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B. De Schutter

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Control Systems Engineering
Faculty of Information Technology and Systems
Delft University of Technology
Delft, The Netherlands
Current URL: https://www.dcsc.tudelft.nl

# On the ultimate behavior of the sequence of consecutive powers of a matrix in the max-plus algebra

Bart De Schutter*

**Abstract**

We study the sequence of consecutive powers of a matrix in the max-plus algebra, which has maximum and addition as its basic operations. If the matrix is irreducible then it is well known that the ultimate behavior of the sequence is cyclic. For reducible matrices the ultimate behavior is more complex, but it is also cyclic in nature. We will give a detailed characterization of the rates and periods of the ultimate behavior for a general matrix.

**Keywords:** max-plus algebra, sequence of matrix powers, ultimate behavior

## 1 Introduction

We consider the sequence of consecutive powers of a matrix in the max-plus algebra, which has maximum and addition as basic operations. The ultimate behavior of such a sequence has already been studied in detail by several authors if the matrix is irreducible (see [1, 5, 7, 8] and the references therein). For reducible matrices it has been shown that the ultimate behavior is periodic [8, 9]. We will extend these results (and correct the results of [7] and [13]) by completely characterizing the rates and periods of the ultimate behavior of the entries of the sequence of consecutive powers of a general max-plus-algebraic matrix.

Our main motivation for studying this problem lies in the max-plus-algebraic system theory for discrete event systems. Typical examples of discrete event systems are flexible manufacturing systems, telecommunication networks, parallel processing systems, traffic control systems and logistic systems. The class of discrete event systems essentially consists of man-made systems that contain a finite number of resources (e.g. machines, communications channels, or processors) that are shared by several users (e.g. product types, information packets, or jobs) all of which contribute to the achievement of some common goal (e.g. the assembly of products, the end-to-end transmission of a set of information packets, or a parallel computation) [1]. There are many modeling techniques for discrete event systems, such as (extended) state machines, max-plus algebra, formal languages, automata, temporal logic, generalized semi-Markov processes, Petri nets, computer simulation models and so on (see [1, 4, 12, 11] and the references cited therein). In general models that describe the behavior of a discrete event system are nonlinear in conventional algebra. However, there is a class

*Control Laboratory, Faculty of Information Technology and Systems, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands, email: b.deschutter@its.tudelft.nl

of discrete event systems – the max-plus-linear discrete event systems – that can be described by a model that is "linear" in the max-plus algebra [1, 5, 6]. The model of a max-plus-linear discrete event system can be characterized by a triple of matrices $(A, B, C)$, which are called the system matrices of the model. The ultimate behavior of the system matrix $A$ determines the ultimate behavior of the max-plus-linear discrete event system [1, 6].

This paper is organized as follows. In Section 2 we introduce some notation and we give a short introduction to the max-plus algebra and to graph theory. We also discuss the connection between max-plus-algebraic matrix operations and graph theory. In Section 3 we characterize the rates and periods of the ultimate behavior of the entries of the sequence of consecutive powers of a general max-plus-algebraic matrix. Finally we present some conclusions in Section 4.

# 2 Notation and definitions

## 2.1 Notation

The set of the real numbers is denoted by $\mathbb{R}$, the set of the nonnegative integers by $\mathbb{N}$, and the set of the positive integers by $\mathbb{N}_0$. The number of elements of a set $\gamma$ is denoted by $\#\gamma$. The least common multiple of the elements of a set $\gamma$ of positive integers is denoted by lcm $\gamma$.

Let $A \in \mathbb{R}^{m \times n}$. The entry on the $i$th row and the $j$th column of $A$ is denoted by $a_{ij}$ or $(A)_{ij}$. If $\alpha \subseteq \{1, \ldots, m\}$ and $\beta \subseteq \{1, \ldots, n\}$, then $A_{\alpha\beta}$ is the submatrix of $A$ obtained by removing all rows not indexed by $\alpha$ and all columns not indexed by $\beta$.

## 2.2 Max-plus algebra

The basic operations of the max-plus algebra are the maximum (represented by $\oplus$) and the addition (represented by $\otimes$):

$$\begin{aligned} x \oplus y &= \max(x, y) \\ x \otimes y &= x + y \end{aligned}$$

with $x, y \in \mathbb{R} \cup \{-\infty\}$. Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The operations $\oplus$ and $\otimes$ are extended to matrices as follows. If $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ and $C \in \mathbb{R}_\varepsilon^{n \times p}$ then we have

$$\begin{aligned} (A \oplus B)_{ij} &= a_{ij} \oplus b_{ij} \\ (A \otimes C)_{ij} &= \bigoplus_{k=1}^{p} a_{ik} \otimes c_{kj} \end{aligned}$$

for all $i, j$. Note that these definitions resemble the definitions of the sum and the product of matrices in linear algebra but with $\oplus$ instead of $+$ and $\otimes$ instead of $\times$. This analogy is one of the reasons why we call $\oplus$ the max-plus-algebraic addition and $\otimes$ the max-plus-algebraic multiplication. For more information on the analogies and differences between max-plus algebra and linear algebra the interested reader is referred to [1, 6].

The matrix $\mathcal{E}_{m \times n}$ is the $m \times n$ max-plus-algebraic zero matrix: $(\mathcal{E}_{m \times n})_{ij} = \varepsilon$ for all $i, j$. The matrix $E_n$ is the $n \times n$ max-plus-algebraic identity matrix: $(E_n)_{ii} = 0$ for all $i$ and $(E_n)_{ij} = \varepsilon$ for all $i, j$ with $i \neq j$. If we permute the rows or the columns of $E_n$, we obtain a max-plus-algebraic permutation matrix. For a max-plus-algebraic permutation matrix $P \in \mathbb{R}_\varepsilon^{n \times n}$ we have $P \otimes P^T = P^T \otimes P = E_n$.

Let $k \in \mathbb{N}$. The $k$th max-plus-algebraic power of $x \in \mathbb{R}$ is denoted by $x^{\otimes^k}$ and corresponds to $kx$ in conventional algebra. If $k > 0$ then $\varepsilon^{\otimes^k} = \varepsilon$. We have $\varepsilon^{\otimes^0} = 0$ by definition. The max-plus-algebraic matrix power of $A \in \mathbb{R}_\varepsilon^{n \times n}$ is defined as follows:

$$A^{\otimes^0} = E_n \quad \text{and} \quad A^{\otimes^k} = A \otimes A^{\otimes^{k-1}} \quad \text{for } k = 1, 2, \ldots$$

Consider sequences $h_i = \{(h_i)_k\}_{k=0}^{\infty}$ for $i = 1, \ldots, m$ with $(h_i)_k \in \mathbb{R}_\varepsilon$ for all $i, k$. The max-plus-algebraic sum $g = h_1 \oplus \ldots \oplus h_m$ is defined by $g_k = (h_1)_k \oplus \ldots \oplus (h_m)_k$. The max-plus-algebraic product $g = h_1 \otimes \ldots \otimes h_m$ is defined by

$$g_k = \bigoplus_{\substack{k_1, \ldots, k_m \in \mathbb{N} \\ k_1 + \ldots + k_m = k}} (h_1)_{k_1} \otimes \ldots \otimes (h_m)_{k_m} \ .$$

**Definition 2.1 (Ultimately geometric sequence)** *We say that the sequence $\{g_k\}_{k=0}^{\infty}$ is ultimately geometric if*

$$\exists K \in \mathbb{N},\ \exists c \in \mathbb{N}_0,\ \exists \lambda \in \mathbb{R}_\varepsilon \text{ such that } \forall k \geqslant K : \ g_{k+c} = \lambda^{\otimes^c} \otimes g_k \ . \tag{1}$$

The term "ultimately geometric" was introduced by Gaubert in [8, 9]. Note that "geometric" has to be understood in the max-plus-algebraic sense: the terms of the sequence are max-plus-multiplied by a constant factor $c\lambda$. If $g$ is an ultimately geometric sequence then the smallest possible $c$ for which (1) holds is called the *period* of $g$. The smallest possible corresponding $\lambda$ is then called the *rate* of $g$. Note that $\{\varepsilon\}_{k=0}^{\infty}$ has period 1 and rate $\varepsilon$.

**Definition 2.2 (Ultimately periodic sequence)** *We say that the sequence $\{g_k\}_{k=0}^{\infty}$ is ultimately periodic if*

$$\begin{aligned} &\exists K \in \mathbb{N},\ \exists c \in \mathbb{N}_0,\ \exists \lambda_0, \ldots, \lambda_{c-1} \in \mathbb{R}_\varepsilon \text{ such that} \\ &\qquad g_{kc+c+s} = \lambda_s^{\otimes^c} \otimes g_{kc+s} \text{ for all } k \geqslant K \text{ and for } s = 0, \ldots, c-1 \,. \end{aligned} \tag{2}$$

If $g$ is an ultimately periodic sequence then the smallest possible $c$ for which (2) holds is called the *period* of $g$. The smallest possible corresponding $\lambda_s$s are called the *rates* of $g$. In general the max-plus-algebraic sum of ultimately geometric sequences is ultimately periodic. The reverse also holds: every ultimately periodic sequence can be considered as the max-plus-algebraic sum of ultimately geometric sequences [8, 9].

Let us now illustrate the concepts defined above by an example.

**Example 2.3** Consider the sequence

$$g = \{g_k\}_{k=0}^{\infty} = 0, 0, 0, 2, \varepsilon, 0, 8, \varepsilon, 6, 14, \varepsilon, 12, 20, \varepsilon, 18, \ldots$$

This sequence is ultimately geometric with rate $\lambda = 2$ and period $c = 3$ since $g_{k+3} = 2^{\otimes^3} \otimes g_k = 6 \otimes g_k = 6 + g_k$ for all $k \geqslant 3$. The sequence

$$h = \{h_k\}_{k=0}^{\infty} = \varepsilon, 0, 1, 0, 2, 0, 3, 0, 4, \ldots$$

is ultimately periodic with period $c = 2$ and rates $\lambda_0 = \dfrac{1}{2}$ and $\lambda_1 = 0$ since $h_{2k+2} = \left(\dfrac{1}{2}\right)^{\otimes^2} \otimes h_{2k} = 1 \otimes h_{2k}$ and $h_{2k+2+1} = 0^{\otimes^2} \otimes h_{2k+1} = h_{2k+1}$ for all $k \geqslant 1$.
It is easy to verify that $h$ can be written as the max-plus-algebraic sum of the ultimately geometric sequences $h_1 = \varepsilon, \varepsilon, 1, \varepsilon, 2, \varepsilon, 3, \varepsilon, 4. \ldots$ and $h_2 = \varepsilon, 0, \varepsilon, 0, \varepsilon, 0, \varepsilon, 0, \varepsilon, \ldots$ Note that the rates of $h_1$ and $h_2$ are respectively $\dfrac{1}{2}$ and 0, and that their period is equal to 2. □

## 2.3 Max-plus-algebra and graph theory

We assume that the reader is familiar with basic concepts of graph theory such as directed graph, loop, circuit, elementary circuit and so on (see e.g. [1, 14]).

If we have a directed graph $\mathcal{G}$ with vertex set $\mathcal{V} = \{1, 2, \ldots, n\}$ and if we associate a real number $w_{ij}$ with each arc $(j, i)$ of $\mathcal{G}$, then we say that $\mathcal{G}$ is a weighted directed graph. We call $w_{ij}$ the weight of the arc $(j, i)$. Note that the first subscript of $w_{ij}$ corresponds to the final (and not the initial) vertex of the arc $(j, i)$. With every weighted directed graph $\mathcal{G}$ with vertex set $\mathcal{V} = \{1, 2, \ldots, n\}$ there corresponds a matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ such that $a_{ij} = w_{ij}$ if there is an arc $(j, i)$ in $\mathcal{G}$ with weight $w_{ij}$ and $a_{ij} = \varepsilon$ if there is no arc $(j, i)$ in $\mathcal{G}$. We say that $\mathcal{G}$ is the precedence graph of $A$, denoted by $\mathcal{G}(A)$.

Let $A \in \mathbb{R}_\varepsilon^{n\times n}$. The weight of a path $i_1 \to i_2 \to \cdots \to i_l$ in $\mathcal{G}(A)$ is defined as the sum of the weights of the arcs that compose the path. Let us now give a graph-theoretic interpretation of the max-plus-algebraic matrix power. If $k \in \mathbb{N}_0$ then we have

$$\begin{aligned}(A^{\otimes^k})_{ij} &= \bigoplus_{i_1,\ldots,i_{k-1}} a_{ii_1} \otimes a_{i_1 i_2} \otimes \ldots \otimes a_{i_{k-1}j} \\ &= \max_{i_1,\ldots,i_{k-1}} \left( a_{ii_1} + a_{i_1 i_2} + \cdots + a_{i_{k-1}j} \right)\end{aligned}$$

for all $i, j$. Hence, $(A^{\otimes^k})_{ij}$ is the maximal weight of all paths of $\mathcal{G}(A)$ of length $k$ that have $j$ as their initial vertex and $i$ as their final vertex — where the maximal weight is equal to $\varepsilon$ by definition if there does not exist a path of length $k$ from $j$ to $i$.

The average weight of a circuit is defined as the weight of the circuit divided by its length. A circuit is called critical if it has maximum average weight. The critical graph $\mathcal{G}^{\mathrm{c}}(A)$ of $A$ consists of those vertices and arcs of $\mathcal{G}(A)$ that belong to some critical circuit of $\mathcal{G}(A)$.

We say that $\mathcal{G}(A)$ is strongly connected if for any two *different*[1] vertices $v_i$, $v_j$ of $\mathcal{G}(A)$ there exists a path from $v_i$ to $v_j$. A maximal strongly connected subgraph (m.s.c.s.) $\mathcal{G}_{\text{sub}}$ of $\mathcal{G}(A)$ is a strongly connected subgraph that is maximal, i.e. if we add an extra vertex (and some extra arcs) of $\mathcal{G}(A)$ to $\mathcal{G}_{\text{sub}}$ then $\mathcal{G}_{\text{sub}}$ is no longer strongly connected. The matrix $A$ is called irreducible if $\mathcal{G}(A)$ is strongly connected.

The cyclicity of an irreducible matrix $A$ is equal to the greatest common divisor of the lengths of all the elementary circuits of the $\mathcal{G}^{\mathrm{c}}(A)$. If the graph $\mathcal{G}^{\mathrm{c}}(A)$ contains no circuits then the cyclicity is equal to 1 by definition. Note that the $1 \times 1$ max-plus-algebraic zero matrix $[\,\varepsilon\,]$ is the only max-plus-algebraic zero matrix that is irreducible and that its cyclicity is equal to 1. The following theorem gives a relation between the cyclicity of an irreducible matrix $A$ and the ultimate behavior of the sequence $\{A^{\otimes^k}\}_{k=0}^{\infty}$.

**Theorem 2.4** *If $A \in \mathbb{R}_\varepsilon^{n\times n}$ is irreducible, then*

$$\exists \lambda \in \mathbb{R}_\varepsilon, \exists k_0 \in \mathbb{N} \text{ such that } \forall k \geqslant k_0 \;:\; A^{\otimes^{k+c}} = \lambda^{\otimes^c} \otimes A^{\otimes^k}$$

*where $c$ is the cyclicity of $A$.*

**Proof:** See e.g. Theorem 1.2.3 of [8]. ■

[1] Most authors do not add the extra condition that the vertices should be different. However, this definition, which was taken from [1], makes some of the subsequent definitions, theorems and proofs easier to formulate.

The number $\lambda$ that appears in Theorem 2.4 is called the max-plus-algebraic eigenvalue of $A$ and it corresponds to the maximal average weight over all elementary circuits of $\mathcal{G}(A)$.

The following theorem is the max-plus-algebraic analogue of a well-known result from matrix algebra that states that any square matrix can be transformed into a block upper diagonal matrix with irreducible blocks by simultaneously reordering the rows and columns of the matrix (see e.g. [1, 3, 10] for the proof of this theorem and for its interpretation in terms of graph theory):

**Theorem 2.5** *If $A \in \mathbb{R}_\varepsilon^{n\times n}$ then there exists a max-plus-algebraic permutation matrix $P \in \mathbb{R}_\varepsilon^{n\times n}$ such that the matrix $\hat{A} = P \otimes A \otimes P^T$ is a max-plus-algebraic block upper triangular matrix of the form*

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \ldots & \hat{A}_{1l} \\ \mathcal{E} & \hat{A}_{22} & \ldots & \hat{A}_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{E} & \mathcal{E} & \ldots & \hat{A}_{ll} \end{bmatrix} \tag{3}$$

*with $l \geqslant 1$ and where the matrices $\hat{A}_{11}, \ldots, \hat{A}_{ll}$ are square and irreducible. The matrices $\hat{A}_{11}, \ldots, \hat{A}_{ll}$ are uniquely determined to within simultaneous permutation of their rows and columns, but their ordering in (3) is not necessarily unique.*

The form in (3) is called the max-plus-algebraic Frobenius normal form of $A$. If $A$ is irreducible then there is only one block in (3) and then $A$ is a max-plus-algebraic Frobenius normal form of itself. Each diagonal block of $\hat{A}$ corresponds to an m.s.c.s. of $\mathcal{G}(\hat{A})$. If $\hat{A} = P \otimes A \otimes P^T$ is the max-plus-algebraic Frobenius normal form of $A$, then we have $A = P^T \otimes \hat{A} \otimes P$ since $P$ is a max-plus-algebraic permutation matrix. Hence,

$$A^{\otimes^k} = (P^T \otimes \hat{A} \otimes P)^{\otimes^k} = P^T \otimes \hat{A}^{\otimes^k} \otimes P \qquad \text{for all } k \in \mathbb{N}\,. \tag{4}$$

## 3 The ultimate behavior of the sequence $\{A^{\otimes^k}\}_{k=0}^\infty$

If $A \in \mathbb{R}_\varepsilon^{n\times n}$ is irreducible the ultimate behavior of $\{A^{\otimes^k}\}_{k=0}^\infty$ is characterized by Theorem 2.4. For a general matrix $A$ it has already been shown in [8] that the sequences $\{(A^{\otimes^k})_{ij}\}_{k=0}^\infty$ are ultimately periodic. An essentially equivalent result has been obtained independently by Bonnier-Rigny and Krob in [2] for the structure $(\mathbb{N} \cup \{+\infty\}, \min, +)$, which is called the tropical semiring and which is strongly related to the max-plus-algebra. Now we will extend these results and give a detailed characterization of rates and periods of the entries of the sequence $\{A^{\otimes^k}\}_{k=0}^\infty$ for a general matrix $A$.

The following two technical lemmas will be used in the proof of the main theorem. Their proofs can be found in the appendix.

**Lemma 3.1** *Consider $m$ ultimately geometric sequences $h_1, \ldots, h_m$ with rates different from $\varepsilon$. Let $c_i$ be the period of $h_i$ and let $\lambda_i$ be the rate of $h_i$ for $i = 1, \ldots, m$. If $g = h_1 \oplus \cdots \oplus h_m$ and if $c = \operatorname{lcm}(c_1, \ldots, c_m)$ then*

$$\exists K \in \mathbb{N},\ \exists \gamma_0, \ldots, \gamma_{c-1} \in \{\lambda_1, \ldots, \lambda_m\} \text{ such that}$$

$$g_{kc+c+s} = {\gamma_s}^{\otimes^c} \otimes g_{kc+s} \quad \text{for all } k \geqslant K \text{ and for } s = 0, \ldots, c-1 \,. \tag{5}$$

*Furthermore, there exists at least one index $s \in \{0, \ldots, c-1\}$ such that the smallest $\gamma_s$ for which (5) holds is equal to $\bigoplus_{i=1}^{m} \lambda_i$.*

**Lemma 3.2** *Consider $m$ ultimately geometric sequences $h_1, \ldots, h_m$ with rates different from $\varepsilon$. Let $c_i$ be the period of $h_i$ and let $\lambda_i$ be the rate of $h_i$ for $i = 1, \ldots, m$. If $g = h_1 \otimes \ldots \otimes h_m$ and if $c = \mathrm{lcm}(c_1, \ldots, c_m)$ then*

$$\begin{aligned} &\exists K \in \mathbb{N},\ \exists \gamma_0, \ldots, \gamma_{c-1} \in \mathbb{R}_\varepsilon \text{ such that} \\ &\qquad g_{kc+c+s} = {\gamma_s}^{\otimes^c} \otimes g_{kc+s} \quad \text{for all } k \geqslant K \text{ and for } s = 0, \ldots, c-1 \,. \end{aligned} \tag{6}$$

*There exists at least one index $s \in \{0, 1, \ldots, c-1\}$ such that the smallest $\gamma_s$ for which (6) holds is equal to $\bigoplus_{i=1}^{m} \lambda_i$. Moreover, for $k^*$ large enough $\{g_k\}_{k=k^*}^{\infty}$ can be written as a* finite *sum of ultimately geometric sequences with rates $\lambda_i$ and periods $c_i$.*

If $\hat{A}$ is a max-plus-algebraic normal form of $A$, then it follows from (4) that we may consider the sequence $\{\hat{A}^{\otimes^k}\}_{k=0}^{\infty}$ instead of $\{A^{\otimes^k}\}_{k=0}^{\infty}$ if we want to study the ultimate behavior of the sequence of consecutive powers of $A$. The following theorem, which is an extension of Theorem 2.4 and a corrected version of Lemma 4 of [13] and of Lemma C.1.4 of [7] characterizes the rates and periods of the ultimate behavior of $\{\hat{A}^{\otimes^k}\}_{k=0}^{\infty}$:

**Theorem 3.3** *Let $\hat{A} \in \mathbb{R}_\varepsilon^{n \times n}$ be a matrix of the form (3) where the matrices $\hat{A}_{11}, \ldots, \hat{A}_{ll}$ are square and irreducible. Let $\lambda_i$ and $c_i$ be respectively the max-plus-algebraic eigenvalue and the cyclicity of $\hat{A}_{ii}$ for $i = 1, \ldots, l$. Define sets $\alpha_1, \ldots, \alpha_l$ such that $\hat{A}_{\alpha_i \alpha_j} = \hat{A}_{ij}$ for all $i, j$ with $i \leqslant j$.*
*Define*

$$S_{ij} = \Big\{ \{i_0, \ldots, i_s\} \subseteq \{1, \ldots, l\} \Bigm| i = i_0 < i_1 < \ldots < i_s = j \text{ and } \hat{A}_{i_r i_{r+1}} \neq \mathcal{E} \text{ for } r = 0, \ldots, s-1 \Big\}$$

$$\Gamma_{ij} = \bigcup_{\gamma \in S_{ij}} \gamma$$

$$\Lambda_{ij} = \begin{cases} \{\lambda_t | t \in \Gamma_{ij}\} & \text{if } \Gamma_{ij} \neq \emptyset \,, \\ \{\varepsilon\} & \text{if } \Gamma_{ij} = \emptyset \,, \end{cases}$$

$$c_{ij} = \begin{cases} \mathrm{lcm}\{ c_t \,|\, t \in \Gamma_{ij} \} & \text{if } \Gamma_{ij} \neq \emptyset \text{ and } c_t \neq 0 \text{ for some } t \in \Gamma_{ij} \,, \\ 1 & \text{otherwise} \,, \end{cases}$$

*for all $i, j$ with $i < j$. We have*

$$\forall i, j \in \{1, \ldots, l\} \text{ with } i > j : \left(\hat{A}^{\otimes^k}\right)_{\alpha_i \alpha_j} = \mathcal{E}_{n_i \times n_j} \quad \text{for all } k \in \mathbb{N} \,. \tag{7}$$

*Moreover, there exists an integer $K \in \mathbb{N}$ such that*

$$\forall i \in \{1,\dots,l\} : \left(\hat{A}^{\otimes^{k+c_i}}\right)_{\alpha_i\alpha_i} = \lambda_i^{\otimes^{c_i}} \otimes \left(\hat{A}^{\otimes^k}\right)_{\alpha_i\alpha_i} \quad \textit{for all } k \geqslant K \tag{8}$$

*and*

$$\forall i,j \in \{1,\dots,l\} \textit{ with } i<j, \forall p \in \alpha_i, \forall q \in \alpha_j, \exists \gamma_0,\dots,\gamma_{c_{ij}-1} \in \Lambda_{ij} \textit{ such that}$$
$$\left(\hat{A}^{\otimes^{kc_{ij}+c_{ij}+s}}\right)_{pq} = \gamma_s^{\otimes^{c_{ij}}} \otimes \left(\hat{A}^{\otimes^{kc_{ij}+s}}\right)_{pq} \quad \textit{for all } k \geqslant K \textit{ and for } s=0,\dots,c_{ij}-1\,. \tag{9}$$

*Furthermore, for each combination $i,j,p,q$ with $i<j$, $p \in \alpha_i$ and $q \in \alpha_j$, there exists at least one index $s \in \{0,\dots,c_{ij}-1\}$ such that the smallest $\gamma_s$ for which (9) holds is equal to* $\max \Lambda_{ij}$.

**Remark 3.4** Let us give a graphical interpretation of the sets $S_{ij}$ and $\Gamma_{ij}$. Let $C_i$ be the m.s.c.s. of $\mathcal{G}(\hat{A})$ that corresponds to $\hat{A}_{ii}$ for $i=1,\dots,l$. So $\alpha_i$ is the vertex set of $C_i$.
If $\{i_0=i,i_1,\dots,i_s=j\} \in S_{ij}$ then there exists a path from a vertex in $C_{i_r}$ to a vertex in $C_{i_{r-1}}$ for each $r \in \{1,\dots,s\}$. Since each m.s.c.s. $C_i$ of $\mathcal{G}(\hat{A})$ is strongly connected, this implies that there exists a path from a vertex in $C_j$ to a vertex in $C_i$ that passes through $C_{i_{s-1}}$, $C_{i_{s-2}}$, $\dots$, $C_{i_1}$.
If $S_{ij}=\emptyset$ then there does not exist any path from a vertex in $C_j$ to a vertex in $C_i$.
The set $\Gamma_{ij}$ is the set of indices of the m.s.c.s.'s of $\mathcal{G}(\hat{A})$ through which some path from a vertex of $C_j$ to a vertex of $C_i$ passes. $\diamond$

**Proof of Theorem 3.3 :** Since the matrices $\hat{A}_{\alpha_i\alpha_i}$ are irreducible (8) is a direct consequence of Theorem 2.4.
Recall that $(\hat{A}^{\otimes^k})_{ij}$ is equal to the maximal weight over all paths of length $k$ from $j$ to $i$ in $\mathcal{G}(\hat{A})$ where the maximal weight is equal to $\varepsilon$ by definition if there does not exist any path of length $k$ from $j$ to $i$. Let $C_i$ be the m.s.c.s. of $\mathcal{G}(\hat{A})$ that corresponds to $\hat{A}_{ii}$ for $i=1,\dots,l$. Since $\hat{A}_{\alpha_i\alpha_j} = \mathcal{E}_{n_i\times n_j}$ if $i>j$, there are no arcs from any vertex of $C_j$ to a vertex in $C_i$. As a consequence, (7) holds.
Now consider $i,j \in \{1,\dots,l\}$ with $i<j$. We distinguish three cases:

- If $\Gamma_{ij}=\emptyset$ then there does not exist a path from a vertex in $C_j$ to a vertex in $C_i$. Hence, $\left(\hat{A}^{\otimes^k}\right)_{\alpha_i\alpha_j} = \mathcal{E}_{n_i\times n_j}$ for all $k \in \mathbb{N}$. Since in this case we have $\Lambda_{ij}=\{\varepsilon\}$ and $c_{ij}=1$, this implies that (9) and the last statement of the theorem hold if $\Gamma_{ij}=\emptyset$.
- If $\Gamma_{ij} \neq \emptyset$ and $\Lambda_{ij}=\{\varepsilon\}$ then $\hat{A}_{tt}=[\,\varepsilon\,]$ and $c_t=1$ for all $t \in \Gamma_{ij}$. So there exist paths from a vertex in $C_j$ to a vertex in $C_i$, but each path passes only through m.s.c.s.'s that consist of one vertex and contain no loop. Such a path passes through at most $\#\Gamma_{ij}$ of such m.s.c.s.'s ($C_j$ and $C_i$ included). This implies that there does not exist a path with a length larger than or equal to $\#\Gamma_{ij}$ from a vertex in $C_j$ to a vertex in $C_i$. Hence, $\left(A^{\otimes^k}\right)_{\alpha_i\alpha_j} = \mathcal{E}_{n_i\times n_j}$ for all $k \geqslant \#\Gamma_{ij}$. Furthermore, $c_{ij}=1$ since $c_t=1$ for all $t \in \Gamma_{ij}$. Hence, (9) and the last statement of the theorem also hold if $\Gamma_{ij}=\emptyset$ and $\Lambda_{ij}=\{\varepsilon\}$.
- Finally, we consider the case with $\Gamma_{ij} \neq \emptyset$ and $\Lambda_{ij} \neq \{\varepsilon\}$. Select an arbitrary vertex $p$ of $C_i$ and an arbitrary vertex $q$ of $C_j$. For each set $\gamma=\{i_0,\dots,i_s\} \in S_{ij}$ we define

$$\mathcal{S}(\gamma) = \big\{(U,V)\,\big|\,U=\{u_0,\dots,u_s\},\ V=\{v_0,\dots,v_s\},\ u_s=q,\ v_0=p,\ \text{and}$$

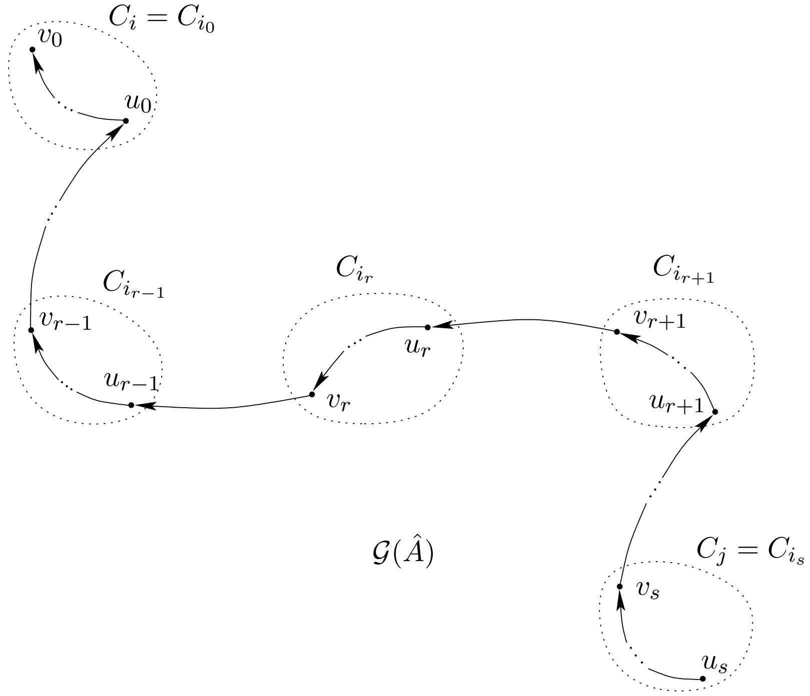


Figure 1: Illustration of the proof of Theorem 3.3. There exists a path from vertex $u_s$ of m.s.c.s. $C_j$ to vertex $v_0$ of m.s.c.s. $C_i$ that passes through the m.s.c.s.'s $C_{i_{s-1}}$, $C_{i_{s-2}}$, …, $C_{i_1}$.

$$u_r \in \alpha_{i_r},\ v_{r+1} \in \alpha_{i_{r+1}} \text{ and } (\hat{A})_{u_r v_{r+1}} \neq \varepsilon \text{ for } r = 0, \ldots, s\big\} \ .$$

So if $(U, V) \in \mathcal{S}(\gamma)$ with $U = \{u_0, \ldots, u_s\}$ and $V = \{v_0, \ldots, v_s\}$ then there exists a path from $q$ to $p$ that passes through m.s.c.s. $C_{i_r}$ for $r = 0, \ldots, s$ and that enters $C_{i_r}$ at vertex $u_r$ for $r = 0, \ldots, s-1$ and that exits from $C_{i_r}$ through vertex $v_r$ for $r = 1, \ldots, s$ (see also Figure 1). Hence, we have

$$\left(\hat{A}^{\otimes^k}\right)_{pq} = \bigoplus_{\gamma \in S_{ij}} \bigoplus_{(U,V) \in \mathcal{S}(\gamma)} g(\gamma, U, V) \qquad \text{for all } k \in \mathbb{N}_0$$

where

$$g(\gamma, U, V) = \bigoplus_{\substack{p_0, \ldots, p_s \in \mathbb{N} \\ p_0 + \ldots + p_s = k - s}} \left(\hat{A}_{i_0 i_0}^{\otimes^{p_0}}\right)_{p u_0} \otimes \left(\hat{A}_{i_0 i_1}\right)_{u_0 v_1} \otimes \left(\hat{A}_{i_1 i_1}^{\otimes^{p_1}}\right)_{v_1 u_1} \otimes \ldots$$
$$\otimes \left(\hat{A}_{i_{s-1} i_s}\right)_{u_{s-1} v_s} \otimes \left(\hat{A}_{i_s i_s}^{\otimes^{p_s}}\right)_{v_s q} \tag{10}$$

with the empty max-plus-algebraic sum equal to $\varepsilon$ by definition. Each term of the max-plus-algebraic sum in (10) represents the maximal weight over all paths from $q$ to $p$ that consist of the concatenation of paths of length $p_r$ from vertex $u_r$ to vertex $v_r$ of $C_{i_r}$ for $r = 0, \ldots, s$ and paths of length 1 from vertex $v_{r+1}$ of $C_{i_{r+1}}$ to vertex $u_r$ of $C_{i_r}$ for $r = 0, \ldots, s$ where by definition the maximal weight is equal to $\varepsilon$ if no such paths exist. Note that if $\lambda_{i_r} = \varepsilon$ for some $r$ then every term in the max-plus-algebraic sum (10) for which $p_r > 0$ will be equal to $\varepsilon$. Furthermore, since $\varepsilon^{\otimes^0} = 0$ by definition, this

means that each factor of the form $\left(\hat{A}_{i_r i_r}^{\otimes^{p_r}}\right)_{u_r v_r}$ for which $\lambda_{i_r} = \varepsilon$ may be removed from the max-plus-algebraic sum (10). Note that indices $t$ for which $\lambda_t = \varepsilon$ or equivalently $c_t = 1$ do not influence the value of $c_{ij}$. Also note that since $\Gamma_{ij} \neq \emptyset$ and $\Lambda_{ij} \neq \{\varepsilon\}$ we have at least one combination $\gamma, U, V$ for which the sequence (10) has a rate $\lambda_{i_r}$ that is different from $\varepsilon$.
Since $\hat{A}_{i_r i_r}$ is irreducible, we have

$$\left(\hat{A}_{i_r i_r}^{\otimes^{k+c_{i_r}}}\right)_{v_r u_r} = {\lambda_{i_r}}^{\otimes^{c_{i_r}}} \otimes \left(\hat{A}_{i_r i_r}^{\otimes^{k}}\right)_{v_r u_r} \qquad \text{for } k \text{ large enough}$$

by Theorem 2.4. Hence, if $g(\gamma, U, V)$ is different from $\varepsilon$, i.e. if it still contains terms after the factors for which $\lambda_{i_r} = \varepsilon$ have been removed, $g(\gamma, U, V)$ is a max-plus-algebraic product of ultimate geometric sequences with rates $\lambda_{i_r} \neq \varepsilon$ and periods $c_{i_r}$. From Lemma 3.2 it follows that $g(\gamma, U, V)$ is an ultimately periodic sequence and that for $k^*$ large enough $\{\left(g(\gamma, U, V)\right)_k\}_{k=k^*}^{\infty}$ can be written as the max-plus-algebraic sum of a finite number of ultimately geometric sequences with rates $\lambda_{i_r} \neq \varepsilon$ and periods $c_{i_r}$. So $\{(\hat{A}^{\otimes^k})_{pq}\}_{k=0}^{\infty}$ is a max-plus-algebraic sum of ultimately geometric sequences with rates $\lambda_{i_r} \neq \varepsilon$ and periods $c_{i_r}$. Hence, it follows from Lemma 3.1 that (9) and the last statement of the theorem hold. ■

The following example shows that the lcm in the definition of $c_{ij}$ in Theorem 3.3 is necessary (Lemma 4 of [13] incorrectly uses max instead of lcm.).

**Example 3.5** Consider the matrix

$$A = \left[\begin{array}{c|cc|ccc|c}
\varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \hline
\varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & 0 \\
\varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \hline
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 0 \\
\varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \hline
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon
\end{array}\right] .$$

This matrix is in max-plus-algebraic Frobenius normal form and its block structure is indicated by the vertical and horizontal lines. The precedence graph of $A$ is represented in Figure 2. The sets and variables of Theorem 3.3 have the following values for $A$: $\alpha_1 = \{1\}$, $\alpha_2 = \{2, 3\}$, $\alpha_3 = \{4, 5, 6\}$, $\alpha_4 = \{7\}$, $\lambda_1 = \lambda_4 = \varepsilon$, $\lambda_2 = \lambda_3 = 0$, $c_1 = c_4 = 1$, $c_2 = 2$ and $c_3 = 3$. Now we consider the ultimate behavior of the sequence $\{(A^{\otimes^k})_{\alpha_1 \alpha_4}\}_{k=0}^{\infty}$. Note that $S_{14} = \{\{2\}, \{3\}\}$, $\Gamma_{14} = \{2, 3\}$, $\Gamma_{23} = \{0\}$, and $c_{14} = \mathrm{lcm}(c_2, c_3) = \mathrm{lcm}(2, 3) = 6$. We have

$$\{(A^{\otimes^k})_{\alpha_1 \alpha_4}\}_{k=0}^{\infty} = \varepsilon, 0, \varepsilon, 0, 0, 0, \varepsilon, 0, \varepsilon, 0, 0, 0, \varepsilon, 0, \varepsilon, 0, 0, 0, \varepsilon, 0, \varepsilon, 0, 0, 0, \ldots$$

The period of this sequence is given by $c_{14} = 6 = \mathrm{lcm}(c_2, c_3)$. Hence, the lcm in the definition of $c_{ij}$ in Theorem 3.3 is really necessary. □

The following example shows that the sequence $\{(\hat{A}^{\otimes^k})_{ij}\}_{k=1}^{\infty}$ is in general not ultimately geometric (Lemma 4 of [13] and Lemma C.1.4 of [7] incorrectly state that if $i < j$ then the matrix sequence $\{\hat{A}^{\otimes^k}\}_{k=0}^{\infty}$ is ultimately geometric).

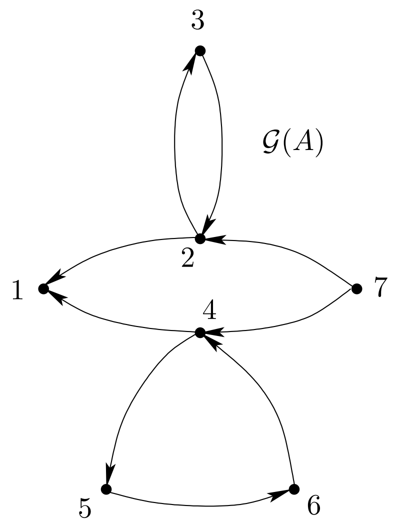


Figure 2: The precedence graph $\mathcal{G}(A)$ of the matrix $A$ of Example 3.5. All the arcs have weight 0.

**Example 3.6** We construct the matrix $\tilde{A}$ from the matrix $A$ of Example 3.5 by replacing $a_{23}$ by 2 and keeping all other entries. Now we have $\lambda_2 = 1$. The values of the other variables and sets of Theorem 3.3 are the same as for the matrix $A$ of Example 3.5. We have

$$\{(\tilde{A}^{\otimes^k})_{\alpha_1\alpha_4}\}_{k=0}^{\infty} = \varepsilon, 0, \varepsilon, 2, 0, 4, \varepsilon, 6, \varepsilon, 8, 0, 10, \varepsilon, 12, \varepsilon, 14, 0, 16, \varepsilon, 18, \varepsilon, 20, 0, 22, \ldots$$

This sequence is ultimately periodic with period $c_{14} = 6$ and with rates $\gamma_0 = \gamma_2 = \varepsilon$, $\gamma_1 = \gamma_3 = \gamma_5 = 1 = \lambda_2$ and $\gamma_4 = 0 = \lambda_4$. So the sequence $\{(\hat{A}^{\otimes^k})_{ij}\}_{k=0}^{\infty}$ is in general not ultimately geometric. $\Box$

**Corollary 3.7** *Let $\hat{A} \in \mathbb{R}_\varepsilon^{n\times n}$ be a matrix of the form (3) where the matrices $\hat{A}_{11}$, $\hat{A}_{22}$, …, $\hat{A}_{ll}$ are square and irreducible. Let $\lambda_i$ and $c_i$ be respectively the max-plus-algebraic eigenvalue and the cyclicity of $\hat{A}_{ii}$ for $i = 1, \ldots, l$. Let $\alpha_i$, $\Lambda_{ij}$ and $c_{ij}$ be defined as in Theorem 3.3. Then there exists an integer $K$ such that*

$$\begin{aligned}
&\forall i, j \in \{1, \ldots, l\} \text{ with } i > j,\ \forall p \in \alpha_i, \forall q \in \alpha_j,\ \exists s \in \{0, \ldots, c_{ij}-1\} \text{ such that} \\
&\left(\hat{A}^{\otimes^{kc_{ij}+s+c_{ij}}} \oplus \hat{A}^{\otimes^{kc_{ij}+s+c_{ij}+1}} \oplus \ldots \oplus \hat{A}^{\otimes^{kc_{ij}+s+2c_{ij}-1}}\right)_{pq} = \\
&\lambda_{ij}{}^{\otimes^{c_{ij}}} \otimes \left(\hat{A}^{\otimes^{kc_{ij}+s}} \oplus \hat{A}^{\otimes^{kc_{ij}+s+1}} \oplus \ldots \oplus \hat{A}^{\otimes^{kc_{ij}+s+c_{ij}-1}}\right)_{pq} \quad \text{for all } k \geqslant K,
\end{aligned}$$

*where* $\lambda_{ij} = \max \Lambda_{ij}$.

**Proof:** This is a direct consequence of the last statement of Theorem 3.3. ■

# 4 Conclusions

In this paper we have given a detailed characterization of the rates and periods of the ultimate behavior of the sequence of consecutive matrix powers of a general max-plus-algebraic matrix as a function of structural parameters of the matrix such as its size, its Frobenius normal form, and the eigenvalues and cyclicities of the diagonal blocks in the Frobenius normal form. This result extends and corrects previously known results. We have only considered the ultimate behavior of the sequence. An important open question and topic for future research is the characterization of both the length and the evolution of the *transient* behavior of the sequence as a function of the structural parameters of the matrix.

**Acknowledgment**

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## Appendix: Proofs of Lemmas 3.1 and 3.2

**Proof of Lemma 3.1 :** In this proof we always assume that $i \in \{1, \ldots, m\}$, $s, s^* \in \{0, \ldots, c-1\}$ and $k, l \in \mathbb{N}$. Since each sequence $h_i$ is ultimately geometric, there exists an integer $K$ such that $(h_i)_{k+c_i} = {\lambda_i}^{\otimes^{c_i}} \otimes (h_i)_k$ for all $k \geqslant K$ and for all $i$. Hence,

$$(h_i)_{k+pc_i} = {\lambda_i}^{\otimes^{pc_i}} \otimes (h_i)_k \qquad \text{for all } p \in \mathbb{N}, \text{ for all } k \geqslant K \text{ and for all } i\,. \tag{11}$$

Since $c = \operatorname{lcm}(c_1, \ldots, c_m)$ there exist positive integers $w_1, \ldots, w_m$ such that $c = w_i c_i$ for all $i$. Select $L \in \mathbb{N}$ with $Lc \geqslant K$. Consider an arbitrary index $s$. Since $Lc + s \geqslant K$, it follows from (11) that

$$(h_i)_{lc+s} = (h_i)_{Lc+s+(l-L)w_i c_i} = {\lambda_i}^{\otimes^{(l-L)w_i c_i}} \otimes (h_i)_{Lc+s} = {\lambda_i}^{\otimes^{(l-L)c}} \otimes (h_i)_{Lc+s} \tag{12}$$

for all $l \geqslant L$ and for all $i$. Define $N_s = \{\, i \,|\, (h_i)_{Lc+s} \neq \varepsilon \,\}$. We consider two cases:

- If $N_s = \emptyset$ then $(h_i)_{Lc+s} = \varepsilon$ for all $i$ and thus also $(h_i)_{lc+s} = \varepsilon$ for all $l \geqslant L$ and for all $i$ by (12). Hence, $g_{lc+s} = \varepsilon$ for all $l \geqslant L$. So if we set $\gamma_s = \lambda_1$ and select $K \geqslant K_s \overset{\text{def}}{=} L$ then (5) holds for this case.
- If $N_s \neq \emptyset$ then we define $\gamma_s = \max\limits_{i \in N_s} \lambda_i$ and $i_s = \arg\max\limits_{i \in N_s} \left\{ (h_i)_{Lc+s} \,\middle|\, \lambda_i = \gamma_s \right\}$. By (12) we have $(h_i)_{lc+s} = (h_i)_{Lc+s} + (l-L)c\lambda_i$ for all $l \geqslant L$ and for all $i$. Furthermore, $\lambda_{i_s} \geqslant \lambda_i$ for all $i$, and $\varepsilon \neq (h_{i_s})_{Lc+s} \geqslant (h_i)_{Lc+s}$ for all $i$ with $\lambda_i = \gamma_s$. So if we define $K_s = L + \max\left(0, \max\limits_{\substack{i \in N_s \\ \lambda_i \neq \gamma_s}} \left( \dfrac{(h_i)_{Lc+s} - (h_{i_s})_{Lc+s}}{c(\gamma_s - \lambda_i)} \right)\right)$ with $\max \emptyset = 0$ by definition, then we have $(h_{i_s})_{lc+s} \geqslant (h_i)_{lc+s}$ for all $l \geqslant K_s$ and all $i$. Hence, $g_{lc+s} = (h_{i_s})_{lc+s}$ for all $l \geqslant K_s$. As a consequence, (5) also holds for this case if we select $K \geqslant K_s$.

So if we define $K = \max(K_0, K_1, \ldots, K_{c-1})$ then (5) holds for all $s$.

Assume $\bigoplus\limits_{i=1}^{m} \lambda_i = \lambda_j$. Since $\lambda_j \neq \varepsilon$, there exists at least one index $s^*$ such that $(h_j)_{Lc+s^*} \neq \varepsilon$. Since $g_{lc+s^*} = (h_{i_{s^*}})_{lc+s^*}$ for all $l \geqslant K$ and since $\lambda = \lambda_{i_{s^*}}$ is the rate of $h_{i_{s^*}}$, i.e. the smallest $\lambda$ for which (1) holds, $\gamma_s = \lambda_{i_{s^*}} = \lambda_j$ is also the smallest $\gamma_s$ for which (5) holds. ■

**Proof of Lemma 3.2 :** For sake of simplicity, we shall only prove the lemma for the case $m = 2$. The proof for $m > 2$ follows similar lines.
In this proof we always assume that $r, s \in \{0, \ldots, c-1\}$, $p, q, i, k \in \mathbb{N}$.
Since $h_1$ and $h_2$ are ultimately geometric, there exists an integer $L$ such that

$$(h_i)_{Lc+pc+s} = {\lambda_i}^{\otimes^{pc}} \otimes (h_i)_{Lc+s} \qquad \text{for all } p, r \text{ and } i = 1, 2 \tag{13}$$

(cf. (12) with $l = L + p$). We have

$$(h_1 \otimes h_2)_k = \bigoplus_{i=0}^{Lc+c-1} (h_1)_i \otimes (h_2)_{k-i} \oplus \bigoplus_{i=Lc+c}^{k-Lc-c} (h_1)_i \otimes (h_2)_{k-i} \oplus \bigoplus_{i=0}^{Lc+c-1} (h_1)_{k-i} \otimes (h_2)_i \tag{14}$$

for all $k \geqslant 2(Lc + c)$. Now we consider an arbitrary term of the second max-plus-algebraic sum of (14). Let $k \geqslant 2(Lc + c)$ and $i \in \{Lc + c, \ldots, k - Lc - c\}$. Select $p, q, r, s$ such that

$i = Lc + pc + r$ and $k - i = Lc + qc + s$. It is easy to verify that we have $\alpha^{\otimes^p} \otimes \beta^{\otimes^q} \leqslant \alpha^{\otimes^{p+q}} \oplus \beta^{\otimes^{p+q}}$ for all $\alpha, \beta \in \mathbb{R}_\varepsilon$ and all $p, q \in \mathbb{N}$. Hence,

$$\begin{aligned}
(h_1)_i \otimes (h_2)_{k-i} &= \lambda_1{}^{\otimes^{pc}} \otimes (h_1)_{Lc+r} \otimes \lambda_2{}^{\otimes^{qc}} \otimes (h_2)_{Lc+s} && \text{(by (13))} \\
&\leqslant \left( \lambda_1{}^{\otimes^{(p+q)c}} \oplus \lambda_2{}^{\otimes^{(p+q)c}} \right) \otimes (h_1)_{Lc+r} \otimes (h_2)_{Lc+s} \\
&\leqslant \lambda_1{}^{\otimes^{(p+q)c}} \otimes (h_1)_{Lc+r} \otimes (h_2)_{Lc+s} \oplus \lambda_2{}^{\otimes^{(p+q)c}} \otimes (h_1)_{Lc+r} \otimes (h_2)_{Lc+s} \\
&\leqslant (h_1)_{Lc+r} \otimes (h_2)_{Lc+(p+q)c+s} \oplus (h_1)_{Lc+(p+q)c+r} \otimes (h_2)_{Lc+s} && \text{(by (13))}.
\end{aligned}$$

Since $Lc{+}r \leqslant Lc{+}c{-}1$ and $Lc{+}r{+}Lc{+}s{+}(p{+}q)c = k$, the term $(h_1)_{Lc+r} \otimes (h_2)_{Lc+(p+q)c+s}$ also appears in the first max-plus-algebraic sum of (14). Similarly, it can be shown that $(h_1)_{Lc+(p+q)c+r} \otimes (h_2)_{Lc+s}$ also appears in the third max-plus-algebraic sum of (14). So the second max-plus-algebraic sum in (14) is redundant and can be omitted.
Now we define the sequences $f_{i,j}$ for $i = 0, \ldots, Lc{+}c$ and $j = 1, 2$ with $(f_{1,i})_k = (h_1)_i \otimes (h_2)_{k-i}$ and $(f_{2,i})_k = (h_1)_{k-i} \otimes (h_2)_i$. The sequences $f_{i,j}$ are ultimately geometric with rate $\lambda_j$ and cyclicity $c_j$. As shown above, the terms of $h_1 \otimes h_2$ coincide with $\bigoplus_{i,j} f_{i,j}$ for $k$ large enough. Therefore, we can now apply Lemma 3.1 provided that we select $K$ such that $K \geqslant 2(Lc{+}c)$. ■

Note that in general we do not have $\gamma_s = \bigoplus_{i=1}^{m} \lambda_i$ for all indices $s$ in Lemma 3.1 as is shown by the following example.

**Example A.1** Consider the ultimately geometric sequences

$$\begin{aligned}
h_1 &= 0, \varepsilon, 1, 3, \varepsilon, 4, 6, \varepsilon, 7, 9, \varepsilon, 10, \ldots \\
h_2 &= 0, \varepsilon, 0, \varepsilon, 0, \varepsilon, 0, \varepsilon, 0, \varepsilon, 0, \varepsilon, \ldots
\end{aligned}$$

with $\lambda_1 = 1$, $c_1 = 3$, $\lambda_2 = 0$ and $c_2 = 2$. We have

$$h_1 \oplus h_2 = 0, \varepsilon, 1, 3, 0, 4, 6, \varepsilon, 7, 9, 0, 10, 12, \varepsilon, 13, 15, 0, 16, \ldots$$

This sequence is ultimately periodic with $c = \mathrm{lcm}(c_1, c_2) = \mathrm{lcm}(2, 3) = 6$. Furthermore, the smallest $\gamma_s$s for which (5) holds are $\gamma_0 = \gamma_2 = \gamma_3 = \gamma_5 = 1$, $\gamma_1 = \varepsilon$ and $\gamma_4 = 0$. Note that $\varepsilon = \gamma_1 \neq \lambda_1 \oplus \lambda_2 = 1 \oplus 0 = 1$ and $\varepsilon = \gamma_1 \notin \{\lambda_1, \lambda_2\} = \{1, 0\}$. □