LINEAR PARAMETER-VARYING CONTROLLER SYNTHESIS USING MATRIX SUM-OF-SQUARES RELAXATIONS

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Abstract— In this paper we provide a new simplified formulation for recently developed matrix sum-of-squares relaxations and prove their asymptotic exactness. It is shown how these techniques can be applied to gain-scheduled controller synthesis for systems which depend on a time-varying parameter. An academic example illustrates the benefits if compared to alternative relaxation schemes.

Keywords— Linear parameter-varying controller synthesis, matrix sum-of-squares relaxations, parameter-dependent linear matrix inequalities

1 Introduction

Control theory of linear parameter-varying (LPV) systems was originally motivated by gain scheduling methods (Rugh and Shamma, 2000) and has been applied to practical problems in e.g. (Trangbaek, 2001; Groot Wassink et al., 2005). An LPV model reads as

$$\begin{aligned} \dot{x} &= A(\delta(t))x + B_1(\delta(t))w + B_2(\delta(t))u \\ z &= C_1(\delta(t))x + D_{11}(\delta(t))w + D_{12}(\delta(t))u \\ y &= C_2(\delta(t))x + D_{21}(\delta(t))w \end{aligned} \tag{1}$$

where $\delta(t)$ is a time-varying parameter. The size and the rate-of-variation of admissible parameter trajectories are assumed to satisfy $(\delta(t), \dot{\delta}(t)) \in \mathcal{R}$ for some compact set $\mathcal{R}$. LPV synthesis amounts to designing $A_k(.), B_k(.), C_k(.), D_k(.)$ such that the controller

$$\begin{aligned} \dot{x}_k &= A_k(\delta(t), \dot{\delta}(t))x_k + B_k(\delta(t))y \\ u &= C_k(\delta(t))x_k + D_k(\delta(t))y \end{aligned} \tag{2}$$

implemented with on-line measured $(\delta(t), \dot{\delta}(t))$ guarantees closed-loop stability and performance for all admissible parameter trajectories.

Existing LPV synthesis results are variations and extensions of two principally different approaches. Developed by (Packard, 1994) and (Apkarian and Gahinet, 1995), the first method based on so-called D-scalings assumes a plant description in the form of a linear fractional representation and constructs a parameter-dependent controller of the same form. Somewhat later, conservatism could be reduced by applying more general classes of scalings or multipliers (Helmersson, 1995; Scorletti and El Ghaoui, 1995; Scherer, 2001; Wu, 2001).

Our contribution relates to the second approach to LPV control (Apkarian and Adams, 1998) which is based on a rather straightforward application of the controller parameter transformation developed for linear time-invariant output feedback synthesis problems (Masubuchi et al., 1998; Scherer et al., 1997). Due to the parameter-dependence of the system and controller descriptions, the resulting linear matrix inequalities (LMIs) will turn into so-called parameter-dependent linear matrix inequalities (PLMIs).

One approach to computationally handle such PLMIs is to just grid the parameter space. However, this could be unreliable since neither stability nor performance can be guaranteed over the whole parameter region. Therefore it is preferable to construct so-called relaxations of PLMIs, which are standard LMI problems whose feasibility guarantees the validity of the original semi-infinite PLMIs. It is particularly interesting to construct families of relaxations of increasing complexity that allow for a gradual reduction of conservatism.

If the parameter-dependence is affine and the parameter region is a polytope or a box, it is easy to see that a PLMI is equivalent to the LMIs obtained at the vertices of the parameter region. The more difficult case of PLMIs that depend quadratically on the parameters has been extensively studied, in particular in the field of hybrid systems, and the key arguments used to construct relaxations are based on multi-convexity (Gahinet et al., 1996; Tuan and Apkarian, 1998; Apkarian and Tuan, 2000).

In this paper we suggest new formulations of so-called matrix sum-of-squares (SOS) relaxations for LPV controller synthesis (Scherer and Hol, 2006). The paper is organized as follows. Section 2 recalls one version of the PLMIs used for LPV controller synthesis. In Section 3 we propose matrix SOS relaxations and discuss their asymptotic exactness, while Section 4 shows how to translate these relaxations into computationally tractable LMI problems. A numerical example in Section 5 reveals the benefits of our approach.

2 LPV Synthesis

Stability and a bound on the L_2-gain of the controlled system (1)-(2) is guaranteed with a Lyapunov function that is quadratic in the state and depends on the parameter δ. A suitable extension of the bounded real lemma for the closed-loop system (Wu et al., 1996; Wu and Dong, 2006) leads to linear matrix inequalities which depend on the frozen parameters $(\delta, \dot{\delta}) \in \mathcal{R}$. It is then possible to eliminate the controller matrices $A_k(.), B_k(.), C_k(.), D_k(.)$ under the following hypotheses:

- The triple $(A(\delta), B_2(\delta), C_2(\delta))$ is stabilizable and detectable for all $(\delta, \dot{\delta}) \in \mathcal{R}$
- $[B_2^T(\delta)\ D_{12}^T(\delta)]$ and $[C_2(\delta)\ D_{21}(\delta)]$ have full row rank for all $(\delta, \dot{\delta}) \in \mathcal{R}$.

For a smooth matrix-valued function $P(\delta)$ let us define

$$\dot{P}(\delta, \dot{\delta}) = \sum_i \frac{\partial P}{\partial \delta_i}(\delta)\dot{\delta}_i.$$

It can then be shown that there exists an LPV controller which renders the closed-loop L_2-gain smaller than γ if there exist smooth symmetric matrix functions $R(\delta)$ and $S(\delta)$ which satisfy

$$U_R(\delta)^T \left[\begin{array}{cc|cc} 0 & R(\delta) & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ R(\delta) & -\dot{R}(\dot{\delta}, \delta) & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \gamma^{-1} I & 0 \\ & & 0 & -\gamma I \end{array}\right] U_R(\delta) \prec 0 \tag{3}$$

$$U_S(\delta)^T \left[\begin{array}{cc|cc} 0 & S(\delta) & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ S(\delta) & \dot{S}(\dot{\delta}, \delta) & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \gamma^{-1} I & 0 \\ & & 0 & -\gamma I \end{array}\right] U_S(\delta) \prec 0 \tag{4}$$

$$\begin{pmatrix} R(\delta) & I \\ I & S(\delta) \end{pmatrix} \succ 0 \tag{5}$$

for all $(\delta, \dot{\delta}) \in \mathcal{R}$. Here we use the abbreviations

$$U_R(\delta) = \left[\begin{array}{cc} A^T(\delta) & C_1^T(\delta) \\ I & 0 \\ \hline B_1^T(\delta) & D_{11}^T(\delta) \\ 0 & I \end{array}\right] \mathcal{N}_R(\delta)$$

$$U_S(\delta) = \left[\begin{array}{cc} A(\delta) & B_1(\delta) \\ I & 0 \\ \hline C_1(\delta) & D_{11}(\delta) \\ 0 & I \end{array}\right] \mathcal{N}_S(\delta)$$

where $\mathcal{N}_R(\delta), \mathcal{N}_S(\delta)$ are basis matrices for the kernels of $[B_2^T(\delta)\ D_{12}^T(\delta)]$ and $[C_2(\delta)\ D_{21}(\delta)]$ respectively. For computational tractability we restrict the search of $R(\delta)$ and $S(\delta)$ to some finite-dimensional subspace. In fact these matrices are parameterized as

$$R(\delta) = T_R(\delta)^T P T_R(\delta), \quad S(\delta) = T_S(\delta)^T Q T_S(\delta)$$

with some fixed polynomial (or rational) matrices $T_R(\delta)$, $T_S(\delta)$ and the to-be-computed symmetric coefficient matrices P, Q. After substituting $R(\delta)$, $S(\delta)$ into (3)-(5) and employing a Schur-complement argument to linearize with respect to γ, these three inequalities admit the natural structure

$$\tilde{U}_i(\delta, \dot{\delta})^T \tilde{F}_i(\gamma, P, Q) \tilde{U}_i(\delta, \dot{\delta}) \prec 0, \quad i = 1, 2, 3 \tag{6}$$

where the inner matrix depends affinely on the decision variables $y = (\gamma, P, Q)$ and only the outer factor is affected by the parameters $x = (\delta, \dot{\delta}) \in \mathcal{R}$. If the system matrices in (1) and $T_R(\delta)$, $T_S(\delta)$ happen to be rational in δ, the columns of the outer factors can be multiplied with scalar polynomials in order to render them polynomial (by performing a congruence transformation on (6)). The problem of minimizing the L_2-gain of the closed-loop LPV system then amounts to minimizing γ subject to the PLMIs (6) for all $(\delta, \dot{\delta}) \in \mathcal{R}$, where the outer factors in (6) are polynomial in $(\delta, \dot{\delta})$. In contrast to many other suggestions, we allow for correlation in the constraints imposed on δ and $\dot{\delta}$, and just assume that $\mathcal{R}$ is described by the solution set of a general polynomial matrix inequality. Any such inequality can be written as $V(\delta, \dot{\delta})^T \tilde{G} V(\delta, \dot{\delta}) \preceq 0$ with a polynomial matrix $V(\delta, \dot{\delta})$ and some symmetric $\tilde{G}$.

In summary, the LPV-synthesis problem has been subsumed to the following general PLMI optimization problem: Minimize $c^T y$ over all y such that

$$U(x)^T \tilde{F}(y) U(x) \succ 0$$
$$\text{for all } x \text{ satisfying } V(x)^T \tilde{G} V(x) \preceq 0.$$

Here $\tilde{F}(y)$ is symmetric-valued and affine in y, the matrix $\tilde{G}$ is symmetric, and $U(x)$, $V(x)$ are matrices with polynomial entries. In the sequel let us use the abbreviations

$$F(x, y) = U(x)^T \tilde{F}(y) U(x), \;\; G(x) = V(x)^T \tilde{G} V(x).$$

3 Matrix SOS Relaxations

A multivariate polynomial matrix $P(x)$ in indeterminants $x = (x_1, \ldots, x_s)$ is called matrix SOS if there exists some (typically tall) polynomial matrix $S(x)$ such that

$$P(x) = S(x)^T S(x).$$

As motivated in Section 2, we consider the following polynomial PLMI problem with optimal value γ_{opt}:

$$\begin{array}{l} \text{infimize } c^T y \\ \text{subject to} \\ \qquad F(x, y) \succ 0 \text{ for all } x \text{ with } G(x) \preceq 0. \end{array} \tag{7}$$

Suppose $F(x,y)$ and $G(x)$ are of dimension $p \times p$ and $q \times q$ respectively. With polynomial matrices $T_1(x), \ldots, T_M(x)$ of dimension $q \times p$ let us consider

$$\begin{aligned} &\text{infimize } c^T y \\ &\text{such that } \; F(x,y) - \epsilon I + \sum_{k=1}^{M} T_k(x)^T G(x) T_k(x) \\ &\qquad\qquad \text{is SOS in } x \text{ and } \epsilon > 0. \end{aligned} \tag{8}$$

As the main goal of this section, let us prove that (8) defines an upper bound on $\gamma_{\rm opt}$, and that (8) is guaranteed to approximate $\gamma_{\rm opt}$ arbitrarily well if increasing the number M as well as the degrees of the multipliers $T_1(x), \ldots, T_M(x)$.

Theorem 1 *The optimal value of (8) is not smaller than γ_{opt}. Suppose that there exists an $r > 0$ and some SOS matrix $R(x)$ such that*

$$r - \|x\|^2 + trace(R(x)G(x)) \;\; \text{is SOS.} \tag{9}$$

Then for all $\varepsilon > 0$ there exists some M and polynomial matrices $T_1(x), \ldots, T_M(x)$ for which the optimal value of (8) is smaller than $\gamma_{opt} + \varepsilon$.

Proof: The first statement is elementary to prove. Indeed suppose that $\hat{y}$ is feasible for (8). This implies that

$$F(x,\hat{y}) - \epsilon I \succeq - \sum_{k=1}^{M} T_k(x)^T G(x) T_k(x) \succeq 0$$

for all x since SOS matrices are globally nonnegative semi-definite. If we now choose an arbitrary x for which $G(x) \preceq 0$, we infer

$$- \sum_{k=1}^{M} T_k(x)^T G(x) T_k(x) \succeq 0$$

and hence $F(x,\hat{y}) \succeq \epsilon I$. Since $\epsilon > 0$, this reveals that $\hat{y}$ is feasible for (7). We have shown that the feasible set of (8) is contained in that of (7), which indeed implies that the optimal value of (8) is not smaller than that of (7).

The proof for the second statement is somewhat more involved and strongly resembles that in (Scherer and Hol, 2006). Given $\varepsilon > 0$, choose some $\hat{y}$ which is feasible for (7) and which satisfies $c^T\hat{y} < \gamma_{\rm opt} + \varepsilon$. This implies $F(x,\hat{y}) \succ 0$ for all x with $G(x) \preceq 0$. As shown in (Scherer and Hol, 2006), there exist unit vectors $v_1, \ldots, v_{N_0}$ such that

$$v_i^T G(x) v_i \leq 0, \;\; i = 1, \ldots, N_0 \;\; \Rightarrow \;\; F(x,\hat{y}) \succ 0.$$

Hence, by an extension of Putinar's scalar representation result (Putinar, 1993) to matrix-valued polynomials (Scherer and Hol, 2006) we infer that there exist SOS matrices $S_1(x), \ldots, S_{N_0}(x)$ and $\epsilon > 0$ for which

$$F(x,\hat{y}) - \epsilon I + \sum_{i=1}^{N_0} S_i(x) v_i^T G(x) v_i \;\; \text{is SOS.}$$

As an SOS matrix, $S_i(x)$ can be written as $S_i(x) = \sum_{j=1}^{r_i} t_j^i(x)(t_j^i(x))^T$ with polynomial column vectors $t_j^i(x)$. This implies that

$$F(x,\hat{y}) - \epsilon I + \sum_{i=1}^{N_0} S_i(x) v_i^T G(x) v_i =$$

$$= F(x,\hat{y}) - \epsilon I + \sum_{i=1}^{N_0} \sum_{j=1}^{r_i} t_j^i(x) v_i^T G(x) v_i (t_j^i(x))^T$$

is SOS. With the $M = r_1 + \ldots + r_{N_0}$ rank-one polynomial matrices $v_i(t_j^i(x))^T$, $i = 1, \ldots, N_0$, $j = 1, \ldots, r_i$ (whose degrees are determined from those of $S_1(x), \ldots S_{N_0}(x)$), we have proved that $\hat{y}$ is feasible for (8). Therefore the optimal value of (8) is not larger than $c^T\hat{y}$ which is in turn smaller than $\gamma_{\rm opt} + \varepsilon$. □

In (Wu and Prajna, 2005), SOS techniques have already been applied to LPV controller synthesis. In that paper, the PLMI is first reduced to a scalar inequality after which more conventional scalar SOS arguments can be employed. More precisely, it is exploited that $F(x,y) \succ 0$ iff

$$z^T F(x,y) z > 0 \;\; \text{for all } z \text{ with } \;\; z^T z = 1, \tag{10}$$

which introduces the variable $z = (z_1, \ldots, z_p)$ with p components. The Positivstellensatz allows to determine relaxations for the scalar inequality in (10) with respect to the variables (x,z) which are as well asymptotically exact. As argued in (Scherer and Hol, 2006), the matrix SOS approach can be interpreted as follows: the constructed relaxations remain asymptotically exact even if restricting the involved polynomials to be of homogenous degree two in the variable z. This key feature will be exploited in building the LMI relaxations without any need for scalarization in Section 4. Moreover, it is particularly relevant for LPV-synthesis problems in which the number of auxiliary variables $(z_1, \ldots, z_p)$ is proportional to the state-dimension of (1). Therefore matrix SOS techniques are expected to be favorable for analysis and synthesis of LPV-systems with a somewhat larger state-dimension.

4 Translation of Relaxations into LMIs

Let us denote by $\Pi_d^{p \times q}$ the space of $p \times q$ matrices with polynomial entries having a total degree of at most d. This section serves to show how one computes the value of (8) by solving a semi-definite program. If we intend to verify whether a polynomial matrix $P(x)$ is SOS, we represent it as $P(x) = W(x)^T \tilde{P} W(x)$ with a symmetric $\tilde{P}$ and some monomial matrix $W(x)$. One can then easily compute a basis of the subspace $\mathcal{K}$ of all symmetric matrices K for which

$$W(x)^T K W(x) \;\; \text{is the zero polynomial matrix.} \tag{11}$$

Then $P(x)$ is SOS if there exists some $K \in \mathcal{K}$ for which $\tilde{P} + K \succeq 0$, which is a standard LMI feasibility problem. If $P(x)$ has dimension $p \times p$ and a total degree of at most $2d$, it suffices to choose $W(x)$ as a tall monomial matrix whose $\mathbb{R}$-left-span equals $\Pi_d^{p\times p}$ in order to guarantee the converse: If $P(x)$ is SOS then $\tilde{P} + K \succeq 0$ is feasible. Newton-polytope techniques (Sturmfels, 1998) allow a priori reductions of the size of $W(x)$ for improved computational complexity.

Let us now consider (8). We choose a bound e on the total degree of $T_1(x), \ldots, T_M(x)$ and parameterize these matrices with a monomial basis $B_1(x), \ldots B_N(x)$ of $\Pi_e^{q\times p}$ as

$$T_k(x) = \sum_{\nu=1}^{N} \alpha_\nu^k B_\nu(x). \tag{12}$$

If we make use of the description $F(x,y) = U(x)^T \tilde{F}(y) U(x)$ and $G(x) = V(x)^T \tilde{G} V(x)$ from Section 2, the crucial constraint in (8) reads as follows:

$$U(x)^T \tilde{F}(y) U(x) - \epsilon I + \sum_{k=1}^{M} T_k(x)^T V(x)^T \tilde{G} V(x) T_k(x) \ \text{ is SOS in } x. \tag{13}$$

With X defined as

$$X = \sum_{k=1}^{M} \begin{pmatrix} \alpha_1^k \\ \vdots \\ \alpha_N^k \end{pmatrix} \begin{pmatrix} \alpha_1^k \\ \vdots \\ \alpha_N^k \end{pmatrix}^T \succeq 0, \tag{14}$$

we infer

$$\sum_{k=1}^{M} T_k(x)^T V(x)^T \tilde{G} V(x) T_k(x) =$$
$$= \sum_{\substack{\nu=1\\ \mu=1}}^{N} \left[B_\nu(x)^T V(x)^T \sum_{k=1}^{M} (\alpha_\nu^k \alpha_\mu^k) \tilde{G} V(x) B_\mu(x) \right] =$$
$$= \begin{pmatrix} V(x)B_1(x) \\ \vdots \\ V(x)B_N(x) \end{pmatrix}^T [X \otimes \tilde{G}] \begin{pmatrix} V(x)B_1(x) \\ \vdots \\ V(x)B_N(x) \end{pmatrix}.$$

Let us denote by d the maximum of the total degrees of $U(x), V(x)B_1(x), \ldots, V(x)B_N(x)$, and let us choose a tall monomial matrix $W(x)$ whose $\mathbb{R}$-left-span is $\Pi_d^{p\times p}$. Then there exist $L_0 = L_0^T$, L_U and L_V such that

$$I = W^T(x) L_0 W(x), \quad U(x) = L_U W(x), \quad \begin{pmatrix} V(x)B_1(x) \\ \vdots \\ V(x)B_N(x) \end{pmatrix} = L_V W(x). \tag{15}$$

Therefore the matrix involved in (13) equals

$$W(x)^T \left(L_U^T \tilde{F}(y) L_U - \epsilon L_0 + L_V^T [X \otimes \tilde{G}] L_V \right) W(x). \tag{16}$$

Let us finally define again the subspace $\mathcal{K}$ of all K satisfying (11). Then the existence of solutions $X \succeq 0$ and $K \in \mathcal{K}$ of the LMI

$$L_U^T \tilde{F}(y) L_U - \epsilon L_0 + L_V^T [X \otimes \tilde{G}] L_V + K \succeq 0 \tag{17}$$

implies that there exist some M and coefficients α_ν^k (obtained by a Cholesky factorization of X according to (14)) such that (12)-(13) hold true.

Remark 2 *Observe that the dimension of X is determined by the length N of the basis $B_1(x), \ldots, B_N(x)$, while the number of terms M appearing in (13) equals the rank of X. It is hence guaranteed that M is bounded by N, while it might be smaller if a lower-rank feasible X is computed.*

We arrive at the following LMI-relaxation of (7):

$$\begin{aligned} &\text{infimize } c^T y \\ &\text{subject to } \ \epsilon > 0, \quad X \succeq 0, \quad K \in \mathcal{K}, \\ &L_U^T \tilde{F}(y) L_U - \epsilon L_0 + L_V^T [X \otimes \tilde{G}] L_V + K \succeq 0. \end{aligned}$$

Due to Theorem 1 its solution provides an upper bound on the value γ_{opt} of the original problem (7). Moreover if the total degree e of (12) increases, the relaxation value decreases and can lead to improved upper bounds. Most importantly, if $G(x)$ satisfies the constraint qualification appearing in Theorem 1, the nontrivial part of this result guarantees that the value of the relaxations converges to γ_{opt} for $e \to \infty$.

Remark 3 *Let us stress again that $W(x)$ can be chosen of smaller size by exploiting the concept of Newton-polytopes (Sturmfels, 1998). We also note that we can choose* arbitrary *polynomial matrices $B_1(x), \ldots, B_N(x)$ and* any *monomial matrix $W(x)$ for which the representations (15) are possible, while still guaranteeing that the constructed LMI-relaxation allows to compute an upper bound of γ_{opt}. Unfortunately, it is unknown how to systematically pick $B_1(x), \ldots, B_N(x)$ in order to obtain good-quality relaxations.*

Remark 4 *If compared to the LMI-relaxations in (Scherer and Hol, 2006), the presented approach is conceptually simpler and leads to semi-definite programs without any affine equation constraints.*

5 Example

Consider the following LPV system

$$\begin{pmatrix} \dot{x} \\ \hline z_u \\ \hline z_p \\ \hline y \end{pmatrix} = M \begin{pmatrix} x \\ \hline w_u \\ \hline w_p \\ \hline u \end{pmatrix} \tag{18}$$

in feedback with

$$w_u = \left(\begin{array}{cc|cc} \delta_1(t) & 0 & 0 & 0 \\ 0 & \delta_1(t) & 0 & 0 \\ \hline 0 & 0 & \delta_2(t) & 0 \\ 0 & 0 & 0 & \delta_2(t) \end{array}\right) z_u \qquad (19)$$

where

$$M = \left(\begin{array}{c|cccc|c|c} -1 & 1 & 1 & 1 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 1 & 0 \\ 0 & 2a & 0 & a & 0 & 0 & 0 \\ 0 & 0 & -2a & 0 & -a & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad a \in [0.2,\ 1.2].$$

The time-varying parameters $\delta_1(t), \delta_2(t)$ are assumed to satisfy

$$|\delta_1(t)| \leq 0.9, \ |\delta_2(t)| \leq 0.9.$$

The goal is to find an LPV controller that stabilizes (18)-(19) and minimizes a bound γ on the $\mathcal{L}_2$-gain of the channel $w_p \to z_p$. For this purpose, the LPV synthesis conditions of Section 2 have been used and three different types of relaxations were computed. The first two are based on an S-procedure argument as in (Wu and Dong, 2006) after which the so-called convex-hull or multi-convexity relaxations as described in (Iwasaki and Shibata, 2001; Scherer, 2001) are applied. In the third type of relaxation we directly use the matrix SOS approach as developed in Sections 3-4 for the PLMI obtained from (3)-(5) and for three difference choices of the monomial bases $B_1(x), \ldots, B_N(x)$ of increasing complexity. The constraints on the parameters have been implemented as

$$V(\delta_i)^T \tilde{G} V_i(\delta_i) = \begin{pmatrix} 1 \\ \delta_i \end{pmatrix}^T \begin{pmatrix} -0.9^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \delta_i \end{pmatrix} \leq 0$$

for $i = 1, 2$. For ease of comparison of the different approaches, we plot the minimal achievable γ-level against the parameter a in Figure 1. The total number of required design variables for each technique is specified in Table 1. In contrast to results obtained in (Scherer, 2001), only the novel SOS relaxations allow to guarantee a finite $\mathcal{L}_2$-gain over the whole interval $a \in [0.2, 1.2]$. Moreover, SOS 1 requires less than half the number of variables if compared to the multiplier-based relaxation while still leading to considerable improvements.

Remark 5 *We emphasize that the suggested matrix SOS relaxation technique can be applied to any of the multitude of problems in robust control which can be translated into the generic formulation (7). In particular, different versions of LPV synthesis conditions can be chosen as a starting-point. For example, as pointed out in (Apkarian and Tuan, 2000), one can employ Finsler's Lemma to obtain PLMIs for synthesis which avoids the projection onto the null-spaces of* $[B_2^T(\delta)\ D_{12}^T(\delta)]$ *and* $[C_2(\delta)\ D_{21}(\delta)]$ *respectively.*

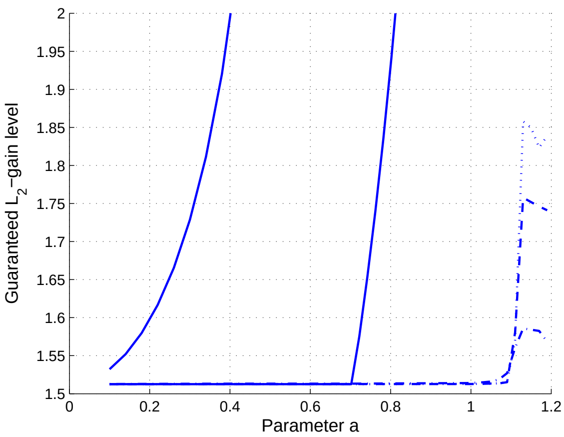

Figure 1: Guaranteed $\mathcal{L}_2$-gain levels for Convex Hull (left solid), Multiconvexity (right solid), SOS 1 (dotted), SOS 2 (dashed), SOS 3 (dash-dotted).

Relaxation	# variables
vertex (convexity)	275
vertex (multi-convexity)	275
SOS 1	135
SOS 2	215
SOS 3	835

Table 1: Number of relaxation variables.

Remark 6 *Approaches based on a plant description in the form of a linear fractional representation which lead to multiplier-based synthesis conditions (Iwasaki and Shibata, 2001; Scherer, 2001) are as well amenable to matrix SOS relaxations. If compared to convex-hull or multi-convexity based techniques, matrix SOS techniques offer a way to systematically reduce conservatism, with the extra benefit of allowing for far more general parameter regions than polytopes or boxes. Moreover, one can apply recently developed test for verifying whether the relaxation does not involve any conservatism at all. A tutorial exposition of these aspects in a unifying framework can be found in (Scherer, 2006).*

6 Conclusions

In this paper we suggest novel relaxations for linear parameter-varying controller synthesis using recently developed matrix sum-of-squares techniques. For generic parameter-dependent LMI problems as they appear in robust control it is shown how to construct families of relaxations which allow to reduce conservatism to an arbitrary degree. In an academic example we illustrate the merits of this new method in terms of improved approximation power and a reduction in computational cost.

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