty.

In general, models that describe the behavior of a DES are nonlinear, but there exists a class of DESs — the max-linear DESs — for which the model becomes "linear" when we formulate it in the max-plus algebra [1, 2]. Loosely speaking we could say that this class corresponds to the class of deterministic DESs in which only synchronization and no concurrency occurs.

There exists a remarkable analogy between the basic operations of the max-plus algebra (maximization and addition) on the one hand, and the basic operations of conventional algebra (addition and multiplication) on the other hand. As a consequence, many concepts and properties of conventional algebra and linear system theory also have a max-plus-algebraic analogue [1]. This analogy also allows us to translate many concepts, properties and techniques from conventional linear system theory to max-plus-algebraic system theory. However, there are also some major differences that prevent a straightforward translation of properties, concepts and algorithms from conventional linear algebra and linear system theory to max-plus algebra and max-plus-algebraic system theory.

One of the open problems in the max-plus-algebraic system theory for DESs is the minimal realization problem: given the impulse response of a max-linear DES, determine a model of smallest possible size the impulse response of which coincides with the given impulse response. In this paper we shall consider a simplified version of the general minimal realization problem: the boolean minimal realization problem. We derive a lower bound for the minimal system order of a boolean max-linear DES as a function of the length of the transient part of its impulse response. Furthermore, we show that the problem of deciding whether or not a boolean realization with a given order of a given impulse response exists, is decidable. We also show that the boolean minimal realization problem can be solved in a number of operations that is bounded from above by an exponential of the square of (any upper bound of) the minimal system order.

MAX-PLUS ALGEBRA

The basic operations of the max-plus algebra are the maximum (represented by $\oplus$) and the addition (represented by $\otimes$):

$$\begin{aligned} x \oplus y &= \max(x, y) \\ x \otimes y &= x + y \end{aligned}$$

with $x, y \in \mathbb{R}$. The reason for choosing the symbols $\oplus$ and $\otimes$ to represent respectively maximization and addition is that many properties from conventional linear

[1]This research was carried out while the first author was working at ESAT-SISTA, K.U.Leuven, Belgium.

algebra can be translated to the max-plus algebra simply by replacing $+$ by $\oplus$ and $\times$ by $\otimes$. Therefore, we call $\oplus$ the max-plus-algebraic sum and $\otimes$ the max-plus-algebraic product.

Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The structure $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ is called the max-plus algebra. Note that in the max-plus algebra ε and 0 play the role of respectively 0 and 1 in conventional algebra: ε is the identity element for $\oplus$ and is absorbing for $\otimes$, and 0 is the identity element for $\otimes$.

The operations $\oplus$ and $\otimes$ are extended to matrices as follows. If $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ then $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$ for all i, j. If $A \in \mathbb{R}_\varepsilon^{m \times p}$ and $B \in \mathbb{R}_\varepsilon^{p \times n}$ then $(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj}$ for all i, j. Note that these definitions resemble the definitions of matrix sum and matrix product in conventional algebra but with $+$ replaced by $\oplus$ and $\times$ replaced by $\otimes$. The matrix $\mathcal{E}_{m \times n}$ is the max-plus-algebraic zero matrix: $(\mathcal{E}_{m \times n})_{ij} = \varepsilon$ for all i, j. The matrix E_n is the max-plus-algebraic identity matrix: we have $(E_n)_{ii} = 0$ for all i and $(E_n)_{ij} = \varepsilon$ for all i, j with $i \neq j$.

The kth max-plus-algebraic matrix power of a matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ with $k \in \mathbb{N}$ is defined as follows:

$$A^{\otimes^0} = E_n, \quad A^{\otimes^k} = \underbrace{A \otimes A \otimes \ldots \otimes A}_{k \text{ times}} \quad \text{if } k > 0\,.$$

Define $\mathbb{B} = \{0, \varepsilon\}$. A matrix with entries in $\mathbb{B}$ is called a max-plus-algebraic boolean matrix.

In order to define the cyclicity of a max-plus-algebraic matrix we first need some definitions from graph theory. A directed graph is called strongly connected if for any two different vertices v_i, v_j there exists a path from v_i to v_j. A maximal strongly connected subgraph (m.s.c.s.) of a directed graph is a strongly connected subgraph that is maximal.

The cyclicity of an m.s.c.s. is the greatest common divisor of the lengths of all the circuits of the given m.s.c.s. If an m.s.c.s. or a graph contains no circuits then its cyclicity is equal to 0 by definition. The *cyclicity* $c(\mathcal{G})$ of a graph $\mathcal{G}$ is the least common multiple of the nonzero cyclicities of its m.s.c.s.'s.

Consider $A \in \mathbb{R}_\varepsilon^{n \times n}$. The precedence graph of A, denoted by $\mathcal{G}(A)$, is a weighted directed graph with set of vertices $\{1, \ldots, n\}$ and an arc (j, i) with weight a_{ij} for each $a_{ij} \neq \varepsilon$. The average weight of a path in $\mathcal{G}(A)$ is defined as the sum of the weights of the arcs that compose the path divided by the length of the path. An elementary circuit of $\mathcal{G}(A)$ is called critical if it has maximum average weight among all circuits. The critical graph $\mathcal{G}^c(A)$ consists of those nodes and arcs of $\mathcal{G}(A)$ that belong to a critical circuit of $\mathcal{G}(A)$.

The cyclicity of a matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ is denoted by $c(A)$ and is equal to the cyclicity of the critical graph of the precedence graph of A. So $c(A) = c(\mathcal{G}^c(A))$. If $A \in \mathbb{B}^{n \times n}$ then every circuit in $\mathcal{G}(A)$ is critical, which implies that $c(A) = c(\mathcal{G}^c(A)) = c(\mathcal{G}(A))$. For the cyclicity of a general matrix we have the following upper bound [4]:

Lemma 1 *If* $A \in \mathbb{R}_\varepsilon^{n \times n}$ *then* $c(A) \leq \exp\left(\dfrac{n}{e}\right)$.

Remark 2 Another upper bound that is more complex but tighter for large n can be found in [12] (See also [4]). $\diamond$

For max-plus-algebraic boolean matrices we have [8]:

Theorem 3 *If* $A \in \mathbb{B}^{n \times n}$ *and if* c *is the cyclicity of* A, *then* $A^{\otimes^{k+c}} = A^{\otimes^k}$ *for all* $k \geq 2n^2 - 3n + 2$.

MAX-PLUS-ALGEBRAIC SYSTEM THEORY

There is a class of DESs that can be modeled by a max-plus-algebraic model of the following form [1, 2]:

$$x(k+1) = A \otimes x(k) \oplus B \otimes u(k) \qquad (1)$$

$$y(k) = C \otimes x(k)\ . \qquad (2)$$

The vector x represents the state, u is the input vector and y is the output vector of the system. Since the model (1) – (2) closely resembles the state space model for linear time-invariant discrete-time systems, a DES that can be modeled by (1) – (2) will be called a *max-linear time-invariant DES* (MLTI DES).

The number of components of the state vector x is the order of the state space model. The matrices A, B and C are called the system matrices of the model. We shall characterize a model of the form (1) – (2) by the triple (A, B, C) of system matrices. A system with one input and one output is called a single-input single-output (SISO) system. A system with more than one input and more than one output is called a multi-input multi-output (MIMO) system.

A max-plus-algebraic unit impulse is a sequence $\{e_k\}_{k=0}^{\infty}$ defined by: $e_0 = 0$ and $e_k = \varepsilon$ for $k = 1, 2, \ldots$ If we apply a max-plus-algebraic unit impulse to the ith input of the system, and if we assume that $x(0) = \mathcal{E}_{n \times 1}$, we get $y(k) = C \otimes A^{\otimes^{k-1}} \otimes B_{.,i}$ as the output of the DES, where $B_{.,i}$ is the ith column of B. This output is called the impulse response due to a max-plus-algebraic impulse at the ith input. Since $y(k)$ corresponds to the ith column of the matrix $G_{k-1} \stackrel{\text{def}}{=} C \otimes A^{\otimes^{k-1}} \otimes B$, the sequence $\{G_k\}_{k=0}^{\infty}$ is called the *impulse response* of the DES. In the remainder of this paper we shall use the symbol G as a abbreviated notation for $\{G_k\}_{k=0}^{\infty}$.

The impulse response of an MLTI DES can be characterized as follows:

Theorem 4 *If* G *is the impulse response of an MLTI DES with* m *inputs and* l *outputs then we have*

$$\begin{aligned}
&\forall i \in \{1, \ldots, l\}, \forall j \in \{1, \ldots, m\}, \exists c \in \mathbb{N}_0, \\
&\quad \exists \lambda_1, \ldots, \lambda_c \in \mathbb{R}_\varepsilon, \exists k_0 \in \mathbb{N} \text{ such that } \forall k \in \mathbb{N}: \\
&\qquad (G_{k_0+kc+c+s-1})_{ij} = \lambda_s{}^{\otimes^c} \otimes (G_{k_0+kc+s-1})_{ij} \qquad (3)\\
&\qquad \text{for } s = 1, \ldots, c\,.
\end{aligned}$$

Proof: This is a direct consequence of Corollary 1.1.9 of [9, p. 166] or of Proposition 1.2.2 of [10]. $\square$

A sequence G that exhibits a behavior of the form (3) is called *ultimately periodic*. The smallest possible c for which (3) holds is called the *period* of G. If G is the impulse response of an MLTI DES and if the triple (A, B, C) is a state space realization of the DES, then the period of G is a divisor of the cyclicity of A.

Proposition 5 *A sequence G with $G_k \in \mathbb{R}_\varepsilon^{l\times m}$ for all k is the impulse response of an MLTI DES if and only if it is ultimately periodic.*

Proof: See [1, 9, 10] for the SISO case and [3, 6] for the MIMO case. □

In order to get a concise, unique representation of an ultimately periodic sequence we now introduce a new concept, the so-called *canonical representation* of an ultimately periodic sequence. We shall only do this for a sequence of real numbers. The extension to sequences of matrices is straightforward. Consider an ultimately periodic sequence of real numbers $g = \{g_k\}_{k=0}^{\infty}$. First we determine the smallest possible $c \in \mathbb{N}_0$ for which (3) holds. The λ_s's are then defined uniquely[2] (up to a circular permutation of the indices). Next, we determine the smallest possible $k_0 \in \mathbb{N}$ such that (3) holds for all $k \geq 0$. Now we can uniquely represent the sequence g by the $(k_0 + 2c + 1)$-tuple $(c, \lambda_1, \ldots, \lambda_c, g_0, \ldots, g_{k_0+c-1})$. The subsequence $g_0, \ldots, g_{k_0-1}$ will be called the *transient part* of g.

Now consider the following problem:

> Given an ultimately periodic sequence G with $G_k \in \mathbb{R}_\varepsilon^{l\times m}$ for all k and an integer r, find, if possible, matrices $A \in \mathbb{R}_\varepsilon^{r\times r}$, $B \in \mathbb{R}_\varepsilon^{r\times m}$ and $C \in \mathbb{R}_\varepsilon^{l\times r}$ such that (A, B, C) is a realization of G, i.e., $G_k = C \otimes A^{\otimes^k} \otimes B$ for all $k \in \mathbb{N}$.

This problem is called the *state space realization problem.* If we make r as small as possible, then the problem is called the *minimal* state space realization problem and the resulting value of r is called the minimal system order.

The minimal state space realization problem for MLTI DESs has been studied by many authors and for some specific cases the problem has been solved [3, 7, 11, 13]. However, at present it is still an open problem whether there exist tractable methods to solve the general minimal state space realization problem. Since the general minimal realization problem is still an open problem, we consider a simplified version of this problem in the next section.

BOOLEAN MINIMAL REALIZATION

An MLTI DES for which all the terms of the impulse response are max-plus-algebraic boolean matrices is called a *boolean* MLTI DES. It is easy to verify that if we have an rth order state space realization (A, B, C) of a boolean MLTI DES where the entries of A, B, C belong to $\mathbb{R}_\varepsilon$, then there also exists an rth order state space realization $(\tilde{A}, \tilde{B}, \tilde{C})$ such that the entries of $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ belong to $\mathbb{B}$.
The following corollaries are direct consequences of Theorem 3.

Corollary 6 *Consider a boolean MLTI DES with minimal system order n and impulse response G. Let c be the period of G. Then we have $G_{k+c} = G_k$ for all $k \geq 2n^2 - 3n + 2$.*

[2] Provided that for a subsequence of the form $\varepsilon, \varepsilon, \varepsilon, \ldots,$ we take λ_s equal to ε.

Corollary 7 *Let G and F be impulse responses of boolean MLTI DESs with minimal system order less than or equal to n. Let c be the maximum of the period of G and the period of F. If $G_k = F_k$ for $k = 0, \ldots, 2n^2 - 3n + 1 + c$ then $G_k = F_k$ for all $k \in \mathbb{N}$.*

Corollary 7 gives an upper bound on the number of terms that two boolean impulse responses should have in common in order to coincide completely.

A lower bound for the minimal system order

At present there do not exist efficient (i.e., polynomial time) algorithms to compute a non-trivial lower bound for the minimal system order for a given ultimately periodic sequence. However, for a *boolean* impulse response the following lemma provides an easily computable lower bound:

Lemma 8 *Let G be the impulse response of a boolean MLTI DES with minimal system order n. Let c be the period of G. Let L be the length of the transient part of G, i.e., L is equal to the smallest integer K for which we have $G_{k+c} = G_k$ for all $k \geq K$. If $L \geq 2$ then $n \geq \dfrac{3 + \sqrt{8L-7}}{4}$.*

Proof: From Corollary 6 it follows that

$$L \leq 2n^2 - 3n + 2 \ . \qquad (4)$$

If is easy to verify that this condition holds for every $n \in \mathbb{N}$ if $L = 0$ or if $L = 1$. So from now on we assume that $L \geq 2$. The zeros of the function f defined by $f(n) = 2n^2 - 3n + 2 - L$ are $n_1 = \dfrac{3 + \sqrt{8L-7}}{4}$ and $n_2 = \dfrac{3 - \sqrt{8L-7}}{4}$. Since $n_2 \leq 0$ if $L \geq 2$ and since n is always positive, the function f will be nonnegative if $n \geq n_1$. Hence, condition (4) will only be satisfied if $n \geq n_1$. □

Complexity of the boolean minimal realization problem

Let us now consider the following two problems:

- the boolean realization decision problem (BRDP): Given an ultimately periodic sequence G with $G_k \in \mathbb{B}^{l\times m}$ in its canonical representation and an integer r, does there exist an rth order boolean state space realization of G? This problem will be denoted by BRDP(G,r).
- the boolean minimal realization problem (BMRP): Given an ultimately periodic sequence G with $G_k \in \mathbb{B}^{l\times m}$ in its canonical representation, compute a minimal state space realization of G. This problem will be denoted by BMRP(G).

Proposition 9 *Let G be an ultimately periodic sequence with $G_k \in \mathbb{B}^{l\times m}$ for all k and let $r \in \mathbb{N}$. The problem BRDP(G,r) is decidable using a finite number of elementary operations (such as addition, subtraction, multiplication, division, maximum, minimum and comparison).*

Proof: Since G is an ultimately periodic sequence, it corresponds to the impulse response of a *boolean* MLTI DES. Let n be the minimal system order of this system. In [5] we have shown that an upper bound n_u for n can be computed in a finite number of steps. If $r \geq n_\mathrm{u}$ then there exists an rth order state space realization of G and then the answer to the BRDP(G,r) is affirmative.
From now on we assume that $r \leq n_\mathrm{u}$. Let c be the period of G. Define $K = 2n_\mathrm{u}^2 - 3n_\mathrm{u} + 1 + c$. If we have an rth order state space realization characterized by the triple of system matrices (A, B, C) and if $C \otimes A^{\otimes^k} \otimes B = G_k$ for all $k \leq K$ then it follows from Corollary 7 that (A, B, C) is an rth order state space realization of G.
This implies that the BRDP(G,r) is equivalent to checking whether or not the following system of equations has a solution:

$$C \otimes A^{\otimes^k} \otimes B = G_k \qquad \text{for } k = 0, \ldots K \ , \tag{5}$$

with $A \in \mathbb{B}^{r \times r}$, $B \in \mathbb{B}^{r \times m}$ and $C \in \mathbb{B}^{l \times r}$. If we write out (5), we get

$$\bigoplus_{p,q=1}^{r} \bigoplus_{s=1}^{r^{k-1}} c_{ip} \otimes \bigotimes_{u,v=1}^{r} a_{uv}{}^{\otimes^{\gamma_{kpqsuv}}} \otimes b_{qj} \ = \ (G_k)_{ij}$$

where γ_{kpqsuv} is the number of times that a_{uv} appears in the sth term of $(A^{\otimes^k})_{pq}$. If we put the entries of A, B and C in one large column vector x of length $L = (r+m+l)r$, if we put the entries of the G_k's in one large column vector d of length $M = lm(K+1)$ and if we reformulate everything in conventional algebra, we obtain

$$\max_i \left(\alpha_{ki1}\, x_1 + \alpha_{ki2}\, x_2 + \cdots + \alpha_{kiL}\, x_L \right) = d_k \ \ . \tag{6}$$

The system of equations (6) with $k = 0, \ldots, M$ can be solved using an exhaustive search method: First we select for the first equation a term for which the maximum is reached, and we eliminate a variable if possible. Then we select for the second equation a term for which the maximum is reached, and so on, until we either find a solution or reach an inconsistent system of equations. In the latter case we backtrack and select another candidate for the maximizing term in the equation where the last choice was made. This continues until we either find a solution (which yields an rth order state space realization of G), or have exhausted all possible choices, in which case the system cannot be solved (which implies that no rth order state space realization of G exists). Hence, we can give an answer to BRDP(G,r) using a finite number of elementary operations. □

Remark 10 In the formulation of Proposition 9 we have used the concept "decidability" in a rather loose and informal way. However, it can be verified that our use of decidability corresponds to the formal concept of decidability in the Turing machine sense. ◇

If $x \in \mathbb{R}$ then $\lceil x \rceil$ is the smallest integer that is larger than or equal to x.

Proposition 11 *Let G be an ultimately periodic sequence with $G_k \in \mathbb{B}^{l \times m}$ for all k. Let n_u be an upper bound*[3] *for the minimal system order of the MLTI DES the impulse response of which coincides with G. Then BMRP(G) can be solved in a number of elementary operations that is bounded from above by the function f defined by*

$$f(n_\mathrm{u}, l, m) \ = \ 2Km \sum_{r=1}^{n_\mathrm{u}} r\,(l+r)\, 2^{r^2 + r(m+l)} \tag{7}$$

with $K = \left\lceil 2n_\mathrm{u}^2 - 3n_\mathrm{u} + 2 + \exp\left(\dfrac{n_\mathrm{u}}{e}\right) \right\rceil$. *Moreover,*

$$f(n_\mathrm{u}, l, m) \leq \gamma^{n_\mathrm{u}^2} \tag{8}$$

with $\gamma = 3^{m+l+3}$.

Proof: Since G is ultimately periodic it corresponds to the impulse response of an MLTI DES. Furthermore, since all the entries of the G_k's are in $\mathbb{B}$, G also corresponds to the impulse response of a *boolean* MLTI DES. Assume that the minimal system order of the boolean MLTI DES we are looking for is equal to n. Let n_l be a lower bound for the minimal system order (that is, e.g., obtained by using Lemma 8).
If c is the period of G, then $c \leq \exp\left(\dfrac{n}{e}\right)$ by Lemma 1. Hence, $c \leq \exp\left(\dfrac{n_\mathrm{u}}{e}\right)$. If we have a sequence F that is the impulse response of an rth order boolean MLTI DES with $r \leq n_\mathrm{u}$, then by Corollary 7 it suffices to check whether the first K terms of F and G are equal in order to decide whether F and G coincide.
Now we can apply the following procedure which is combination of an incremental search procedure[4] (for the system order) combined with an enumerative procedure (for the entries of the system matrices). We start with a guess r for the minimal system order that is equal to n_l. Then we consider all possible triples (A, B, C) with $A \in \mathbb{B}^{r \times r}$, $B \in \mathbb{B}^{r \times m}$ and $C \in \mathbb{B}^{l \times r}$. For each triple we consider the finite sequence $F = \left\{ C \otimes A^{\otimes^k} \otimes B \right\}_{k=0}^{K-1}$. If the terms of this sequence are equal to the first K terms of G, then the triple (A, B, C) is a minimal state space realization of G and r is the minimal system order. Otherwise, we consider the next triple (A, B, C). Note that the number of triples that should be considered is less than or equal to $2^{r^2 + r(m+l)}$. For each triple (A, B, C) we have to compute at most K terms of the sequence F and compare them with the corresponding term of G. It is easy to verify that this can be done using a number of additions or comparisons that is less than or equal to

$$\begin{aligned} Klm(2r-1) &+ (K-1)rm(2r-1) + Klm \\ &= Klm(2r) + (K-1)rm(2r-1) \\ &\leq Klm2r + Krm2r \\ &\leq 2Kmr(r+l) \ \ . \end{aligned}$$

If all rth order triples have been considered and no state space realization of G has been found yet, we augment r and repeat the procedure described above.
Since n_u is an upper bound for the minimal system order, this procedure will ultimately lead to a minimal state space realization of G. Note that in the worst case r ranges from 1 to n_u.

[3] See [5, 10] for a finite upper bound for the minimal system order that can be computed efficiently.

[4] We could also have used a binary search procedure.

As a consequence, the number of elementary operations that is needed to solve BMRP(G) in bounded from above by the function f defined by (7).
Furthermore, it can be verified that (8) holds for all $n_{\mathrm{u}}, l, m \in \mathbb{N}_0$. $\square$

It is still an open problem whether there exist polynomial time algorithms to solve the BRDP and the BMRP.

CONCLUSIONS

In the paper we have considered the minimal state space realization problem for max-linear time-invariant discrete event systems. More specifically we have directed our attention to the boolean minimal realization problem. First we have derived an efficiently computable lower bound for the minimal system order of a boolean max-linear discrete event system. Next we have shown that the decision problem that corresponds to the boolean minimal realization problem is decidable. Finally we have shown that the boolean minimal realization problem can be solved in a number of operations that is bounded from above by an exponential of the square of the minimal system order.

In our future research we hope to extend some of the results of this paper to general max-linear time-invariant discrete event systems.

Acknowledgments

The algorithm used in the proof of Proposition 9 was suggested to the first author by S. Gaubert. The algorithm can be thought of as a Tarski-Seidenberg elimination method for the max-plus algebra.
This research was sponsored by the Belgian program on interuniversity attraction poles (IUAP P4-02 and IUAP P4-24), by the Concerted Action Project of the Flemish Community, entitled "Model-based Information Processing Systems" (GOA-MIPS), and by the ALAPEDES project of the European Community Training and Mobility of Researchers Program.
Bart De Schutter is a senior research assistant with the FWO (Fund for Scientific Research – Flanders) and Bart De Moor is a research associate with the FWO.

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