Robust Stability

- Robust stability against time-invariant and time-varying uncertainties
- Parameter dependent Lyapunov functions
- Semi-infinite LMI problems
- From nominal to robust performance
Consider linear time-invariant system

\[ \dot{x}(t) = A(\delta)x(t) \]

where \( A(.) \) is a continuous function of the parameter vector

\[ \delta = (\delta_1, ..., \delta_p) \]

which is only known to be contained in uncertainty set

\[ \delta \subset \mathbb{R}^p. \]

Robust stability analysis

Is system asymptotically stable for all possible parameters \( \delta \in \delta \)?

Example: Mass- or load-variation of controlled mechanical system.
Example

Academic example with **rational** parameter-dependence:

\[
\dot{x} = \begin{pmatrix}
-1 & 2\delta_1 & 2 \\
\delta_2 & -2 & 1 \\
3 & -1 & \frac{\delta_3 - 10}{\delta_1 + 1}
\end{pmatrix} x.
\]

The parameters \(\delta_1, \delta_2, \delta_3\) are bounded as

\[
\delta_1 \in [-0.5, 1], \quad \delta_2 \in [-2, 1], \quad \delta_3 \in [-0.5, 2].
\]

Hence \(\delta\) is actually a polytope (box) with eight generators:

\[
\delta = [-0.5, 1] \times [-2, 1] \times [-0.5, 2] =
\]

\[
= \text{co} \left\{ \begin{pmatrix}
\delta_1 \\
\delta_2 \\
\delta_3
\end{pmatrix} : \delta_1 \in \{-0.5, 1\}, \quad \delta_2 \in \{-2, 1\}, \quad \delta_3 \in \{-0.5, 2\} \right\}.
\]
Relation to Optimization

Spectral abscissa of square matrix $A$: 
$$\rho_a(A) = \max_{\lambda \in \lambda(A)} \frac{1}{2}(\lambda + \bar{\lambda}).$$

**Obvious fact:** $A(\delta)$ is Hurwitz for all $\delta \in \delta$ if and only if
$$\rho_a(A(\delta)) < 0 \ \text{for all} \ \delta \in \delta.$$

Two main sources for trouble:
- $\rho_a(A(\delta))$ is far from convex/concave in the variable $\delta$.
- Inequality has to hold at infinitely many points.

Hence typically computational approaches fail:
- Cannot find global maximum of $\rho_a(A(\delta))$ over $\delta$.
- Even if $\delta$ is a polytope, not sufficient to check the generators.
- Even more trouble if $\delta$ is no polytope.
**Quadratic Stability**

The uncertain system \( \dot{x} = A(\delta)x \) with \( \delta \in \delta \) is defined to be **quadratically stable** if there exists \( X \succ 0 \) with

\[
A(\delta)^TX +XA(\delta) \prec 0 \quad \text{for all} \quad \delta \in \delta.
\]

**Why name?** \( V(x) = x^TXx \) is quadratic Lyapunov function.

**Why relevant?** Implies that \( A(\delta) \) is Hurwitz for all \( \delta \in \delta \).

**How to check?** Easy if \( A(\delta) \) is affine in \( \delta \) and \( \delta = \text{co}\{\delta^1, \ldots, \delta^N\} \) is a polytope with moderate number of generators: Verify whether

\[
X \succ 0, \quad A(\delta^k)^TX +XA(\delta^k) \prec 0, \quad k = 1, \ldots, N
\]

is feasible.
Example

If $A(\delta)$ is not affine in $\delta$, a parameter transformation often helps!

In example introduce $\delta_4 = \frac{\delta_3 - 10}{\delta_1 + 1} + 12$. Test quadratic stability of

$$
\begin{pmatrix}
-1 & 2\delta_1 & 2 \\
\delta_2 & -2 & 1 \\
3 & -1 & \delta_4 - 12
\end{pmatrix}, \quad (\delta_1, \delta_2, \delta_4) \in \delta = [-0.5, 1] \times [-2, 1] \times [-9, 8].
$$

LMI-Toolbox: System quadratically stable for

$$
(\delta_1, \delta_2, \delta_4) \in r\delta \quad \text{with largest possible factor} \quad r \approx 0.45.
$$

Quadratically stable for shrunk set $0.45\delta$. Not for $r\delta$ with $r > 0.45$.

The critical factor is called **quadratic stability margin**.
Time-Varying Parametric Uncertainties

Now assume that the parameters $\delta(t)$ vary with time, and that they are known to satisfy $\delta(t) \in \delta$ for all $t$. Check stability of

$$\dot{x}(t) = A(\delta(t))x(t), \quad \delta(t) \in \delta.$$ 

The uncertain system with time-varying parametric uncertainties is exponentially stable if there exists $X \succ 0$ with

$$A(\delta)^T X + XA(\delta) \prec 0 \quad \text{for all } \delta \in \delta.$$

Proof will be given for a more general result in full detail.

Therefore quadratic stability does in fact imply robust stability for arbitrary fast time-varying parametric uncertainty. If bounds on velocity of parameters are known, this test is conservative.
Rate-Bounded Parametric Uncertainties

Let us hence assume that the parameter-curves $\delta(.)$ are continuously differentiable and are only known to satisfy

$$\delta(t) \in \delta \text{ and } \dot{\delta}(t) \in v \text{ for all time instances.}$$

Here $\delta \subset \mathbb{R}^p$ and $v \subset \mathbb{R}^p$ are given compact sets (e.g. polytopes).

**Robust stability analysis**

Verify whether the linear time-varying system

$$\dot{x}(t) = A(\delta(t))x(t)$$

is exponentially stable for all parameter-curves $\delta(.)$ that satisfy the above described bounds on value and velocity.

Search for suitable Lyapunov function.
Main Stability Result

Theorem
Suppose $X(\delta)$ is continuously differentiable on $\delta$ and satisfies

$$X(\delta) \succ 0, \quad \sum_{k=1}^{p} \partial_k X(\delta) v_k + A(\delta)^T X(\delta) + X(\delta) A(\delta) \prec 0$$

for all $\delta \in \delta$ and $v \in v$. Then there exist constants $K > 0$, $a > 0$ such that all trajectories of the uncertain time-varying system satisfy

$$\|x(t)\| \leq Ke^{-a(t-t_0)}\|x(t_0)\| \quad \text{for all } t \geq t_0.$$

- Covers many tests in literature. Study the proof to derive variants!
- Condition is in general sufficient only!
- Is also necessary in case that $v = \{0\}$: Time-invariant uncertainty.
Proof

Continuity & compactness ... exist $\alpha, \beta, \gamma > 0$ such that for all $\delta \in \delta$, $v \in v$:

$$\alpha I \preceq X(\delta) \preceq \beta I, \quad \sum_{k=1}^{p} \partial_k X(\delta) v_k + A(\delta)^T X(\delta) + X(\delta) A(\delta) \preceq -\gamma I.$$ 

Suppose that $\delta(t)$ is any admissible parameter-curve and let $x(t)$ denote a corresponding state-trajectory of the system. Here is the crucial point:

$$\frac{d}{dt} x(t)^T X(\delta(t)) x(t) = x(t)^T \left[ \sum_{k=1}^{p} \partial_k X(\delta(t)) \dot{\delta}_k(t) \right] x(t) +$$

$$+ x(t)^T \left[ A(\delta(t))^T X(\delta(t)) + X(\delta(t)) A(\delta(t)) \right] x(t).$$

Since $\delta(t) \in \delta$ and $\dot{\delta}(t) \in v$ we can hence conclude

$$\alpha \| x(t) \|^2 \leq x(t)^T X(\delta(t)) x(t) \leq \beta \| x(t) \|^2, \quad \frac{d}{dt} x(t)^T X(\delta(t)) x(t) \leq -\gamma \| x(t) \|^2.$$
Proof

The proof is now finished as the one for LTI systems given earlier.

Define $\xi(t) := x(t)^T X(\delta(t)) x(t)$ to infer that

$$\|x(t)\|^2 \leq \frac{1}{\alpha} \xi(t), \quad \xi(t) \leq \beta \|x(t)\|^2, \quad \dot{\xi}(t) \leq -\frac{\gamma}{\beta} \xi(t).$$

The latter inequality leads to

$$\xi(t) \leq \xi(t_0) e^{-\frac{\gamma}{\beta} (t-t_0)} \text{ for all } t \geq t_0.$$

With the former inequalities we infer

$$\|x(t)\|^2 \leq \frac{\beta}{\alpha} e^{-\frac{\gamma}{\beta} (t-t_0)} \|x(t_0)\|^2 \text{ for all } t \geq t_0.$$

Can choose $K = \sqrt{\beta/\alpha}$ and $a = \gamma/(2\beta)$. 
Extreme Cases

- Parameters are time-invariant: \( v = \{0\} \).

  Have to find \( X(\delta) \) satisfying
  \[
  X(\delta) \succ 0, \quad A(\delta)^T X(\delta) + X(\delta) A(\delta) \prec 0 \quad \text{for all} \quad \delta \in \delta.
  \]

- Parameters vary arbitrarily fast:

  Have to find parameter-independent \( X \) satisfying
  \[
  X \succ 0, \quad A(\delta)^T X + X A(\delta) \prec 0 \quad \text{for all} \quad \delta \in \delta.
  \]

  Is identical to quadratic stability test!

Can apply subsequently suggested numerical techniques in both cases!
Specializations

Have derived general results based on Lyapunov functions which still depend \textbf{quadratically} on the state (which is restrictive) but which allow for non-linear (smooth) dependence on the uncertain parameters.

Is pure algebraic test and does not involve system- or parameter-trajectory.

Not easy to apply:

- Have to find \textbf{function} satisfying partial differential LMI
- Have to make sure that inequality holds for all $\delta \in \delta$, $\nu \in \nu$.

Allows to easily derive specializations which are or can be implemented with LMI solvers. We just consider a couple of examples.
Example: Affine System - Affine Lyapunov Matrix

Suppose $A(\delta)$ depends affinely on parameters:

$$A(\delta) = A_0 + \delta_1 A_1 + \cdots + \delta_p A_p.$$ 

Parameter- and rate-constraints are boxes:

$$\delta = \{ \delta \in \mathbb{R}^p : \delta_k \in [\underline{\delta}_k, \bar{\delta}_k] \}, \quad \nu = \{ \nu \in \mathbb{R}^p : \nu_k \in [\underline{\nu}_k, \bar{\nu}_k] \}$$ 

These are the convex hulls of

$$\delta_g = \{ \delta \in \mathbb{R}^p : \delta_k \in \{\underline{\delta}_k, \bar{\delta}_k\} \}, \quad \nu_g = \{ \nu \in \mathbb{R}^p : \nu_k \in \{\underline{\nu}_k, \bar{\nu}_k\} \}$$ 

Search for affine parameter dependent $X(\delta)$:

$$X(\delta) = X_0 + \delta_1 X_1 + \cdots + \delta_p X_p \quad \text{and hence} \quad \partial_k X(\delta) = X_k.$$
Example: Affine System - Affine Lyapunov Matrix

With $\delta_0 = 1$ observe that

$$
\sum_{k=1}^{p} \partial_k X(\delta)v_k + A(\delta)^T X(\delta) + X(\delta) A(\delta) = \\
= \sum_{k=1}^{p} X_k v_k + \sum_{\nu=0}^{p} \sum_{\mu=0}^{p} \delta_\nu \delta_\mu (A^T_\nu X_\mu + X_\mu A_\nu).
$$

Is affine in $X_1, \ldots, X_p$ and $v_1, \ldots, v_p$ but quadratic in $\delta_1, \ldots, \delta_p$.

**Relaxation:** Include additional constraint $A^T_\nu X_\nu + X_\nu A_\nu \succeq 0$.

- Implies that it suffices to guarantee required inequality at generators. Why? Partially convex function on box!
- Extra condition renders test stronger. **Still sufficient for RS!**
Partially Convex Function on Box

Suppose that $S \subset \mathbb{R}^n$. The Hermitian-valued function $F : S \rightarrow \mathbb{H}^m$ is said to be **partially convex** if the one-variable mapping

$$t \rightarrow F(x_1, \ldots, x_{l-1}, t, x_{l+1}, \ldots, x_n)$$

defined on $\{t \in \mathbb{R} : (x_1, \ldots, x_{l-1}, t, x_{l+1}, \ldots, x_n) \in S\}$

is convex for all $x \in S$ and $l = 1, \ldots, n$.

Suppose $S = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and that $F : S \rightarrow \mathbb{H}^m$ is partially convex. Then $F(x) \prec 0$ for all $x \in S$ if and only if

$$F(x) \prec 0 \quad \text{for all} \quad x \in \{a_1, b_1\} \times \cdots \times \{a_n, b_n\}.$$

Proof is simple exercise. Various variants suggested in literature.
Example: Affine System - Affine Lyapunov Matrix

Robust exponential stability guaranteed if

There exist $X_0, \ldots, X_p$ with $A^T \nu X_\nu + X_\nu A_\nu \succeq 0$, $\nu = 1, \ldots, p$, and

$$
\sum_{k=0}^p X_k \delta_k \succ 0, \quad \sum_{k=1}^p X_k v_k + \sum_{\nu=0}^p \sum_{\mu=0}^p \delta_\nu \delta_\mu (A^T \nu X_\mu + X_\mu A_\nu) \prec 0
$$

for all $\delta \in \delta_g$ and $\nu \in \nu_g$ and $\delta_0 = 1$.

- This test is implemented in LMI-Toolbox.
  
  For rate-bounded uncertainties often much less conservative than quadratic stability test.

- Understand the arguments in proof to derive your own variants.
General Recipe to Reduce to Finite Dimensions

**Restrict** the search to a chosen finite-dimensional subspace.

For example choose scalar continuously differentiable basis functions $b_1(\delta), \ldots, b_N(\delta)$ and search for the coefficient matrices $X_1, \ldots, X_N$ in the expansion

$$X(\delta) = \sum_{\nu=1}^{N} X_{\nu} b_{\nu}(\delta)$$

with

$$\partial_k X(\delta) = \sum_{\nu=1}^{N} X_{\nu} \partial_k b_{\nu}(\delta).$$

Have to guarantee for all $\delta \in \delta$ and $\nu \in \nu$ that

$$\sum_{\nu=1}^{N} X_{\nu} b_{\nu}(\delta) \succ 0, \quad \sum_{\nu=1}^{N} \left( \sum_{k=1}^{p} X_{\nu} \partial_k b_{\nu}(\delta) v_k + [A(\delta)^T X_{\nu} + X_{\nu} A(\delta)] b_{\nu}(\delta) \right) \prec 0.$$

Is finite dimensional but still semi-infinite LMI problem!
Remarks

• If systematically extending the set of basis functions one can improve the sufficient stability conditions. Example: **Polynomial basis**

\[ b_{(k_1,...,k_p)}(\delta) = \delta_1^{k_1} \cdots \delta_p^{k_p}, \quad k_\nu = 0, 1, 2, \ldots, \quad \nu = 1, \ldots, p. \]

If \( \delta \) is star-shaped one can prove: If the partial differential LMI has a continuously differentiable solution then it also has a polynomial solution (of possibly high degree).

Polynomial basis is a generic choice with guaranteed success.

• One can just grid \( \delta \) and \( \nu \) to arrive at **finite** system of LMI’s.

**Trouble:** Huge LMI system. No guarantees at points not in grid.
Semi-Infinite LMI-Constraints

Nominal stability could be reduced to an LMI feasibility test: Does there exist a solution \( x \in \mathbb{R}^n \) of some LMI

\[
F_0 + x_1 F_1 + \cdots + x_n F_n \prec 0.
\]

We have now seen that testing robust stability can be reduced to the following question: Does there exist some \( x \in \mathbb{R}^n \) with

\[
F_0(\delta) + x_1 F_1(\delta) + \cdots + x_n F_n(\delta) \prec 0 \quad \text{for all } \delta \in \delta
\]

where \( F_0(\delta), \ldots, F_n(\delta) \) are Hermitian-valued functions of \( \delta \in \delta \).

Is a generic formulation for the **robust counterpart** of an LMI feasibility test in which the data matrices are affected by uncertainties.
From Nominal to Robust Performance

- Recall how we obtained from nominal stability characterizations the corresponding robust stability tests against time-varying rate-bounded parametric uncertainties.

- As a major beauty of the dissipation approach, this generalization works without any technical delicacies for performance as well!

- Only for time-invariant uncertainties the frequency-domain characterizations make sense. For time-varying uncertainties (and time-varying systems) we have to rely on the time-domain interpretations.

We provide an illustration for quadratic performance!
Robust Quadratic Performance

**Uncertain** input-output system described as

\[
\begin{align*}
\dot{x}(t) &= A(\delta(t))x(t) + B(\delta(t))w(t) \\
z(t) &= C(\delta(t))x(t) + D(\delta(t))w(t).
\end{align*}
\]

with continuously differential parameter-curves \(\delta(.)\) that satisfy

\[
\delta(t) \in \delta \quad \text{and} \quad \dot{\delta}(t) \in v \quad (\delta, \ v \subset \mathbb{R}^p \ \text{compact}).
\]

**Robust quadratic performance:** Exponential stability and existence of \(\epsilon > 0\) such that for \(x(0) = 0\), for all parameter curves and for all trajectories

\[
\int_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T \begin{pmatrix} Q_p & S_p \\
S_p^T & R_p \end{pmatrix} \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} \ dt \leq -\epsilon \|w\|_2^2.
\]

\(L_2\)-gain, passivity, \ldots
Characterization of Robust Quadratic Performance

Suppose there exists a continuously differentiable Hermitian-valued $X(\delta)$ such that $X(\delta) \succ 0$ and

$$
\begin{pmatrix}
\sum_{k=1}^{p} \partial_k X(\delta)v_k + A(\delta)^T X(\delta) + X(\delta)A(\delta) & X(\delta)B(\delta) \\
B(\delta)^T X(\delta) & 0
\end{pmatrix} + 
\begin{pmatrix}
0 & I \\
C(\delta) & D(\delta)
\end{pmatrix}^T P_p 
\begin{pmatrix}
0 & I \\
C(\delta) & D(\delta)
\end{pmatrix} \prec 0
$$

for all $\delta \in \delta$, $v \in v$. Then the uncertain system satisfies the robust quadratic performance specification.

Numerical search for $X(\delta)$: Same as for stability!
Extends to other LMI performance specifications!
Sketch of Proof

**Exponential stability:** Left-upper block is

\[
\sum_{k=1}^{p} \partial_k X(\delta)v_k + A(\delta)^T X(\delta) + X(\delta)A(\delta) + C(\delta)^T R_p C(\delta) \preceq 0.
\]

Can hence just apply our general result on robust exponential stability.

**Performance:** Add to right-lower block \(\epsilon I\) for some small \(\epsilon > 0\) (compactness). Left- and right-multiply inequality with \(\text{col}(x(t), w(t))\) to infer

\[
\frac{d}{dt} x(t)^T X(\delta(t)) x(t) + \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T P_p \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} + \epsilon w(t)^T w(t) \leq 0.
\]

Integrate on \([0, T]\) and use \(x(0) = 0\) to obtain

\[
x(T)^T X(\delta(T)) x(T) + \int_0^T \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T P_p \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt \leq -\epsilon \int_0^T w(t)^T w(t) dt.
\]

Take limit \(T \to \infty\) to arrive at required quadratic performance inequality.