Preprint: LMI Relaxations in Robust Control

C.W. Scherer*
Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft, The Netherlands
c.w.scherer@dcsc.tudelft.nl

Abstract

The purpose of this tutorial paper is to discuss the important role of robust linear matrix inequalities with rational dependence on uncertainties in robust control. We review how various classical relaxations based on the S-procedure can be subsumed to a unified framework. Based on Lagrange duality for semi-definite programs, we put particular emphasis on a clear understanding under which conditions such relaxations can be verified to be exact. We finally address the systematic construction of families of relaxations which can be shown to be asymptotically exact, based on recent results on the sum-of-squares representation of polynomial matrices.

Notation. $A \prec B$ ($A < B$) means that A, B are Hermitian (real) and that $A - B$ is negative definite (has only negative elements). $'$ denotes conjugation and transposition of matrices, while * is the Hilbert adjoint of linear mappings between inner product spaces, with inner products generically denoted as $\langle .,. \rangle$. $\mathbb{C}_0^\infty$ equals the extended imaginary axis. $\mathrm{diag}_{j=1}^n(A_j)$ denotes the diagonal matrix with blocks $A_1, \ldots, A_n$.

1 Introduction

During the past fifteen years, there has been a tremendous activity devoted to identifying control problems that can be translated into linear semi-definite programs (SDP's) or linear matrix inequalities (LMI's). With the decision variable $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, with real numbers $c_1, \ldots, c_n$ and Hermitian matrices $A_0, A_1, \ldots, A_n$ defining cost and constraints, these problems are formulated as

$$\begin{array}{rl} \text{infimize} & c_1x_1 + \cdots + c_nx_n \\ \text{subject to} & A_0 + A_1x_1 + \cdots + A_nx_n \prec 0. \end{array} \tag{1}$$

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This optimization problem differs from standard linear programs just by the interpretation of the inequality: $A_0 + A_1 x_1 + \cdots + A_n x_n$ is required to be negative definite. It is by now well-known that many classical and advanced problems in control, such as H_2- and H_∞-optimal controller synthesis, have been successfully subsumed - possibly with the help of some tricky transformations - to such formulations [11, 26].

In contrast to *nominal* stability or performance analysis on the basis of the nominal LMI formulation (1), questions in *robust* stability or performance analysis or synthesis lead to *robust* LMI problems with the generic formulation

$$\begin{array}{rl} \text{infimize} & c_1 x_1 + \cdots + c_n x_n \\ \text{subject to} & A_0(\delta) + A_1(\delta) x_1 + \cdots + A_n(\delta) x_n \prec 0 \ \text{ for all } \ \delta \in \boldsymbol{\delta}. \end{array} \tag{2}$$

Here δ is some real or complex uncertainty parameter, which is only known to be contained in the set $\boldsymbol{\delta}$, and which affects the data matrices $A_0(\delta), A_1(\delta), \ldots, A_n(\delta)$ in a nonlinear fashion.

This paper is meant as a tutorial on the emergence of robust linear matrix inequalities in which the data matrices are rational in the uncertainties. We consider rather classical techniques how to construct linear SDP's which allow to compute upper bounds on (2), and which are called relaxations. Based on a unified formulation and using Lagrange duality for semi-definite programs, we put particular emphasis on a clear understanding under which conditions such relaxations can be verified to be exact. Moreover, it will be pointed out how one can systematically construct families of relaxations, whose values can be shown to converge to that of (2). This asymptotic exactness relies on recent results about the sum-of-squares representation of positive definite polynomial matrices on sets described by polynomial matrix inequalities.

After a summarizing exposition of Lagrange duality for convex programs and linear matrix inequalities in Section 2, we address in Section 3 how classical frequency domain inequalities can be approached from the point of view of robust LMI's, with the frequency being the uncertain parameter. It will be shown how to find upper bound LMI relaxations for (2), and we discuss in quite some detail how to understand whether they are actually exact; this reveals under which conditions inherently non-convex frequency domain inequalities can indeed be equivalently reformulated as linear SDP's, with variants of the Kalman-Yakubovich-Popov (KYP) Lemma receiving the main attention. Section 4 serves to illustrate the construction of such relaxations, if δ consists of more than just one (frequency) variable, with a focus on applications in robust performance analysis for uncertain systems having a linear fractional representation. Section 5 indicates that dissipation theory lays ground for subsuming robust stability and performance analysis for time-varying uncertainties to the robust LMI framework. In this section, we address one of the few available results for a priori guarantees on the quality of a suitable relaxation for testing quadratic stability [8], and we show that this relaxation is actually equivalent to the standard (D, G)-scalings upper bound relaxation known from classical μ-theory. In the core Section 6 of this paper, we provide insights into computing lower bounds of (2) by uncertainty sampling, as well as upper bounds by generic relaxation families. We point out a novel relation of the corresponding Lagrange duals, which leads to algorithms for extracting worst-case uncertainties for (2). In the final Section 7, we address how a matrix extension of a classical theorem of Pólya or recent matrix-versions of sum-of-squares representations allow to systematically compose families of upper bound relaxations with guaranteed convergence to the optimal value of (2).

2 A glimpse of Lagrange duality theory

2.1 Problem formulation

In control, one is typically confronted with multiple matrix variables and, despite always possible, it is not advisable to transform such problems to the standard form (1) since this might destroy insightful structure. Let us hence turn to a slightly more abstract and flexible formulation, as used throughout this paper. We start with a rather general nonlinear SDP on the *real inner product spaces* $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ of possibly complex matrices with fixed dimension. It simplifies the exposition, if x, y, z appearing below are tacitly assumed to belong to $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ whenever not explicitly specified. Moreover, recall that the usual inner product in these spaces is defined by $\langle A, B\rangle = \mathrm{Re}(\mathrm{tr}(A'B))$, which simplifies to $\langle A, B\rangle = \mathrm{tr}(A^TB)$ if A, B are real.

The decision variable is $x \in \mathcal{X}$, subject to an abstract constraint $x \in \mathcal{S}$ with some subset $\mathcal{S} \subset \mathcal{X}$, and subject to two concrete inequality constraints

$$G_0 + G(x) \preceq 0 \quad \text{and} \quad H_0 + H(x) \leq 0.$$

The first matrix inequality is defined with a (possibly nonlinear) Hermitian-valued map $G : \mathcal{S} \to \mathcal{Y}$ and a fixed Hermitian matrix $G_0 \in \mathcal{Y}$, while the second elementwise constraint is described with a *real-linear* and real-valued mapping $H : \mathcal{X} \to \mathcal{Z}$ and some fixed real matrix $H_0 \in \mathcal{Z}$. With some cost-function $f : \mathcal{S} \to \mathbb{R}$, we consider the following nonlinear semi-definite program with optimal value p_{opt}:

$$\begin{array}{rl} \text{infimize} & f(x) \\ \text{subject to} & x \in \mathcal{S}, \quad G_0 + G(x) \preceq 0, \quad H_0 + H(x) \leq 0. \end{array} \tag{3}$$

Let us briefly motivate our choice of this seemingly particular but in fact very general description of a nonlinear SDP:

- In many practical circumstances $\mathcal{S}$ just equals $\mathcal{X}$.
- $G_0 + G(x) \preceq 0$ defines the genuine nonlinear constraints, which allows to combine standard scalar-valued nonlinear inequality constraints with Hermitian-valued semi-definite constraints by diagonal augmentation.

  The program is said to be *convex*, if $\mathcal{S}$ is convex and f, G are convex in the following (generalized) sense:

  $$\left.\begin{array}{r} f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) \\ G(\lambda x_1 + (1-\lambda)x_2) \preceq \lambda G(x_1) + (1-\lambda) G(x_2) \end{array}\right\} \text{ for all } \lambda \in [0,1], \quad x_1, x_2 \in \mathcal{S}.$$

  It is said to be an LMI-problem if $\mathcal{S} = \mathcal{X}$ and f, G are *real-linear* mappings.
- $H_0 + H(x) \leq 0$ comprises all scalar affine inequality or equation constraints as they appear in standard linear programming. In this fashion we can constrain $x \in \mathcal{S}$ to live in some affine manifold, or even in some polytope or (non-compact) polyhedron which is described by linear equations and/or inequalities

2.2 What is the Lagrange dual?

As usual, feasible points for minimization problems allow to determine upper bounds on $p_{\rm opt}$. A much less simple question is the determination of lower bounds on the optimal value, which is our starting point for the construction of the Lagrange dual problem. For this purpose, let us fix $Y \succeq 0$ and $Z \geq 0$. For all x that satisfy the constraints of (3), it is then trivial to conclude that

$$f(x) \geq f(x) + \langle Y, G_0 + G(x)\rangle + \langle Z, H_0 + H(x)\rangle =: L(x, Y, Z).$$

If we infimize over all x satisfying the constraints of (3) we infer

$$p_{\rm opt} \geq \inf_{x\in\mathcal{S},\, G_0+G(x)\preceq 0,\, H_0+H(x)\leq 0} L(x, Y, Z) \geq \inf_{x\in\mathcal{S}} L(x, Y, Z) =: l(Y, Z). \tag{4}$$

The lower bound $l(Y, Z)$ of $p_{\rm opt}$ can be improved by maximization and we infer

$$p_{\rm opt} \geq \sup_{Y\succeq 0,\, Z\geq 0} l(Y, Z). \tag{5}$$

In other words, by introducing the *Lagrange multipliers* Y, Z and defining the *Lagrangian* L, we have shown that the value of the optimization problem $\inf_{x\in\mathcal{S}} L(x, Y, Z)$, called *Lagrange dual cost* l, defines a lower bound on $p_{\rm opt}$, and that the best lower bound $d_{\rm opt}$ is obtained by solving the *Lagrange dual problem*

$$d_{\rm opt} = \sup_{Y\succeq 0,\, Z\geq 0} \left(\inf_{x\in\mathcal{S}} f(x) + \langle Y, G_0 + G(x)\rangle + \langle Z, H_0 + H(x)\rangle \right). \tag{6}$$

The general inequality $p_{\rm opt} \geq d_{\rm opt}$ is called *weak duality* and $p_{\rm opt} - d_{\rm opt}$ is the *duality gap*.

Apart from weak duality, it is of crucial interest to understand under which conditions the duality gap vanishes. This is obvious if $p_{\rm opt} = -\infty$, since weak duality then implies $d_{\rm opt} = -\infty$; this just means that

$$\inf_{x\in\mathcal{S}} f(x) + \langle Y, G_0 + G(x)\rangle + \langle Z, H_0 + H(x)\rangle = -\infty \quad \text{for all} \quad Y \succeq 0,\ Z \geq 0.$$

2.3 Lagrange duality theorem

In general, it cannot be expected that the duality gap vanishes. Therefore we continue under the hypothesis that (3) is a convex program (and hence $\mathcal{S}$ and f, G are convex). In the following fundamental and nontrivial result from convex analysis, we formulate an easy-to-handle condition on the constraint functions for guaranteeing the absence of a duality gap. We need to recall the following definition: $x_0 \in \mathcal{S}$ is a relative interior point of the convex set $\mathcal{S}$ if there exists some $\epsilon > 0$ such that

$$\{x \in \mathcal{L} :\ \|x - x_0\| < \epsilon\} \subset \mathcal{S};$$

here $\mathcal{L}$ is the smallest of all affine manifolds in $\mathcal{X}$ which satisfy $\mathcal{S} \subset \mathcal{L}$. If $\mathcal{L} = \mathcal{X}$ then relative interior points are the same as interior points, and if $\mathcal{S} = \mathcal{X}$ then all points in $\mathcal{S}$ are (relative) interior points.

Theorem 1 *Let the program (3) be convex and satisfy Slater's constraint qualification: There exists some relative interior point* $x_0 \in \mathcal{S}$ *with* $G_0 + G(x_0) \prec 0$, $H_0 + H(x_0) \leq 0$. *Then strong duality holds: There exist* $Y \succeq 0$, $Z \geq 0$ *with* $p_{opt} = l(Y, Z) = d_{opt}$.

The inequality $G_0 + G(x) \prec 0$ is often called the strict version of $G_0 + G(x) \preceq 0$, and x_0 as appearing in Slater's constraint qualification is said to satisfy the strict inequality, or to be a strictly feasible point. Note that the constraint qualification does not require the elementwise inequalities to be strict, which is the main motivation for our separation of the two types of inequalities.

Remark 2 *For convex programs which satisfy Slater's constraint qualification, the freedom of the decision variable can be restricted to the (strictly feasible) Slater points without changing the optimal value. Indeed, if* x *is feasible for (3) and* x_0 *is a Slater point, we infer*

$$\left.\begin{array}{r} x + \lambda(x_0 - x) = (1-\lambda)x + \lambda x_0 \text{ is in the relative interior of } \mathcal{S} \\ G(x + \lambda(x_0 - x)) + G_0 \preceq (1-\lambda)(G(x) + G_0) + \lambda(G(x_0) + G_0) \prec 0 \\ f(x + \lambda(x_0 - x)) \leq (1-\lambda)f(x) + \lambda f(x_0) \end{array}\right\} \text{ for all } \lambda \in (0, 1].$$

We can hence choose strictly feasible points arbitrarily close to x *(small* λ*), while keeping the increase of the cost as well arbitrarily small.*

Slater's constraint qualification implies feasibility and hence clearly $p_{\text{opt}} < \infty$. Moreover, both aspects of strong duality, the absence of a duality gap and the existence of an optimal solution of the Lagrange dual problem, can be explicitly and compactly expressed as

$$p_{\text{opt}} = \max_{Y \succeq 0,\, Z \geq 0} \; \inf_{x \in \mathcal{S}} f(x) + \langle Y, G_0 + G(x) \rangle + \langle Z, H_0 + H(x) \rangle = d_{\text{opt}}. \tag{7}$$

Let us finally emphasize that Theorem 1 does neither require the infimum to be attained, nor does it allow to draw any conclusion about this question. It is an often used trick to identify (3) as the Lagrange dual of some other convex program that is strictly feasible; then Theorem 1 allows to draw the conclusion that the value of (3) is attained.

2.4 Linear semi-definite programs

From now on let us consider a genuine LMI problem

$$\begin{array}{rl} \text{infimize} & \langle c, x \rangle \\ \text{subject to} & x \in \mathcal{X}, \quad G_0 + G(x) \preceq 0, \quad H_0 + H(x) \leq 0 \end{array} \tag{8}$$

with $c \in \mathcal{X}$ and $G : \mathcal{X} \to \mathcal{Y}$ being real-linear. Recall that the validity of Slater's constraint qualification just means strict feasibility. As the essential point, it is possible to compute the adjoints $G^* : \mathcal{Y} \to \mathcal{X}$ and $H^* : \mathcal{Z} \to \mathcal{X}$ which are defined by the requirement

$$\langle Y, G(x) \rangle = \langle G^*(Y), x \rangle, \quad \langle Z, H(x) \rangle = \langle H^*(Z), x \rangle \quad \text{for all} \quad x \in \mathcal{X},\ Y \in \mathcal{Y},\ Z \in \mathcal{Z}.$$

This allows to sort for the variable x in the Lagrangian as

$$L(x,Y,Z) = \langle c,x\rangle + \langle Y, G_0 + G(x)\rangle + \langle Z, H_0 + H(x)\rangle = \\ = \langle c + G^*(Y) + H^*(Z), x\rangle + \langle Y, G_0\rangle + \langle Z, H_0\rangle.$$

Therefore, (6) can be rephrased as

$$d_{\rm opt} = \sup_{Y\succeq 0,\, Z\geq 0,\, c+G^*(Y)+H^*(Z)=0} \langle Y, G_0\rangle + \langle Z, H_0\rangle, \tag{9}$$

if we just observe that $\inf_{x\in\mathcal{X}}\langle M,x\rangle > -\infty$ iff $M=0$. Linearity makes it hence possible to explicitly describe the Lagrange dual problem (cost and constraints) in terms of G^* and H^*. It is relevant to stress that the construction of the dual program always proceeds along these routine lines, even if the original primal problem is described differently.

Even without constraint qualification, $p_{\rm opt} = -\infty$ implies $d_{\rm opt} = -\infty$ which means that (9) is not feasible. Let us now assume that (8) is strictly feasible (Slater). Then strong duality holds. If $p_{\rm opt} > -\infty$, this means that (9) is feasible and its value $d_{\rm opt}$ is both attained and equal to $p_{\rm opt}$; in other words there exist dually optimal Lagrange multipliers Y, Z with

$$Y \succeq 0,\quad Z \geq 0,\quad c + G^*(Y) + H^*(Z) = 0,\quad \langle Y, G_0\rangle + \langle Z, H_0\rangle = p_{\rm opt}.$$

We can also conclude that $d_{\rm opt} = -\infty$ (infeasibility of (9)) implies $p_{\rm opt} = -\infty$.

2.5 Farkas alternative

What happens if Slater's constraint qualification does not hold? If E denotes the all-ones matrix, one can easily verify that (8) is not strictly feasible iff

$$0 \leq \inf_{G_0+G(x)\preceq tI,\, H_0+H(x)\leq tE} t. \tag{10}$$

Note that (10) is always strictly feasible, since $G_0 + G(x) - tI$ and $H_0 + H(x) - tE$ can be rendered negative definite and elementwise negative by choosing t sufficiently large. By strong duality, (10) holds iff there exist $Y \succeq 0$ and $Z \geq 0$ with

$$0 \leq \inf_{x,t} t + \langle Y, G_0 + G(x) - tI\rangle + \langle Z, H_0 + H(x) - tE\rangle = \\ = \inf_{x,t} t(1 - \langle Y, I\rangle - \langle Z, E\rangle) + \langle Y, G_0\rangle + \langle Z, H_0\rangle + \langle G^*(Y) + H^*(Z), x\rangle,$$

which is in turn equivalent to $\langle Y, I\rangle + \langle Z, E\rangle = 1$ and

$$Y \succeq 0,\quad Z \geq 0,\quad G^*(Y) + H^*(Z) = 0,\quad \langle Y, G_0\rangle + \langle Z, H_0\rangle \geq 0. \tag{11}$$

We have proved a pretty general version of Farkas' theorem on alternatives: The LMI problem (8) is not strictly feasible iff there exists an infeasibility certificate, a nonzero pair (Y, Z) satisfying (11).

2.6 Summary

With the mere exception of Theorem 1, elementary arguments allowed us to arrive at precise formulations of the following results:

- In full generality, the value of the Lagrange dual (6) is a lower bound on p_{opt}.
- There is no duality gap if $p_{\text{opt}} = -\infty$.
- If (3) is convex and satisfies Slater's constraint qualification, the duality gap vanishes and the Lagrange dual problem admits an optimal solution.

For an LMI problem (8), the Lagrange dual (cost and constraints) can be explicitly determined to be (9). Moreover:

- Suppose (8) is strictly feasible. Then $p_{\text{opt}} = -\infty$ iff the Lagrange dual (9) is infeasible. If $p_{\text{opt}} > -\infty$ then the value of (9) is attained and equal to p_{opt}.
- Finally, (8) is not strictly feasible iff there exists an infeasibility certificate, a nonzero pair (Y, Z) which satisfies (11).

Remark 3 *If we replace the inequality constraint $H_0 + H(x) \leq 0$ by the equation constraint $H_0 + H(x) = 0$, all results hold true if we just substitute the condition $Z \geq 0$ by $Z \in \mathcal{Z}$. A proof is based on replacing the equation constraint by $H_0 + H(x) \leq 0$, $-H_0 - H(x) \leq 0$, which introduces two multipliers $Z_1, Z_2 \geq 0$ that can be combined to $Z = Z_1 - Z_2$, a Lagrange multiplier for the* equation constraint $H_0 + H(x) = 0$ without sign condition.

Among the numerous applications of Lagrange dualization in convex analysis, we pick out the following two: existence of subgradients and separation of convex sets from points.

2.7 Existence of subgradients

If $f : \mathcal{S} \to \mathbb{R}$ is convex and x_0 is a relative interior point of $\mathcal{S}$, then f has a subgradient at x_0: There exists some $f_0 \in \mathcal{X}$ with

$$f(x) - f(x_0) \geq \langle f_0, x - x_0 \rangle \quad \text{for all} \quad x \in \mathcal{S}.$$

Proof. Just consider the following convex program

$$\inf_{x \in \mathcal{S},\, x - x_0 = 0} f(x) - f(x_0).$$

Since the only feasible point is x_0, its value is zero. By hypothesis on x_0 to be a relative interior point of $\mathcal{S}$, Slater's constraint qualification is satisfied. By strong duality (and Remark 3) there exists $f_0 \in \mathcal{X}$ with $0 = \inf_{x \in \mathcal{S}} f(x) - f(x_0) + \langle f_0, x - x_0 \rangle$. ∎

2.8 Separation

Suppose that $p \in \mathcal{X}$ is not contained in the convex set $\mathcal{S}$. Then there exists some nonzero $\hat{x} \in \mathcal{X}$ with

$$\langle \hat{x}, p \rangle \geq \langle \hat{x}, x \rangle \ \text{ for all } \ x \in \mathcal{S}.$$

Geometrically, $\mathcal{S}$ is completely contained in the half-space $\{x \in \mathcal{X} : \ \langle \hat{x}, p-x \rangle \geq 0\}$, while p itself is contained in the hyperplane $\{x \in \mathcal{X} : \ \langle \hat{x}, p-x \rangle = 0\}$.

Proof. Let x_0 be any relative interior point of $\mathcal{S}$, and consider the convex program

$$\inf_{t \in \mathbb{R},\, x \in \mathcal{S},\, x = p + t(x_0 - p)} t.$$

Since $p+t(x_0-p)$ is contained in $\mathcal{S}$ for $t = 1$ and not contained in $\mathcal{S}$ for $t \leq 0$ (by convexity of $\mathcal{S}$), we infer that the value of this program is finite and non-negative. Moreover Slater's constraint qualification is satisfied. Therefore there exists some $\hat{x}$ with

$$0 \leq \inf_{t \in \mathbb{R},\, x \in \mathcal{S}} t + \langle \hat{x}, p + t(x_0 - p) - x \rangle = \inf_{t \in \mathbb{R},\, x \in \mathcal{S}} t(1 + \langle \hat{x}, x_0 - p \rangle) + \langle \hat{x}, p - x \rangle.$$

Hence $\langle \hat{x}, p - x \rangle \geq 0$ for all $x \in \mathcal{S}$, and $1 + \langle \hat{x}, x_0 - p \rangle = 0$ implying $\hat{x} \neq 0$. ■

3 Frequency domain inequalities

3.1 From frequency domain inequalities to robust LMI's

The role of the L_∞-norm

$$\|F\|_\infty = \max_{\omega \in \mathbb{R} \cup \{\infty\}} \sigma_{\max}(F(i\omega))$$

for a proper transfer matrix F without pole on the imaginary axis is by now very well-established [78, 68] within robustness analysis and loop-shaping controller synthesis. Computing this norm through its very definition requires the solution of a non-convex optimization problem.

Let us, instead, follow an alternative path. For $x \in \mathbb{R}$ and $\omega \in \mathbb{R} \cup \{\infty\}$, we recall that $\sigma_{\max}(F(i\omega))^2 < x$ is equivalent to $F(i\omega)'F(i\omega) \prec xI$, which in turn implies

$$\sigma_{\max}(F(i\omega))^2 < x \ \text{ iff } \ \begin{pmatrix} I \\ F(i\omega) \end{pmatrix}' \begin{pmatrix} -xI & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ F(i\omega) \end{pmatrix} \prec 0.$$

Hence $\|F\|_\infty^2$ coincides with the infimal $x \in \mathbb{R}$ for which the frequency domain inequality

$$\begin{pmatrix} I \\ F(i\omega) \end{pmatrix}' \begin{pmatrix} -xI & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ F(i\omega) \end{pmatrix} \prec 0 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}$$

is satisfied. From now on let us tacitly assume $x \in \mathbb{R}$ and introduce the abbreviations

$$c = 1, \quad J(x) = \begin{pmatrix} -xI & 0 \\ 0 & I \end{pmatrix} \ \text{ as well as } \ \boldsymbol{\delta} = \mathbb{C}_0^\infty$$

for the functional to be optimized, the performance index, and the frequency set. We have subsumed the computation of $\|F\|_\infty^2$ to the solution of the *robust LMI problem*

$$p_{\rm opt} = \inf\left\{\langle c, x\rangle : \begin{pmatrix} I \\ F(\delta)\end{pmatrix}' J(x) \begin{pmatrix} I \\ F(\delta)\end{pmatrix} \prec 0 \ \text{ for all } \ \delta \in \boldsymbol{\delta}\right\}. \tag{12}$$

The left-hand side of the constraint inequality depends affinely on x, but in contrast to a genuine linear SDP, we are confronted with infinitely many linear matrix inequalities that are parameterized by $\delta \in \boldsymbol{\delta}$.

3.2 Removal of rational dependence

As one of the crucial difficulties in (12), the transfer matrix $F(\delta)$ is rational in δ. A fundamental step towards convexification is to determine a *linear fractional representation* (LFR) [78], which amounts to finding matrices A, B, C, D such that

$$F(\delta) = D + C\Delta(\delta)(I - A\Delta(\delta))^{-1}B \ \text{ with } \ \Delta(\delta) = \delta I. \tag{13}$$

Such a representation can be extracted from a state-space realization $D + C(sI - A)^{-1}B$ of $F(1/s)$. Moreover, because F does not have poles on the extended imaginary axis, we can make sure that the LFR is *well-posed* on $\boldsymbol{\delta}$:

$$\det(I - A\Delta(\delta)) \neq 0 \ \text{ for all } \ \delta \in \boldsymbol{\delta} \setminus \{\infty\} \ \text{ and } \ \det(A) \neq 0 \ \text{ for } \ \delta = \infty. \tag{14}$$

As the major benefit of this representation, we can apply the following variant of the S-procedure which is elementary to prove.

Lemma 4 *Let J be a Hermitian matrix. Then*

$$\begin{pmatrix} I \\ F(\delta)\end{pmatrix}' J \begin{pmatrix} I \\ F(\delta)\end{pmatrix} \prec 0 \ \textit{ for all } \ \delta \in \boldsymbol{\delta} \tag{15}$$

if there exists a Hermitian multiplier P which satisfies

$$\begin{pmatrix} I & 0 \\ A & B\end{pmatrix}' P \begin{pmatrix} I & 0 \\ A & B\end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D\end{pmatrix}' J \begin{pmatrix} 0 & I \\ C & D\end{pmatrix} \prec 0 \tag{16}$$

and which is related to $\boldsymbol{\delta}$ by

$$\begin{pmatrix} \Delta(\delta) \\ I\end{pmatrix}' P \begin{pmatrix} \Delta(\delta) \\ I\end{pmatrix} \succeq 0 \ \textit{ for } \ \delta \in \boldsymbol{\delta} \setminus \{\infty\}, \quad \begin{pmatrix} I \\ 0\end{pmatrix}' P \begin{pmatrix} I \\ 0\end{pmatrix} \succeq 0 \ \textit{ for } \ \infty \in \boldsymbol{\delta}. \tag{17}$$

Proof. Let us observe that

$$\left(\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D\end{array}\right) \begin{pmatrix} \Delta(\delta)(I - A\Delta(\delta))^{-1}B \\ I\end{pmatrix} = \left(\begin{array}{c} \begin{pmatrix} \Delta(\delta) \\ I\end{pmatrix}(I - A\Delta(\delta))^{-1}B \\ \hline \begin{pmatrix} I \\ F(\delta)\end{pmatrix}\end{array}\right). \tag{18}$$

Therefore (16) implies

$$[(I - A\Delta(\delta))^{-1}B]' \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}' P \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} (I - A\Delta(\delta))^{-1}B + \begin{pmatrix} I \\ F(\delta) \end{pmatrix}' J \begin{pmatrix} I \\ F(\delta) \end{pmatrix} \prec 0.$$

We can make use of (17) for both finite $\delta \in \boldsymbol{\delta}$ and for $\delta = \infty$ to infer (15). ∎

Remark 5 *Without trying to provide a detailed system theoretic interpretation, we stress that the formulation of this result is strongly motivated by the classical S-procedure for static or dynamic quadratic forms [77, 13], which can in turn be related to integral quadratic constraints [47], with the theory of dissipative dynamical systems [75, 76] as the major underlying concept. Actually it can be shown that 'iff' holds for compact sets* $\boldsymbol{\delta}$ *[43, 61] which is not exploited in the present paper.*

Let us now consider the optimization problem

$$\inf\left\{ \langle c, x\rangle : P \text{ satisfies (17)}, \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}' P \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}' J(x) \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \prec 0 \right\}. \quad (19)$$

By Lemma 4, its value is not smaller than $p_{\rm opt}$. Still, (19) typically involves infinitely many LMI constraints for P. As a conceptual benefit, the semi-infinite constraints (17) depend *quadratically* on $\Delta(\delta)$, and they do not involve the parameters A, B, C, D describing $F(\delta)$.

3.3 How to relax semi-infinite constraints?

Let us now exploit the fact that $\boldsymbol{\delta}$ is the extended imaginary axis. Hence, finite $\delta \in \boldsymbol{\delta}$ are characterized by $\delta \in \mathbb{C}$ satisfying the equation constraint $\delta' + \delta = 0$, or

$$\begin{pmatrix} \delta \\ 1 \end{pmatrix}' P_0 \begin{pmatrix} \delta \\ 1 \end{pmatrix} = 0 \text{ with } P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In complete analogy to our discussion of weak Lagrange duality in Section 2.2, let us choose an arbitrary Hermitian Lagrange multiplier Y and observe that the global non-negativity property

$$\begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}' P \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} - \begin{pmatrix} \delta \\ 1 \end{pmatrix}' P_0 \begin{pmatrix} \delta \\ 1 \end{pmatrix} Y \succeq 0 \text{ for all } \delta \in \mathbb{C} \quad (20)$$

implies that P satisfies (17) (also for $\delta = \infty$). Now note, by elementary Kronecker calculus or by direct calculation, that

$$\begin{pmatrix} \delta \\ 1 \end{pmatrix}' P_0 \begin{pmatrix} \delta \\ 1 \end{pmatrix} Y = \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}' (P_0 \otimes Y) \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}.$$

We conclude that (20), and hence (17), hold for all $P = P_0 \otimes Y$ with arbitrary Hermitian Y. This leads to an explicitly parameterized family of multipliers which satisfy (17), but we clearly might not describe *all possible* P in this fashion.

If we restrict the search of P in (19) to this subset, we arrive at the genuine LMI problem

$$p_{\rm rel} = \inf \left\{ \langle c, x\rangle : Y = Y', \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}' H(Y) \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}' J(x) \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \prec 0 \right\}, \tag{21}$$

where we used the abbreviation $H(Y) = P_0 \otimes Y$. Since we restricted the feasible set if compared to (19), $p_{\rm rel}$ is an upper bound of (19) which implies $\|F\|_\infty^2 = p_{\rm opt} \leq p_{\rm rel}$.

If we write out the LMI in (21) explicitly, we obtain

$$\begin{pmatrix} A'Y + YA + C'C & YB + C'D \\ B'Y + D'C & D'D - xI \end{pmatrix} \prec 0.$$

All this reveals how one arrives (for particular choices of $J(x)$ and $\boldsymbol{\delta}$ described by P_0) at the standard bounded real lemma linear matrix inequality.

3.4 How to prove exactness?

How is it possible to show the well-known fact that, actually, $p_{\rm rel}$ is equal to $p_{\rm opt}$? For this purpose, fix some x_0 such that

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}' H(Y) \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & I \\ C & D \end{pmatrix}' J(x_0) \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}}_{J_0} \prec 0 \ \text{ is not feasible.} \tag{22}$$

Therefore, there exists an infeasibility certificate (Section 2.5), some $M \neq 0$ with

$$M \succeq 0, \ \ \langle M, J_0\rangle \geq 0, \ \ H^*\left(\begin{pmatrix} I & 0 \\ A & B \end{pmatrix} M \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}' \right) = 0. \tag{23}$$

Let us now exploit the particular structure $H(Y) = P_0 \otimes Y$. With the abbreviations $U = (I \ \ 0)$ and $V = (A \ \ B)$, it is elementary to verify that

$$H^*\left(\begin{pmatrix} U \\ V \end{pmatrix} M \begin{pmatrix} U \\ V \end{pmatrix}' \right) = \begin{pmatrix} U' \\ V' \end{pmatrix}' (P_0 \otimes M) \begin{pmatrix} U' \\ V' \end{pmatrix}.$$

The resulting explicit formulation of (23) makes it possible to apply Lemma 5 in [62] in order to draw the following essential conclusion: *there exists as well an infeasibility certificate of rank one*, i.e., a vector m with

$$m \neq 0, \ \ \langle mm', J_0\rangle \geq 0, \ \ \begin{pmatrix} m'U' \\ m'V' \end{pmatrix}' P_0 \begin{pmatrix} m'U' \\ m'V' \end{pmatrix} = 0. \tag{24}$$

If $m'V' \neq 0$, it is elementary to prove ([58, Corollary 4] for our particular P_0 and [63, Lemma A.3] for general P_0) that there exist $\delta_0 \in \mathbb{C}$ with

$$m'U' = \delta_0(m'V') \ \text{ and } \ \begin{pmatrix} \delta_0' \\ 1 \end{pmatrix}' P_0 \begin{pmatrix} \delta_0' \\ 1 \end{pmatrix} = \begin{pmatrix} \delta_0 \\ 1 \end{pmatrix}' P_0 \begin{pmatrix} \delta_0 \\ 1 \end{pmatrix} = 0.$$

If $m'V' = 0$ choose $\delta_0 = \infty$. Then $\delta_0 \in \boldsymbol{\delta}$. If we partition $m' = (\xi' \;\; w')$ and recall the definitions of U, V, we conclude $\xi = \Delta(\delta_0)(A\xi + Bw)$ or $\xi = \Delta(\delta_0)(I - A\Delta(\delta_0))^{-1}Bw$ (also if $\delta_0 = \infty$). Since $m \neq 0$ we have $w \neq 0$. Due to (13) we get $C\xi + Dw = F(\delta_0)w$. This finally leads us to

$$0 \leq \langle mm', J_0 \rangle = w' \begin{pmatrix} I \\ F(\delta_0) \end{pmatrix}' J(x_0) \begin{pmatrix} I \\ F(\delta_0) \end{pmatrix} w,$$

which reveals that x_0 is *not feasible for (12)*.

We have just proved that any x_0 which is feasible for (12) is as well feasible for (21), which in turn implies $p_{\rm rel} \leq p_{\rm opt}$, and hence actually equality.

3.5 Discussion

Given the problem (12) with optimal value $p_{\rm opt}$, it was not difficult to construct *some relaxation* (21) whose optimal value $p_{\rm rel}$ is an upper bound of $p_{\rm opt}$. After introducing an LFR and applying the S-procedure, the main task consisted of finding a family of multipliers which satisfies (17) for the parameter set $\boldsymbol{\delta}$ under consideration. We determined one such family by relying on an elementary weak duality argument, a technique that is often called Lagrange-relaxation.

The given proof of exactness, $p_{\rm rel} = p_{\rm opt}$, has been based on Farkas' theorem of alternatives (Lagrange duality) for the LMI constraint involved in computing $p_{\rm rel}$, with the following essential step: Whenever an infeasibility certificate exits, we showed that there exists as well an infeasibility certificate *of rank one*.

A careful inspection shows that such an argument is at the heart of the optimization-based proofs of the Kalman-Yakubovich-Popov Lemma in [58, 3] and its extension to more general frequency sets $\boldsymbol{\delta}$ [48, 41, 62]. However, it also emerges in numerous other contexts, such as in showing exactness of D-scaling and (D, G)-scaling μ-upper bound computations for simple block structures [51, 48], or in proofs around the S-procedure and its extensions [28, 7, 69, 12], see also [8, 9]. A unifying exposition with an emphasis on structured singular value theory can be found in [63].

In this paper, it is one of our main intentions to review the power of this approach to construct LMI-relaxations for rather general robust LMI-problems and parameter sets. In general, these relaxations are not expected to be lossless for *all* possible problem instances. Still, for some concrete practical problem, it can very well happen that a relaxation is exact, and we will discuss techniques how to actually detect this fact numerically.

3.6 Variations and extensions

Obvious adaptations are related to modifications of the *performance index* $J(x)$ without violating $p_{\rm rel} = p_{\rm opt}$, since we only exploited that it is Hermitian-valued. If $J(x)$ depends on some x in an arbitrary finite dimensional real vector space, then (21) amounts to solving a standard SDP in the variables x and Y. The reformulation could as well be useful for general $J(x)$ being convex or even non-convex in x, as long as (21) is tractable by global or local optimization techniques. Note that $J(x)$ might not depend on x at all. Then our approach covers conventional feasibility problems, with the strict version of the general KYP lemma as the most prominent example [58, 48, 3]. From now on let us assume that $J(x)$ is real-affine in x.

F was only required to be a rational function without pole in $\boldsymbol{\delta}$ which admits an LFR in terms of A, B, C, D. We stress that F can be non-proper, or even just a polynomial in case that ∞ is not contained in $\boldsymbol{\delta}$.

A last crucial variation is related to modifications for the constraints on frequency, which amounts to choosing sets $\boldsymbol{\delta}$ with a more sophisticated description. For example, given two Hermitian 2×2-matrices P_0, P_1 with negative determinant (to avoid trivialities), let us consider

$$\boldsymbol{\delta} := \left\{ \delta \in \mathbb{C} : \begin{pmatrix} \delta \\ 1 \end{pmatrix}' P_0 \begin{pmatrix} \delta \\ 1 \end{pmatrix} = 0, \ \begin{pmatrix} \delta \\ 1 \end{pmatrix}' P_1 \begin{pmatrix} \delta \\ 1 \end{pmatrix} \geq 0 \right\}.$$

Quadratic equations describe circles in $\mathbb{C}$, while quadratic inequalities capture the interior or exterior of some disc, both covering the degenerate cases of lines and half-planes; if the set is unbounded we assume that $\infty \in \boldsymbol{\delta}$. Again, by Lagrange relaxation (or as easily verified directly), (17) holds true for all P equal to

$$H(Y_0, Y_1) = P_0 \otimes Y_0 + P_1 \otimes Y_1 \ \text{ with arbitrary } \ Y_0 = Y_0', \ \ Y_1 \succeq 0.$$

Moreover, $p_{\rm opt}$ is not larger than

$$\inf \left\{ \langle c, x \rangle : Y_1 \succ 0, \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}' H(Y_0, Y_1) \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}' J(x) \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \prec 0 \right\}.$$

Note that the upper bound is defined, as throughout the paper, with strict LMI's in order to guarantee the absence of a duality gap (see Section 2). Since one can determine the adjoint of H and apply [63, Lemmas A.3 and A.7] literally as in Section 3.4, we can conclude that *equality holds*. This recovers [41, 62] (with engineering motivations in [42]) as well as generalizations of the KYP lemma to spectral mask constraints [21, 30, 1, 60].

Example 6 *The segment* $\boldsymbol{\delta} = [i\omega_1, i\omega_2] \subset \mathbb{C}_0^\infty$ *is captured with*

$$P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} -1 & m \\ m' & r^2 - |m|^2 \end{pmatrix} \quad \text{for } \ m = \frac{1}{2}(\omega_1 + \omega_2)i, \quad r = \frac{1}{2}|\omega_2 - \omega_1|,$$

since the P_0*-equation constraint describes the imaginary axis, while the* P_1*-inequality constraint is the disk with center* m *and radius* r*. With* $P_0 = \mathrm{diag}(-1, 1)$*, one describes arcs on the unit circle to model discrete-time performance criteria.*

For the intersection of r disks or half-planes

$$\boldsymbol{\delta} = \left\{ \delta \in \mathbb{C} : \begin{pmatrix} \delta \\ 1 \end{pmatrix}' P_j \begin{pmatrix} \delta \\ 1 \end{pmatrix} \geq 0, \ j = 1, \ldots, r \right\},$$

(where P_j are 2×2 Hermitian matrices with negative determinant), choose

$$y = (Y_1, \ldots, Y_r), \quad H(y) := P_1 \otimes Y_1 + \cdots + P_r \otimes Y_r, \quad G(y) = \mathrm{diag}(-Y_1, \ldots, -Y_r)$$

to infer that $G(y) \preceq 0$ implies the validity of (17) for $P = H(y)$. Therefore,

$$p_{\rm opt} \leq \inf \left\{ \langle c, x\rangle : G(y) \prec 0, \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}' H(y) \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}' J(x) \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \prec 0 \right\}, \quad (25)$$

but now *equality cannot be proved in general.* Still, in addition to computing the upper bound, one can as well determine a Lagrange dual optimal solution of (25). If the optimal Lagrange multiplier for the second LMI has rank one it is again possible to conclude that equality holds. After illustrating this exactness test in the example below, we will discuss and extend it to general robust LMI problems in Section 6.

Remark 7 *It is somewhat surprising that, even for $r = 1$, equality does not hold in general (since the* quadratic equation *constraint is absent). If the right-lower block of $J(x)$ is positive semi-definite (as often true for practically relevant indices like those characterizing H_∞-norm bounds or positive realness), exactness can be easily recovered [51, 63].*

Example 8 *Choose*

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{ccc|c} -1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 5 & -50 \\ \hline -1 & -2 & 5 & -50 \end{array}\right), \ P_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, P_2 = \begin{pmatrix} 1 & -0.2 \\ -0.2 & -0.21 \end{pmatrix}, P_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1-\rho^2 \end{pmatrix}$$

with uncertainty regions for $\rho = 1.7$, $\rho = 1.5$, $\rho = 0.5$ depicted in Figure 1. For $\rho = 1.7$ we determine $\sqrt{p_{opt}} = 18.3$, and we can confirm exactness since the computed Lagrange multiplier has rank one. One could as well have concluded exactness as follows: The P_2-constraint is redundant; due to the maximum modulus theorem, $F(\delta)$ achieves its maximum at the boundary of $\boldsymbol{\delta}$; hence either the P_1- or the P_3-constraint is active, and the general exactness result for one quadratic equation and inequality applies.

For $\rho = 1.5$ we obtain $\sqrt{p_{opt}} = 19.9$, but the computed Lagrange multiplier has rank two. We could still extract the worst-case uncertainties $0.43 \pm 0.44i$ (as sketched in Section 6.5), which can be used to confirm exactness. For $\rho = 0.5$, the computed bound is $\sqrt{p_{opt}} = 27.4$, which is not exact since the actual value is estimated to be smaller than 22 by gridding.

3.7 General LMI-regions

Let us recall the concept of an *LMI-region* [19, 54], which is described with some Hermitian $P_0 \in \mathbb{C}^{2p \times 2p}$ partitioned into the $p \times p$ blocks Q_0, S_0, R_0 as

$$\boldsymbol{\delta} = \left\{ \delta \in \mathbb{C} : \begin{pmatrix} \delta I_p \\ I_p \end{pmatrix}' P_0 \begin{pmatrix} \delta I_p \\ I_p \end{pmatrix} \succeq 0 \right\} = \{\delta \in \mathbb{C} : \ \delta' \delta Q_0 + \delta' S_0 + \delta S_0' + R_0 \succeq 0\}.$$

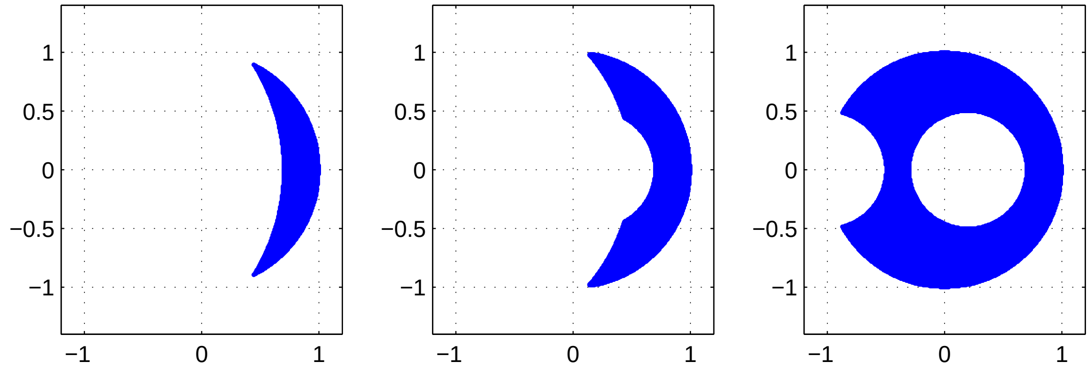

Figure 1: Uncertainty regions for $\rho = 1.7$ (left), $\rho = 1.5$ (middle), $\rho = 0.5$ (right).

It is well-established how to describe circles, sectors and possibly non-convex regions in the complex plane and their intersection in this fashion. In contrast to the examples discussed so far, $\boldsymbol{\delta}$ involves a genuine *matrix* inequality for its description. The Lagrange relaxation technique requires to determine a family of real-linear $L(.)$ which map positive semi-definite matrices of size $p \times p$ into positive semi-definite matrices of size $n \times n$ and which satisfy

$$\begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}' P \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} - L\left(\begin{pmatrix} \delta I_p \\ I_p \end{pmatrix}' P_0 \begin{pmatrix} \delta I_p \\ I_p \end{pmatrix}\right) \succeq 0 \ \text{ for all } \ \delta \in \mathbb{C}.$$

For this purpose we define [39]

$$(A, B)_q := \mathrm{tr}_q(A'(I_q \otimes B)) \in \mathbb{C}^{q \times q} \ \text{ for } \ A \in \mathbb{C}^{qp \times qp} \ \text{ and } \ B \in \mathbb{C}^{p \times p}$$

where

$$\mathrm{tr}_q(C) := \begin{pmatrix} \mathrm{tr}(C_{11}) & \cdots & \mathrm{tr}(C_{1q}) \\ \vdots & \ddots & \vdots \\ \mathrm{tr}(C_{q1}) & \cdots & \mathrm{tr}(C_{qq}) \end{pmatrix} \ \text{ with } \ C \in \mathbb{C}^{qp \times qp} \ \text{ partitioned into } \ C_{jk} \in \mathbb{C}^{p \times p}.$$

It is elementary to verify by inspection that

$$\mathrm{tr}_q(A'(I_q \otimes B)) = \mathrm{tr}_q((I_q \otimes B)A') \ \text{ for all } \ A \in \mathbb{C}^{qp \times qp} \ \text{ and } \ B \in \mathbb{C}^{p \times p}. \tag{26}$$

The following auxiliary result will be essential.

Lemma 9 *$(A, B)_q \in \mathbb{C}^{q \times q}$ is conjugate linear in $A \in \mathbb{C}^{qp \times qp}$ and linear in $B \in \mathbb{C}^{p \times p}$, and it satisfies*

$$(A, B)_q \succeq 0 \ \textit{ if } \ A \succeq 0 \ \textit{ and } \ B \succeq 0. \tag{27}$$

Proof. If $C \succeq 0$ then $\mathrm{tr}_q(C) \succeq 0$. Indeed, for arbitrary complex scalars $x_1, \ldots, x_q$ we have

$$\begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix}' \begin{pmatrix} \operatorname{tr}(C_{11}) & \cdots & \operatorname{tr}(C_{1q}) \\ \vdots & \ddots & \vdots \\ \operatorname{tr}(C_{q1}) & \cdots & \operatorname{tr}(C_{qq}) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix} = \sum_{j,k=1}^{q} x_j' \operatorname{tr}(C_{jk}) x_k =$$

$$= \operatorname{tr}\left(\sum_{j,k=1}^{q} x_j' C_{jk} x_k \right) = \operatorname{tr} \begin{pmatrix} x_1 I_p \\ \vdots \\ x_q I_p \end{pmatrix}' \begin{pmatrix} C_{11} & \cdots & C_{1q} \\ \vdots & \ddots & \vdots \\ C_{q1} & \cdots & C_{qq} \end{pmatrix} \begin{pmatrix} x_1 I_p \\ \vdots \\ x_q I_p \end{pmatrix} \succeq 0.$$

Note that this also follows from complete positivity of the trace map [20]. Let us now show (27): If $B \succeq 0$ then $B = DD'$ for some D; if $A \succeq 0$ we infer $(I \otimes D')A(I \otimes D) \succeq 0$; then (26) allows to conclude the proof as

$$(A, B)_q = \operatorname{tr}_q(A'(I \otimes D)(I \otimes D')) = \operatorname{tr}_q((I \otimes D')A(I \otimes D)) \succeq 0.$$

■

Let us now define

$$G(y) = -y \quad \text{and} \quad H(y) = \begin{pmatrix} (y, Q_0)_n & (y, S_0)_n \\ (y, S_0')_n & (y, R_0)_n \end{pmatrix} \quad \text{for} \quad y \in \mathbb{C}^{np \times np}.$$

If $G(y) \preceq 0$, we conclude that $P = H(y)$ satisfies (17). Indeed, if $\delta \in \boldsymbol{\delta}$, we infer with Lemma 9 the first inequality in

$$\begin{aligned} 0 \preceq (y, \delta'\delta Q_0 + \delta' S_0 + \delta S_0' + R_0)_n = \\ = \operatorname{tr}_n(y'(\delta'\delta(I_n \otimes Q_0) + \delta'(I_n \otimes S_0) + \delta(I_n \otimes S_0') + (I_n \otimes R_0)) = \\ = \delta'\delta(y, Q_0)_n + \delta'(y, S_0)_n + \delta(y, S_0')_n + (y, R_0)_n = \\ = \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}' \begin{pmatrix} (y, Q_0)_n & (y, S_0)_n \\ (y, S_0')_n & (y, R_0)_n \end{pmatrix} \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}. \end{aligned}$$

With this choice of linear mappings G, H, we have found a new upper bound LMI-relaxation (25) for frequency domain inequalities on general LMI-regions.

3.8 Well-posedness

Until now we have assumed that the employed LFR is well-posed (see (14)). In particular for multivariable uncertainty sets, as considered in the next section, it is a nontrivial task to verify this property. The following elementary variation of Lemma 4 is the key to construct suitable numerical tests, along the lines as discussed for performance throughout Section 3.

Lemma 10 *Well-posedness (14) holds if there exists a Hermitian P with (17) and*

$$\begin{pmatrix} I \\ A \end{pmatrix}' P \begin{pmatrix} I \\ A \end{pmatrix} \prec 0. \tag{28}$$

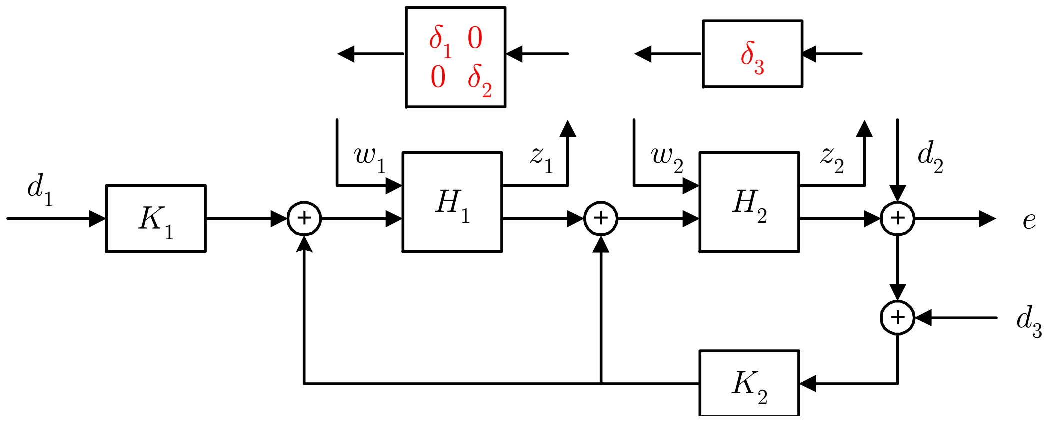

Figure 2: Uncertain interconnection.

Proof. Suppose (14) does not hold for $\delta \neq \infty$. Then there exists some vector $z \neq 0$ with $(I - A\Delta(\delta))z = 0$. If we set $w = \Delta(\delta)z$, we infer $z = Aw$ and hence $w \neq 0$. Moreover we can use (17) and (28) to conclude

$$0 \leq z' \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}' P \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} z = w' \begin{pmatrix} I \\ A \end{pmatrix}' P \begin{pmatrix} I \\ A \end{pmatrix} w \prec 0,$$

a contradiction. The same arguments apply for $\delta = \infty$. ∎

Again 'iff' holds if $\boldsymbol{\delta}$ is compact [40]. Let us finally observe that the left-upper block of (16) reads as

$$\begin{pmatrix} I \\ A \end{pmatrix}' P \begin{pmatrix} I \\ A \end{pmatrix} + \begin{pmatrix} 0 \\ C \end{pmatrix}' J \begin{pmatrix} 0 \\ C \end{pmatrix} \prec 0.$$

If the right-lower block of J is positive semi-definite, we infer from Lemma 10 that all multiplier relaxations based on Lemma 4 *do automatically include a test of well-posedness* of the LFR.

4 Robust performance against LTI uncertainties

So far, we considered uncertainty sets $\boldsymbol{\delta}$ for robust LMI problems which are one-dimensional curve segments, disks or half-planes in $\mathbb{C}$ with explicit exactness results. For more general sets we had to give up exactness, but it was still possible to construct powerful relaxations for upper bound computations in a systematic fashion. A substantial next step consists in the extension to *multivariable* uncertainties. As an important field of application, let us address robust performance of LTI systems against structured LTI uncertainties.

In a general controlled uncertain system, the first step consists of separating the uncertainties from the interconnection of known components such as illustrated in Figure 2. This leads to the following linear fractional representation

$$\begin{pmatrix} z \\ e \end{pmatrix} = \underbrace{\begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}}_{N} \begin{pmatrix} w \\ d \end{pmatrix}, \quad w = \Delta z, \tag{29}$$

where N is a given proper and stable transfer matrix (including frequency dependent weights for uncertainty and performance shaping) and Δ denote the proper and stable uncertainty transfer matrices. Let us recall at this point that LFR's are very naturally tied to system interconnections and to the generalized plant framework within robust control. Once simple LFR's for system components have been constructed, it is routine to determine the representing matrices for an LFR of the overall interconnection, such as implemented in the practically very reliable μ-toolbox [4].

One easily verifies that (29) implies

$$e = (\Delta \star N)d \;\text{ with the star-product }\; \Delta \star N := N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}.$$

Recall that Δ typically admits a block-diagonal structure and its H_∞-norm is bounded in size by one. In somewhat more generality, constraints on both the structure and the size of Δ can be described by

$$\Delta(s) \in \mathbf{\Delta} \;\text{ for all }\; s \in \mathbb{C}_0^\infty,$$

where $\mathbf{\Delta}$ is some set of complex matrices which is star-shaped with star-center zero. It is well-known [51, 78, 68] that robust stability of the interconnection is guaranteed if

$$\det(I - N_{11}(s)\Delta) \neq 0 \;\text{ for all }\; \Delta \in \mathbf{\Delta} \tag{30}$$

and for all frequencies $s \in \mathbb{C}_0^\infty$ on the extended imaginary axis. Since we can translate transfer matrix norm bounds into robust matrix inequalities as clarified in Section 3.1, the squared H_∞-norm of $d \to e$ is bounded by x for all uncertainties if

$$\begin{pmatrix} I \\ \Delta \star N(s) \end{pmatrix}' J(x) \begin{pmatrix} I \\ \Delta \star N(s) \end{pmatrix} \prec 0 \;\text{ for all }\; \Delta \in \mathbf{\Delta} \tag{31}$$

and all $s \in \mathbb{C}_0^\infty$.

4.1 Computation by frequency gridding

If we fix $s_0 \in \mathbb{C}_0^\infty$, the validity of (30) and (31) for $s = s_0$ is guaranteed if there exists a Hermitian P_0 with

$$\begin{pmatrix} \Delta \\ I \end{pmatrix}' P_0 \begin{pmatrix} \Delta \\ I \end{pmatrix} \succeq 0 \;\text{ for all }\; \Delta \in \mathbf{\Delta}, \tag{32}$$

$$\begin{pmatrix} * \\ * \end{pmatrix}' P_0 \begin{pmatrix} I & 0 \\ N_{11}(s_0) & N_{12}(s_0) \end{pmatrix} + \begin{pmatrix} * \\ * \end{pmatrix}' J(x) \begin{pmatrix} 0 & I \\ N_{21}(s_0) & N_{22}(s_0) \end{pmatrix} \prec 0. \tag{33}$$

The proof is identical to that of Lemma 4/Lemma 10, since the right-lower block of $J(x)$ is positive semi-definite. If the latter property fails, one has to check (30) at $s = s_0$ by an independent test as described in Section 3.8. Note again that 'iff' is true if $\mathbf{\Delta}$ is compact.

Typically, $\mathbf{\Delta}$ consists of block-diagonal matrices $\mathrm{diag}(\Delta_1, \ldots, \Delta_f, \delta_{f+1}I, \ldots, \delta_{f+r}I)$, where Δ_k, δ_k are unstructured complex *matrices* and *scalars* satisfying $\Delta_k \in \boldsymbol{\Delta_k}$ and $\delta_k \in \boldsymbol{\delta_k}$

respectively. For repeated blocks $\delta_k I$, we have already discussed for various sets $\boldsymbol{\delta_k}$ (in Sections 3.6 and 3.7) how to construct linear mappings $G_k(y_k)$ and $H_k(y_k)$ such that

$$G_k(y_k) \preceq 0 \ \text{ implies } \ \begin{pmatrix} \delta_k I \\ I \end{pmatrix}' \underbrace{\begin{pmatrix} Q_k(y_k) & S_k(y_k) \\ S_k(y_k)' & R_k(y_k) \end{pmatrix}}_{H_k(y_k)} \begin{pmatrix} \delta_k I \\ I \end{pmatrix} \succeq 0 \ \text{ for all } \ \delta_k \in \boldsymbol{\delta_k}.$$

Very similar ideas apply to full blocks. For example, suppose

$$\boldsymbol{\Delta_k} = \left\{ \Delta_k : \begin{pmatrix} \Delta_k \\ I \end{pmatrix}' H_k^j \begin{pmatrix} \Delta_k \\ I \end{pmatrix} \succeq 0 \ \text{ for } \ j = 1, \ldots, r_k \right\} \ \text{ with fixed } \ (H_k^j)' = H_k^j.$$

It is then trivial that

$$G_k(y_k) \preceq 0 \ \text{ implies } \ \begin{pmatrix} \Delta_k \\ I \end{pmatrix}' \underbrace{\begin{pmatrix} Q_k(y_k) & S_k(y_k) \\ S_k(y_k)' & R_k(y_k) \end{pmatrix}}_{H_k(y_k)} \begin{pmatrix} \Delta_k \\ I \end{pmatrix} \succeq 0 \ \text{ for all } \ \Delta_k \in \boldsymbol{\Delta_k},$$

if we choose $y_k = (y_k^1, \ldots, y_k^{r_k}) \in \mathbb{R}^{r_k}$ and

$$G_k(y_k) = \text{diag}(-y_k^1, \ldots, -y_k^{r_k}), \quad H_k(y_k) = y_k^1 H_k^1 + \cdots + y_k^{r_k} H_k^{r_k}.$$

After having determined suitably parameterized classes of multipliers for the individual blocks, it is now straightforward to combine them to a multiplier for the full structured uncertainty block by diagonal augmentation. This amounts to combining $y := (y_1, \ldots, y_{f+r})$ and defining

$$G(y) = \operatorname*{diag}_{k=1}^{f+r}(G_k(y_k)), \quad H(y) = \begin{pmatrix} \operatorname*{diag}_{k=1}^{f+r}(Q_k(y_k)) & \operatorname*{diag}_{k=1}^{f+r}(S_k(y_k)) \\ \operatorname*{diag}_{k=1}^{f+r}(S_k(y_k)') & \operatorname*{diag}_{k=1}^{f+r}(R_k(y_k)) \end{pmatrix}.$$

Then (32) is indeed true for $P_0 = H(y)$ if y is chosen with $G(y) \preceq 0$. Therefore, (30) and (31) are valid at $s = s_0$ if there exists a solution y of the LMI

$$G(y) \prec 0, \quad \begin{pmatrix} * \\ * \end{pmatrix}' H(y) \begin{pmatrix} I & 0 \\ N_{11}(s_0) & N_{12}(s_0) \end{pmatrix} + \begin{pmatrix} * \\ * \end{pmatrix}' J(x) \begin{pmatrix} 0 & I \\ N_{21}(s_0) & N_{22}(s_0) \end{pmatrix} \prec 0.$$

Moreover, optimal bounds on the worst-case spectral norm of $\Delta \star N(s_0)$ are computed by optimizing x over these LMI constraints.

Example 11 *Consider the interconnection in Figure 2 with*

$$H_1(s) = \begin{pmatrix} \frac{s+1}{s^2+2s+1} & 1 & \frac{-s-1}{s^2+2s+1} \\ \frac{1}{s^2+2s+1} & 0 & \frac{-1}{s^2+2s+1} \\ \frac{s}{s^2+2s+1} & 0 & \frac{-s}{s^2+2s+1} \end{pmatrix}, \quad H_2(s) = \begin{pmatrix} 0 & 1 \\ 1 & \frac{100}{10s+1} \end{pmatrix}, \quad K_1 = 1, \quad K_2 = -\frac{5}{s+2}.$$

The frequency responses of $\delta_1(s)$, $\delta_2(s)$, $\delta_3(s)$ are contained in the subsets of the unit disk as shown in Figure 3, which correspond to the amplitude and phase constraints

$$\begin{pmatrix} \delta_1 \\ 1 \end{pmatrix}' \begin{pmatrix} -2 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} \delta_1 \\ 1 \end{pmatrix} \geq 0, \ \begin{pmatrix} \delta_2 \\ 1 \end{pmatrix}' \begin{pmatrix} -2 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} \delta_2 \\ 1 \end{pmatrix} \geq 0, \ \begin{pmatrix} \delta_2 \\ 1 \end{pmatrix}' \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_2 \\ 1 \end{pmatrix} \geq 0$$

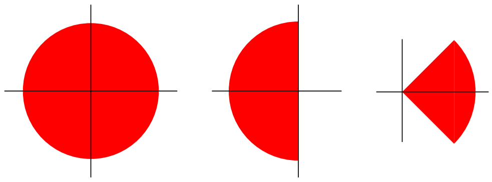

Figure 3: Uncertainty sets for examples 11 and 12.

as well as

$$\begin{pmatrix}\delta_3\\1\end{pmatrix}'\begin{pmatrix}-2&0\\0&0.5\end{pmatrix}\begin{pmatrix}\delta_3\\1\end{pmatrix}\geq 0,\ \begin{pmatrix}\delta_3\\1\end{pmatrix}'\begin{pmatrix}0&1+i\\1-i&0\end{pmatrix}\begin{pmatrix}\delta_3\\1\end{pmatrix}\geq 0,\ \begin{pmatrix}\delta_3\\1\end{pmatrix}'\begin{pmatrix}0&1-i\\1+i&0\end{pmatrix}\begin{pmatrix}\delta_3\\1\end{pmatrix}\geq 0$$

respectively [72]. Although these constraints cannot be easily handled with standard μ-tools [4], it is now routine to construct the following relaxation as described above. The three blocks are viewed as being complex repeated, subject to one, two and three quadratic inequality constraints respectively. Hence we choose $y = (Y_1, Y_2^1, Y_2^2, Y_3^1, Y_3^2, Y_3^3)$ with Hermitian 1×1 components, and $G(y) = -\operatorname{diag}(Y_1, Y_2^1, Y_2^2, Y_3^1, Y_3^2, Y_3^3)$ as well as

$$H(y)=\begin{pmatrix}\begin{pmatrix}-2Y_1&0&0\\0&-2Y_2^1&0\\0&0&-2Y_3^1\end{pmatrix}&\begin{pmatrix}0&0&0\\0&-Y_2^2&0\\0&0&(1+i)Y_3^2+(1-i)Y_3^3\end{pmatrix}\\\begin{pmatrix}0&0&0\\0&-Y_2^2&0\\0&0&(1-i)Y_3^2+(1+i)Y_3^3\end{pmatrix}&\begin{pmatrix}0.5Y_1&0&0\\0&0.5Y_2^1&0\\0&0&0.5Y_3^1\end{pmatrix}\end{pmatrix}.$$

Computed worst-case performance upper bounds with a grid of 100 frequencies are shown as the continuous line in Figure 4, while worst-case uncertainties as extracted according to the procedure in Section 6.5 confirm the exactness of the relaxation.

4.2 Avoiding frequency gridding

If introducing a minimal state-space realization

$$\begin{pmatrix}N_{11}(1/s)&N_{12}(1/s)\\N_{21}(1/s)&N_{22}(1/s)\end{pmatrix}=\begin{pmatrix}C_1\\C_2\end{pmatrix}(sI-A_0)^{-1}\begin{pmatrix}B_1&B_2\end{pmatrix}+\begin{pmatrix}D_{11}&D_{12}\\D_{21}&D_{22}\end{pmatrix}$$

with A_0 of size $n\times n$, let us observe that

$$\Delta\star N(\delta)=\underbrace{\begin{pmatrix}\delta I_n&0\\0&\Delta\end{pmatrix}}_{\Delta_e}\star\underbrace{\begin{pmatrix}A&B\\C&D\end{pmatrix}}_{N_e}\quad\text{where}\quad N_e:=\begin{pmatrix}A&B\\C&D\end{pmatrix}:=\left(\begin{array}{cc|c}A_0&B_1&B_2\\C_1&D_{11}&D_{12}\\\hline C_2&D_{21}&D_{22}\end{array}\right).$$

With the corresponding uncertainty set

$$\boldsymbol{\Delta_e}=\left\{\begin{pmatrix}\delta I_n&0\\0&\Delta\end{pmatrix}:\ \delta\in\mathbb{C}_0^\infty,\ \Delta\in\boldsymbol{\Delta}\right\},$$

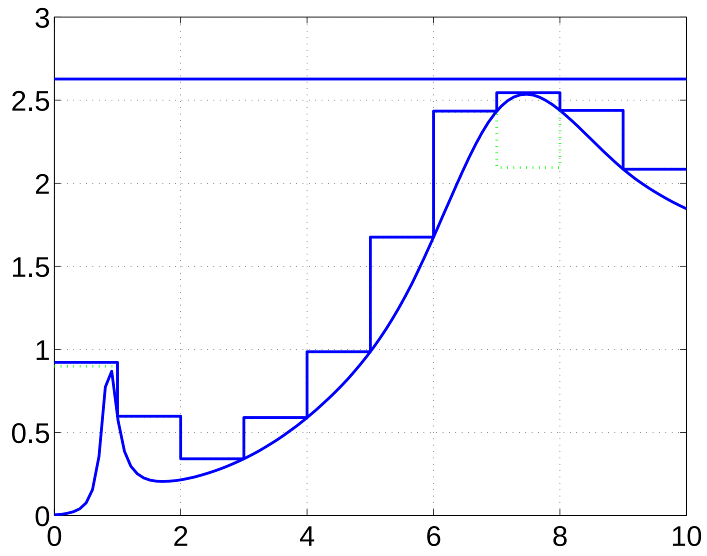

Figure 4: Computed performance level by frequency gridding (curve) and on frequency intervals (staircase). Dotted lines indicate lower bounds that have been determined as explained in Example 12.

both (30) and (31) are equivalent to

$$\det(I - A\Delta_e) \neq 0, \quad \begin{pmatrix} I \\ \Delta_e \star N_e \end{pmatrix}' J(x) \begin{pmatrix} I \\ \Delta_e \star N_e \end{pmatrix} \prec 0 \ \text{ for all } \ \Delta_e \in \boldsymbol{\Delta_e}.$$

We can now apply any of the suggested relaxations to this alternative formulation, which might lead to conservative upper bounds since the uncertainty related to frequency can vary on the whole extended imaginary axis. This source of conservatism can be avoided by covering the imaginary axis with segments $i[\omega_1, \omega_2]$, and constructing relaxations for

$$\boldsymbol{\Delta_e} = \left\{ \begin{pmatrix} \delta I & 0 \\ 0 & \Delta \end{pmatrix} : \ \delta \in i[\omega_1, \omega_2], \Delta \in \boldsymbol{\Delta} \right\},$$

with multipliers for imaginary axis segments as suggested in Example 6 [51, 31]. We stress again that it remains possible to verify relaxation exactness with the computational tools in Section 6.5.

Example 12 *Let us continue with Example 11. We determine guaranteed performance bounds on the frequency interval $i[0, 10]$, and over a partition of ten equidistant subintervals thereof, with results shown in Figure 4. Exactness of the computed bounds cannot be confirmed numerically on the intervals $i[0, 1]$ and $i[7, 8]$ as well as $i[0, 10]$. Still, the algorithm of Section 6.5 returns uncertainties that lead to the lower bounds on the exact optimal value as indicated by dotted lines in Figure 4.*

4.3 Convex hull relaxation

Structured real or complex uncertainty blocks can often be conveniently described by

$$\Delta \in \boldsymbol{\Delta} = \text{co}\{\Delta_1, \ldots, \Delta_q\}.$$

It is easily seen that (32) is implied by

$$\begin{pmatrix} I \\ 0 \end{pmatrix}' P_0 \begin{pmatrix} I \\ 0 \end{pmatrix} \prec 0 \quad \text{and} \quad \begin{pmatrix} \Delta_\nu \\ I \end{pmatrix}' P_0 \begin{pmatrix} \Delta_\nu \\ I \end{pmatrix} \succ 0 \quad \text{for all} \quad \nu = 1, \ldots, q, \tag{34}$$

simply because the first inequality renders the map $\Delta \to (\Delta' \ I)P_0(\Delta' \ I)'$ concave. Even if Δ is structured, the multipliers that are indirectly described through (34) are generally full. This can drastically reduce conservatism if compared to structured multipliers, at the expense of increased computational complexity, with an explicit illustrative example appearing in [61]. Moreover, it is straightforward to subsume the resulting relaxation to the generic formulation of Section 6, which implies that the corresponding exactness characterization are fully applicable. Variations on this theme can be found in [40, 74].

5 LPV systems

Let us now briefly discuss what happens if uncertainties are time-varying. Robustness tests are then formulated in terms of Lyapunov functions (stability) or storage functions (performance) based on dissipation theory [75, 76, 13, 26]. These frameworks allow to reduce trajectory-oriented specifications to algebraic positivity tests. Out of the many possibilities, let us confine the illustrative discussion to *linear-parameter-varying* or LPV systems

$$\begin{pmatrix} \dot\xi(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} A(\delta(t)) & B(\delta(t)) \\ C(\delta(t)) & D(\delta(t)) \end{pmatrix} \begin{pmatrix} \xi(t) \\ w(t) \end{pmatrix} \quad \text{with} \quad \delta(t) \in \boldsymbol{\delta}, \ \dot\delta(t) \in \dot{\boldsymbol{\delta}} \tag{35}$$

for subsets $\boldsymbol{\delta} \subset \mathbb{R}^p$ and $\dot{\boldsymbol{\delta}} \subset \mathbb{R}^p$ which bound values and rate-of-variation of the continuously differentiable parameter trajectory respectively.

5.1 Quadratic stability

The LPV system $\dot\xi(t) = A(\delta(t))\xi(t)$ is quadratically stable if there exists some X with

$$X \succ 0 \quad \text{and} \quad \begin{pmatrix} I \\ A(\delta) \end{pmatrix}' \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} I \\ A(\delta) \end{pmatrix} = A(\delta)'X + XA(\delta) \prec 0 \quad \text{for all} \quad \delta \in \boldsymbol{\delta}. \tag{36}$$

If $\boldsymbol{\delta}$ is compact, it is well known that quadratic stability implies uniform exponential stability of the LPV system for all parameter trajectories with $\delta(t) \in \boldsymbol{\delta}$. If $A(\delta)$ depends affinely on δ and if $\boldsymbol{\delta} = \text{co}\{\delta^1, \ldots, \delta^q\}$, it clearly suffices to just find a solution of the finite LMI system

$$X \succ 0 \quad \text{and} \quad A(\delta^\nu)'X + XA(\delta^\nu) \prec 0 \quad \text{for all} \quad \nu = 1, \ldots, q.$$

Although often used in robust control, this deceptively simple procedure suffers from high computational complexity (as precisely described in [49, 8]) for polytopes with a large number of generators.

For the purpose of illustration, let us consider the unit cube

$$\boldsymbol{\delta} = \{(\delta_1, \dots, \delta_p) \in \mathbb{R}^p : \ -1 \le \delta_j \le 1, \ j = 1, \dots, p\}.$$

With $A(\delta) = D + D_1\delta_1 + \cdots + D_p\delta_p$, we need to find X with

$$X \succ 0 \ \text{ and } \ (D'X + XD) \pm (D_1'X + XD_1) \pm \cdots \pm (D_p'X + XD_p) \prec 0.$$

This requires to check feasibility of $1 + 2^p$ inequalities of size n in $n(n+1)/2$ scalar variables. In [8], the authors suggest the following relaxation, which is easily seen to be *sufficient* for quadratic stability: There exist X and Z_j with

$$X \succ 0, \ \ (D'X + XD) + \sum_{j=1}^{p} Z_j \prec 0, \ \ \pm(D_j'X + XD_j) \prec Z_j \ \text{ for all } \ j = 1, \dots, p. \quad (37)$$

This test only involves $2 + 2p$ inequalities of size n in $n(n+1)(p+1)/2$ scalar variables. A beautiful result in [8] shows that this relaxation is tight in the following sense: If it is not feasible, then quadratic stability fails for the enlarged cube

$$\theta\left(2\max_{j=1}^{p} \ \text{rank}(D_j)\right)\boldsymbol{\delta} \ \text{ with } \theta(\mu) \text{ satisfying } \ \theta(\mu) \le \frac{\pi}{2}\sqrt{\mu}, \ \ \theta(2) = \frac{\pi}{2}.$$

In typical applications, the structure matrices D_j have small rank. This allows to reduce the computational complexity of solving (37) on the basis of the following slight extension of [8, Lemma 3.2].

Lemma 13 *The inequality* $\pm[(XCB)' + (XCB)] \prec Z$ *holds iff there exist* $Q \succ 0$ *and* $S + S' = 0$ *with* $B'QB + (B'S + XC)Q^{-1}(B'S + XC)' \prec Z$.

Proof. The proof is a direct application of the exactness results (one-parameter uncertainty described by one quadratic equation and inequality) in Section 3.6. Indeed $\pm[(XCB)' + (XCB)] \prec Z$ iff

$$(XC[\delta I]B)' + (XC[\delta I]B) - Z = \begin{pmatrix} I \\ XC[\delta I]B \end{pmatrix}' \begin{pmatrix} -Z & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I \\ XC[\delta I]B \end{pmatrix} \prec 0 \ \forall\, \delta \in [-1, 1].$$

Let us now apply the procedure of Section 3.6 to this robust LMI problem, by noting that $\delta \in [-1, 1]$ is equivalent to

$$\begin{pmatrix} \delta \\ 1 \end{pmatrix}' \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta \\ 1 \end{pmatrix} \ge 0, \ \ \begin{pmatrix} \delta \\ 1 \end{pmatrix}' \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \delta \\ 1 \end{pmatrix} = 0.$$

Robust feasibility is, therefore, equivalent to the existence of complex Hermitian matrices $\hat{Q} \succ 0$ and $\hat{S}$ with

$$\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}' \begin{pmatrix} -\hat{Q} & (i\hat{S})' \\ i\hat{S} & \hat{Q} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ XC & 0 \end{pmatrix}' \begin{pmatrix} -Z & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ XC & 0 \end{pmatrix} \prec 0.$$

Since all matrices but $\hat{Q}$, $\hat{S}$ are real, this holds iff there exist real matrices $Q \succ 0$ and $S + S' = 0$ with

$$\begin{pmatrix} -Q & (B'S + XC)' \\ B'S + XC & B'QB - Z \end{pmatrix} \prec 0,$$

and the result follows by taking the Schur complement. ■

Let us now introduce the factorization $D_j = C_j B_j$ where the number of columns/rows of C_j, B_j equals $d_j = \text{rank}(D_j)$. By Lemma 13, (37) holds iff there exist $Q_j \succ 0$, $S_j + S_j' = 0$, $j = 1, \ldots, p$, with

$$(D'X + XD) + \sum_{j=1}^{p} Z_j \prec 0, \;\; B_j'Q_jB_j + (B_j'S_j + XC_j)Q_j^{-1}(B_j'S_j + XC_j)' \prec Z_j, \;\; j = 1, \ldots, p.$$

This obviously implies

$$(D'X + XD) + \sum_{j=1}^{p} B_j'Q_jB_j + (B_j'S_j + XC_j)Q_j^{-1}(B_j'S_j + XC_j)' \prec 0.$$

Even more, the latter inequality implies the former with $Z_j := B_j'Q_jB_j + (B_j'S_j + XC_j)Q_j^{-1}(B_j'S_j + XC_j)' + \epsilon I$ and some sufficiently small $\epsilon > 0$.

In summary, (37) is feasible iff there exist X, Q_j and skew-symmetric S_j with

$$X \succ 0, \;\; \begin{pmatrix} -Q_1 & & 0 & S_1'B_1 + C_1'X \\ & \ddots & & \vdots \\ 0 & & -Q_p & S_p'B_p + C_p'X \\ B_1'S_1 + XC_1 & \cdots & B_p'S_p + XC_p & (D'X + XD) + \sum_{j=1}^{p} B_j'Q_jB_j \end{pmatrix} \prec 0. \quad (38)$$

This involves just two inequalities of size n, $\sum_{j=1}^{p} d_j + n$ in $n(n+1)/2 + \sum_{j=1}^{p} d_j^2$ scalar variables. Since d_j is typically much smaller than n, this equivalent reformulation of the relaxation can be solved much faster than (37) in practice.

Apart from pointing out a simple proof of the relaxations' equivalence, let us finally observe that (38) does actually result from applying the standard procedure to handle real repeated uncertainties in structured singular value theory [27]. Indeed, the full-rank factorization $D_j = C_jB_j$ actually induces the following (minimal sized) LFR

$$A(\delta) = \Delta(\delta) \star \begin{pmatrix} 0 & B \\ C & D \end{pmatrix} \text{ with } \Delta(\delta) := \begin{pmatrix} \delta_1 I_{d_1} & & 0 \\ & \ddots & \\ 0 & & \delta_p I_{d_p} \end{pmatrix}, \;\; \begin{pmatrix} 0 & B \\ C & D \end{pmatrix} := \left(\begin{array}{ccc|c} 0 & & 0 & B_1 \\ & \ddots & & \vdots \\ 0 & & 0 & B_p \\ \hline C_1 & \ldots & C_p & D \end{array}\right).$$

Moreover the second inequality in (38) just reads as

$$\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}' \begin{pmatrix} \text{diag}_{j=1}^{p}(-Q_j) & \text{diag}_{j=1}^{p}(S_j') \\ \text{diag}_{j=1}^{p}(S_j) & \text{diag}_{j=1}^{p}(Q_j) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}' \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \prec 0.$$

This is identical to the outcome of applying the procedure of Section 4.1 to the robust LMI problem (36), similarly as in the proof of Lemma 13.

To summarize, for affine dependence on uncertainties, the tightness results from [8] are indeed valid for the standard real-repeated block upper bound relaxation in classical structured singular value theory. It is not difficult to verify the analogous relations for checking robust performance within the general framework of dissipative dynamical systems [5]. If the uncertainties enter in a rational fashion, it is still possible to routinely construct efficiently computable relaxations along precisely the same lines, while generic tightness results seem out of reach.

5.2 Rate-bounded uncertainties

With the performance index $J(x)$ as defined in Section 3.1, let us assume that there exists a symmetric-valued C^1 function $X(\delta)$ such that

$$X(\delta) \succ 0, \quad \begin{pmatrix} I & 0 \\ A(\delta) & B(\delta) \end{pmatrix}' \begin{pmatrix} \sum_{j=1}^{p} \frac{\partial X}{\partial \delta_j}(\delta) v_j & X(\delta) \\ X(\delta) & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A(\delta) & B(\delta) \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & I \\ C(\delta) & D(\delta) \end{pmatrix}' J(x) \begin{pmatrix} 0 & I \\ C(\delta) & D(\delta) \end{pmatrix} \prec 0 \ \text{ for all } \ \delta \in \boldsymbol{\delta}, \quad (v_1, \ldots, v_p) \in \dot{\boldsymbol{\delta}}. \quad (39)$$

If $\delta(t)$ is an admissible parameter trajectory and $w(t)$ a finite energy disturbance which results in the state-trajectory $\xi(t)$ of (35), we infer

$$\frac{d}{dt}\xi(t)'X(\delta(t))\xi(t) = \dot{\xi}(t)'X(\delta(t))\xi(t) + \xi(t)'X(\delta(t))\dot{\xi}(t) + \xi(t)' \left[\sum_{j=1}^{p} \frac{\partial X}{\partial \delta_j}(\delta(t))\dot{\delta}_j(t) \right] \xi(t).$$

Therefore, (39) implies

$$\xi(t)'X(\delta(t))\xi(t) \geq 0 \ \text{ and } \ \frac{d}{dt}\xi(t)'X(\delta(t))\xi(t) + \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}' J(x) \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} \leq 0.$$

If $\boldsymbol{\delta} \times \dot{\boldsymbol{\delta}}$ is compact and if $w(.) = 0$, elementary arguments allow to conclude exponential stability. If $w(.)$ has finite energy and $x(0) = 0$, we infer by integration that

$$\int_0^\infty z(t)'z(t) - xw(t)'w(t)\, dt = \int_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}' J(x) \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt \leq 0.$$

Therefore the L_2-gain of (35) is not larger than $\sqrt{x}$ for all admissible uncertainty curves.

In a similar fashion, a multitude of other (dynamic) performance criteria translate into an (algebraic) feasibility problem for a partial differential matrix inequality, with a variety of examples found in [26]. We stress that the resulting problem is *infinite-dimensional*, since $X(.)$ has to be found in C^1, and *semi-infinite*, since the inequalities (39) have to hold for infinitely many parameter values. The reduction to a finite-dimensional problem is standard, by just restricting the search for $X(.)$ to some finite dimensional subspace $\text{span}\{X_1(.), \ldots, X_m(.)\}$ with symmetric-valued C^1 basis functions $X_1(.), \ldots, X_m(.)$. We hence substitute

$$X(\delta) = \sum_{k=1}^{m} x_k X_k(\delta) \ \text{ and } \ \frac{\partial X}{\partial \delta_j}(\delta) = \sum_{k=1}^{m} x_k \frac{\partial X_k}{\partial \delta_j}(\delta), \quad j = 1, \ldots, p, \quad (40)$$

in (39) to arrive at a finite-dimensional robust LMI problem in the variables $x_1, \ldots, x_m$. If $A(\delta)$, $B(\delta)$, $C(\delta)$, $D(\delta)$, $X_k(\delta)$, $k = 1, \ldots, m$, admit linear fractional representations, it is routine to subsume the computational problem to the general framework described in Section 6. At this point, it is essential to stress that one has to exploit the particular structure in order to keep the size of the LFR's manageable. As a major advantage, this scenario covers quadratic-in-the-state Lyapunov functions which are constant [6], quadratic [73, 43], or even polynomial or rational in the parameters [24, 23]. Most importantly, it nicely captures as well the design of parameter-dependent controllers for systematic gain-scheduling with rate-bounded parameter-trajectories [50, 67, 31, 2, 61]. It is as well not difficult to handle static or dynamic nonlinearities, with the IQC framework forming the starting point [47]. Finally, it extends to the numerical search for non-quadratic Lyapunov or storage functions [52, 15, 18, 16, 17], if merging with the SOS techniques as considered in Section 7.2.

6 The general framework

Let us now turn again to the generic robust LMI formulation (2). If we define

$$F(\delta) = \frac{1}{2}\begin{pmatrix} A_0(\delta) \\ A_1(\delta) \\ \vdots \\ A_n(\delta) \end{pmatrix} \quad \text{and} \quad J(x) = \begin{pmatrix} 0 & 0 & \cdots & 0 & I \\ 0 & 0 & \cdots & 0 & x_1 I \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & x_n I \\ I & x_1 I & \cdots & x_n I & 0 \end{pmatrix},$$

it can be more compactly formulated as

$$p_{\text{opt}} = \inf \left\{ \langle c, x \rangle : \begin{pmatrix} I \\ F(\delta) \end{pmatrix}' J(x) \begin{pmatrix} I \\ F(\delta) \end{pmatrix} \prec 0 \ \text{ for all } \ \delta \in \boldsymbol{\delta} \right\}. \tag{41}$$

Let us stress that, from now on, it is not required that the decision variable x is contained in $\mathbb{R}^n$ or that $J(x)$ has the above given particular structure.

6.1 Assumptions

The decision variable x belongs to some finite-dimensional real Hilbert space $\mathcal{X}$, while $\delta \in \boldsymbol{\delta}$ is the uncertainty variable that can take its values in some subset $\boldsymbol{\delta}$ of a linear space of matrices. The performance index $J(x)$ is Hermitian-valued, and it can be decomposed as

$$J(x) = J_0 + J_l(x) \ \text{ with real-linear } \ J_l(x) \ \text{ in the decision variable } \ x \in \mathcal{X}.$$

We assume that $F(\delta)$ admits the LFR

$$F(\delta) = D + C\Delta(\delta)(I - A\Delta(\delta))^{-1}B \ \text{ with some linear } \ \Delta(\delta). \tag{42}$$

Due to their intimate relation to system interconnections, we recall that this is a rather natural hypothesis in robust control. In complete generality, LFR's can be constructed in case that $F(\delta)$ is rational and has no pole at zero [25, 70, 78]. Let us finally assume from now on that

$$\det(I - A\Delta(\delta)) \neq 0 \ \text{ for all } \ \delta \in \boldsymbol{\delta} \quad \text{and} \quad p_{\text{opt}} < \infty.$$

Remark 14 *The first property means that the LFR (42) is well-posed on $\boldsymbol{\delta}$, and literally the same remarks as in Section 3.8 apply for testing it in our general situation.*

Remark 15 *The second property means that (41) is feasible. We will exploit in the proof of Theorem 16 that $\prec$ can be replaced by $\preceq$ without changing the optimal value, based on the same reasoning as for standard convex programs (Remark 2). Moreover, if $\boldsymbol{\delta}$ is compact, feasibility of (41) is easily seen to be equivalent to*

$$\inf\left\{t \in \mathbb{R} : \begin{pmatrix} I \\ F(\delta) \end{pmatrix}' J(x) \begin{pmatrix} I \\ F(\delta) \end{pmatrix} \prec tI \ \textit{ for all } \ \delta \in \boldsymbol{\delta}\right\} < 0. \tag{43}$$

As in Section 2.5, testing feasibility of (41) can hence be related to solving this auxiliary (and always feasible) robust LMI problem.

6.2 Lower bounds

It is not difficult to compute lower bounds of p_{opt} as follows: choose finitely many uncertainties $\delta^1, \ldots, \delta^q \in \boldsymbol{\delta}$ and solve the standard SDP

$$p_{\text{lb}} = \inf\left\{\langle c, x\rangle : \ x \in \mathcal{X}, \ \begin{pmatrix} I \\ F(\delta^\nu) \end{pmatrix}' J(x) \begin{pmatrix} I \\ F(\delta^\nu) \end{pmatrix} \prec 0, \ \nu = 1, \ldots, q\right\}. \tag{44}$$

It is a rather straightforward consequence of Helly's theorem in convex analysis that it suffices to take at most $\dim(\mathcal{X})$ uncertainties in order to guarantee that the lower bound is (approximately) exact [14].

Theorem 16 *In general $p_{lb} \leq p_{opt}$. Suppose that there exists some $\delta \in \boldsymbol{\delta}$ for which*

$$X_\delta := \left\{x \in \mathcal{X} : \begin{pmatrix} I \\ F(\delta) \end{pmatrix}' J(x) \begin{pmatrix} I \\ F(\delta) \end{pmatrix} \preceq 0\right\} \quad \textit{is bounded.} \tag{45}$$

Then, for each $\epsilon > 0$ there exist at most $\dim(\mathcal{X})$ uncertainties such that

$$p_{opt} - \epsilon \leq p_{lb}. \tag{46}$$

If $\boldsymbol{\delta}$ is compact, there exist at most $\dim(\mathcal{X})$ uncertainties with $p_{lb} = p_{opt}$.

Remark 17 *From a practical point of view, the additional hypotheses are not restrictive. In particular, robust LMI problems typically comprise nominal constraints which bound the decision variables such that (45) is actually true for all uncertainties.*

Proof. The first part is shown by contradiction. Let us abbreviate $n = \dim(\mathcal{X})$, and assume that there exists some $\epsilon > 0$ such that

$$p_{\rm lb} < p_{\rm opt} - \epsilon \ \ \text{for all} \ \ \delta^1, \ldots, \delta^n \in \boldsymbol{\delta}. \tag{47}$$

Consider the family of closed convex sets

$$\hat{X} := \{x \in \mathcal{X} : \ \langle c, x\rangle \leq p_{\rm opt} - \epsilon\} \ \ \text{and} \ \ X_\delta \ \ \text{for} \ \ \delta \in \boldsymbol{\delta}.$$

At least one of these sets is bounded by hypothesis. Moreover, if we choose $n+1$ arbitrary sets from this family, they have nonempty intersection; if $\hat{X}$ is not among them, this follows from feasibility of (41); if $\hat{X}$ is among them, it follows from (47). By Helly's theorem [71], the intersection $\hat{X} \cap \bigcap_{\delta\in\boldsymbol{\delta}} X_\delta$ is nonempty which implies $p_{\rm opt} \leq p_{\rm opt} - \epsilon$, a contradiction.

The proof of the second part is routine. We can choose a sequence of n uncertainties $(\delta^1(j), \ldots, \delta^n(j))$ such that the corresponding optimal value $p_{\rm lb}^j$ of (44) satisfies $p_{\rm opt} - 1/j \leq p_{\rm lb}^j$. By compactness of $\boldsymbol{\delta}$, we can assume $(\delta^1(j), \ldots, \delta^n(j)) \to (\delta^1, \ldots, \delta^n)$. Now consider (44) for $(\delta^1, \ldots, \delta^n)$ with optimal value $p_{\rm lb}$, and suppose x is a feasible point. Since x is also feasible for (44) with $(\delta^1(j), \ldots, \delta^n(j))$ and all large j, we infer $p_{\rm opt} - 1/j \leq p_{\rm lb}^j \leq \langle c, x\rangle$ for all large j, and hence $p_{\rm opt} \leq \langle c, x\rangle$. This allows to conclude $p_{\rm opt} \leq p_{\rm lb}$. ■

6.3 Upper bounds

We have seen under various circumstances how to construct relaxations for computing an upper bound on $p_{\rm opt}$ as follows. Choose real-linear Hermitian-valued mappings $G(y)$ and $H(y)$ in the auxiliary variable $y \in \mathcal{Y}$ such that

$$y \in \mathcal{Y}, \ \ G(y) \preceq 0 \ \ \text{implies} \ \ \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}' H(y) \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} \succeq 0 \ \ \text{for all} \ \ \delta \in \boldsymbol{\delta}. \tag{48}$$

For any such pair G, H and with the abbreviations

$$U = \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} \ \ \text{and} \ \ V = \begin{pmatrix} 0 & I \\ C & D \end{pmatrix},$$

consider the LMI problem

$$p_{\rm rel} = \inf\left\{\langle c, x\rangle : \ x \in \mathcal{X}, \ y \in \mathcal{Y}, \ G(y) \prec 0, \ U'H(y)U + V'J(x)V \prec 0\right\}. \tag{49}$$

It is elementary to show that $p_{\rm opt} \leq p_{\rm rel}$, just as we did in Lemma 4 by exploiting the crucial identity (18). The upper bound is nontrivial if $p_{\rm rel} < \infty$, which means that (49) is feasible. This is assumed from now on.

6.4 Relation

The main purpose of this section is to show a close relation between the Lagrange duals of the SDP's for upper and lower bound computations. This prepares the next section on the verification of relaxation exactness.

Due to $p_{\rm opt} < \infty$, the SDP (44) is strictly feasible. Therefore it can be dualized without gap. The Lagrange multipliers are taken from the set

$$\mathcal{Z}_q = \{(Z_1, \ldots, Z_q) : \; Z_1 \succeq 0, \ldots, Z_q \succeq 0\}.$$

With the affine map

$$L(Z) = \sum_{\nu=1}^{q} \begin{pmatrix} \Delta(\delta^\nu)(I - A\Delta(\delta^\nu))^{-1}B \\ I \end{pmatrix} Z_\nu \begin{pmatrix} \Delta(\delta^\nu)(I - A\Delta(\delta^\nu))^{-1}B \\ I \end{pmatrix}',$$

we use (18) to see that the Lagrangian of (44) reads as

$$\langle c, x\rangle + \sum_{\nu=1}^{q} \langle Z_\nu, \begin{pmatrix} I \\ F(\delta_\nu) \end{pmatrix}' J(x) \begin{pmatrix} I \\ F(\delta_\nu) \end{pmatrix}\rangle = \langle c + J_l^*(VL(Z)V'), x\rangle + \langle VL(Z)V', J_0\rangle.$$

By strong duality, we conclude that

$$p_{\rm lb} = \max\{\langle VL(Z)V', J_0\rangle : \; Z \in \mathcal{Z}_q, \; c + J_l^*(VL(Z)V') = 0\}. \tag{50}$$

Let us now turn to the upper bound relaxation (49). With multipliers $M \succeq 0$, $N \succeq 0$, the corresponding Lagrangian reads as

$$\begin{aligned} &\langle c, x\rangle + \langle M, U'H(y)U + V'J(x)V\rangle + \langle N, G(y)\rangle = \\ &\qquad = \langle c + J_l^*(VMV'), x\rangle + \langle VMV', J_0\rangle + \langle H^*(UMU') + G^*(N), y\rangle. \end{aligned}$$

Since (49) is (strictly) feasible, strong duality implies

$$p_{\rm rel} = \max\left\{\langle VMV', J_0\rangle : M, N \succeq 0, \; c + J_l^*(VMV') = 0, \; H^*(UMU') + G^*(N) = 0\right\}. \tag{51}$$

The following rather easily established relation between the feasible sets of these duals confirms, again, the general relation $p_{\rm lb} \leq p_{\rm rel}$.

Lemma 18 *If $Z \in \mathcal{Z}_q$ is feasible for (50) there exists an $N \succeq 0$ such that the pair $(M, N) = (L(Z), N)$ is feasible for (51).*

Proof. If we fix $Z \in \mathcal{Z}_q$ which is feasible for (50), it suffices to show that there exists some $N \succeq 0$ with $H^*(UL(Z)U') + G^*(N) = 0$. Equivalently, we will prove

$$\min\{t \in \mathbb{R} : \; H^*(UL(Z)U') + G^*(N) = 0, \; N + tI \succeq 0\} \quad \leq \quad 0. \tag{52}$$

With multipliers y and $Y \succeq 0$, the Lagrangian is

$$\begin{aligned} &t - \langle Y, N + tI\rangle - \langle y, H^*(UL(Z)U') + G^*(N)\rangle = \\ &\qquad = t(1 - \langle Y, I\rangle) - \langle H(y), UL(Z)U'\rangle - \langle G(y) + Y, N\rangle. \end{aligned}$$

Therefore, the Lagrange dual of (52) reads as

$$\sup_{y,\, Y \succeq 0,\, G(y)+Y=0,\, \langle Y, I\rangle = 1} -\langle H(y), UL(Z)U'\rangle = -\inf_{G(y) \preceq 0,\, \langle G(y), I\rangle = -1} \langle H(y), UL(Z)U'\rangle. \tag{53}$$

Since the latter problem is strictly feasible, the value of (52) is indeed attained. Now choose a feasible point y of (53). Since $G(y) \preceq 0$, we exploit (48) to infer

$$\begin{pmatrix} \Delta(\delta^\nu) \\ I \end{pmatrix}' H(y) \begin{pmatrix} \Delta(\delta^\nu) \\ I \end{pmatrix} \succeq 0 \text{ for all } \nu = 1, \ldots, q. \tag{54}$$

If we recall (18), we infer

$$UL(Z)U' = \sum_{\nu=1}^{q} \begin{pmatrix} \Delta(\delta^\nu) \\ I \end{pmatrix} [(I - A\Delta(\delta^\nu))^{-1}B] Z_\nu [(I - A\Delta(\delta^\nu))^{-1}B]' \begin{pmatrix} \Delta(\delta^\nu) \\ I \end{pmatrix}'. \tag{55}$$

We can combine (54), (55) and $Z_\nu \succeq 0$ to conclude $\langle H(y), UL(Z)U' \rangle \geq 0$. Therefore the value of (53) is non-positive, which indeed implies (52). ■

6.5 Characterization of relaxation exactness

We have seen that $p_{\rm lb} \leq p_{\rm opt} \leq p_{\rm rel}$. In practical computations it often happens (as has been already exploited in the examples appearing in this paper) that $p_{\rm opt} = p_{\rm rel}$. The present section serves to discuss how we can numerically detect whether the upper bound relaxation is indeed exact.

For this purpose, let us assume that there exist q (worst-case) uncertainties for which $p_{\rm lb} = p_{\rm opt}$. Under weak hypotheses, this is generally true for $q = \dim(\mathcal{X})$, as seen in Theorem 16, but it is often valid for $q < \dim(\mathcal{X})$, or even for $q = 1$.

Theorem 19 *$p_{opt} = p_{rel}$ iff the relaxation's dual (51) has an optimal solution (M, N) for which there exists $Z \in \mathcal{Z}_q$ with*

$$M = \sum_{\nu=1}^{q} \begin{pmatrix} \Delta(\delta^\nu)(I - A\Delta(\delta^\nu))^{-1}B \\ I \end{pmatrix} Z_\nu \begin{pmatrix} \Delta(\delta^\nu)(I - A\Delta(\delta^\nu))^{-1}B \\ I \end{pmatrix}'. \tag{56}$$

Proof. Suppose $p_{\rm opt} = p_{\rm rel}$. If $Z \in \mathcal{Z}_q$ is optimal for (50) we infer that $\langle VL(Z)V', J_0 \rangle = p_{\rm lb} = p_{\rm opt} = p_{\rm rel}$. By Lemma 18, there exists some $N \succeq 0$ such that $(M, N) = (L(Z), N)$ is feasible for (51); since the cost is $\langle VMV', J_0 \rangle = \langle VL(Z)V', J_0 \rangle = p_{\rm rel}$, this pair is actually optimal for (51). Conversely, suppose $M = L(Z)$ for some $Z \in \mathcal{Z}_q$. Since $(L(Z), N)$ is optimal for (51) with cost $p_{\rm rel}$, it is feasible for (50) with cost $p_{\rm rel}$; hence $p_{\rm lb} \geq p_{\rm rel}$ and therefore $p_{\rm lb} = p_{\rm opt} = p_{\rm rel}$. ■

This motivates the following procedure:

- Solve (49) with a primal-dual solver to determine $p_{\rm rel}$ and some optimal Lagrange multiplier M.
- Solve the equation (56) for $\delta^1, \ldots, \delta^q$ and $Z_1 \succeq 0, \ldots, Z_q \succeq 0$.
  - If (56) holds, then $\delta^1, \ldots, \delta^q$ is guaranteed to be a set of worst case uncertainties which reveal $p_{\rm lb} = p_{\rm opt}$ (as extracted from the given proof), and one has verified that $p_{\rm opt} = p_{\rm rel}$.

– If (56) is only true approximately, the computed $\delta^1, \ldots, \delta^q$ can still be used to determine the value $p_{\rm lb}$ of (44); this leads to a heuristic lower bound $p_{\rm lb} \leq p_{\rm opt}$, which allows to estimate the quality of the upper bound $p_{\rm rel}$ of $p_{\rm opt}$.

The second step requires to solve a nonlinear problem. We point out that a related question in the context of polynomial optimization has been addressed in [37]. If $\boldsymbol{\delta}$ admits an LMI representation, searching a solution $Z \in \mathcal{Z}_q$ and $\delta^1, \ldots, \delta^q$ with (56) reduces to a nonlinear semi-definite program. It is particularly appealing to observe that the to-be-solved SDP is actually *linear* if $q = 1$. Indeed, due to $M \succeq 0$ and (18), it is elementary to show that there exists some $\delta \in \boldsymbol{\delta}$ and $Z \succeq 0$ with

$$M = \begin{pmatrix} \Delta(\delta)(I - A\Delta(\delta))^{-1}B \\ I \end{pmatrix} Z \begin{pmatrix} \Delta(\delta)(I - A\Delta(\delta))^{-1}B \\ I \end{pmatrix}'$$

iff

$$\begin{pmatrix} I & -\Delta(\delta) \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} M = 0 \;\; \text{for some} \;\; \delta \in \boldsymbol{\delta}. \tag{57}$$

The consequence can be summarized as follows.

Corollary 20 *There exists one uncertainty such that $p_{lb} = p_{opt}$ and $p_{opt} = p_{rel}$ iff the upper bound relaxation has a dual optimal solution (M, N) for which (57) holds.*

After having computed M, exactness can then be tested by minimizing the norm of the left-hand side of the equation in (57) over $\boldsymbol{\delta}$, which is a standard LMI problem if $\boldsymbol{\delta}$ is described by LMI constraints. Under quite general circumstances, it can be shown that approximate solutions of the equation lead to approximate worst-case uncertainties. A detailed technical discussion can be found in [63]. Note that this procedure has been applied in Examples 11 and 12 in order to either verify exactness, or to determine the lower bounds as displayed in Figure 4.

Let us finally assume that the mappings G and H, which define the upper bound relaxation, have the following additional property: If w and z are complex vectors that satisfy

$$\begin{pmatrix} w \\ z \end{pmatrix}' H(y) \begin{pmatrix} w \\ z \end{pmatrix} \geq 0 \;\; \text{for all} \;\; y \;\; \text{with} \;\; G(y) \preceq 0,$$

then there exists some $\delta \in \boldsymbol{\delta}$ with $w = \Delta(\delta)z$. Note that this can be shown to be true for all upper bound relaxations discussed so far. The additional property implies that (57) is solvable - and hence $p_{\rm opt} = p_{\rm rel}$ - if the dual optimal multiplier M has rank one [63]. This relates the presented exactness characterization to the basic idea how to prove variants of the KYP Lemma (Section 3.4), and to a whole variety of other rank one principles in the literature (Section 3.5).

7 How to reduce conservatism?

So far we have discussed, with the exception of Section 3.7, rather well-established relaxations and how they can be subsumed to the framework of Section 6.3. In this final part, we address generic techniques for the systematic reduction of conservatism, with an emphasis on methods that can be actually shown to be asymptotically exact. This requires to take the particular description of $\boldsymbol{\delta}$ into account, which can range from polytopes that are either explicitly or implicitly described by their generators or in terms of scalar linear inequalities, to sets that are constrained by LMI's or even by polynomial matrix inequalities. As clarified throughout this paper, the crucial point is to determine inner approximations (subsets) of the set of all multipliers P which satisfy

$$E(\delta, P) := \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}' P \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} \succeq 0 \ \text{ for all } \ \delta \in \boldsymbol{\delta}. \tag{58}$$

Observe that $E(\delta, P)$ is a Hermitian-valued map, which is *linear* in P and polynomial of degree two in δ.

7.1 Pólya relaxation

Let us start with $\boldsymbol{\delta}$ being the explicitly described compact polytope $\text{co}\{\delta^1, \ldots, \delta^q\}$. Then (58) is equivalent to

$$E\left(\sum_{\nu=1}^{q} \lambda_\nu \delta^\nu, P\right) \succeq 0 \ \text{ for all } \ \lambda_\nu \geq 0 \ \text{ with } \ \sum_{\nu=1}^{q} \lambda_\nu = 1. \tag{59}$$

Since the polynomial involved in (59) is homogenous of degree two, its Taylor expansion around zero can be written as

$$E\left(\sum_{\nu=1}^{q} \lambda_\nu \delta^\nu, P\right) = \sum_{\alpha_1 + \cdots + \alpha_q = 2} \lambda_1^{\alpha_1} \cdots \lambda_q^{\alpha_q} \, \Lambda_{(\alpha_1, \ldots, \alpha_q)}(P),$$

where $\Lambda_{(\alpha_1, \ldots, \alpha_q)}(P)$ are Hermitian-valued and real-linear in P (and $\alpha_1, \ldots, \alpha_q$ are tacitly assumed to be non-negative integers). If P satisfies the LMI's

$$\Lambda_{(\alpha_1, \ldots, \alpha_q)}(P) \succeq 0 \ \text{ for all } \ \alpha_1 + \cdots + \alpha_q = 2, \tag{60}$$

it is obvious that (59) and hence also (58) are satisfied. If we compactly rewrite (60) as $G_0(P) \preceq 0$ and choose $H(P) = P$, we have constructed a relaxation as in Section 6.3 which computes an upper bound p^0_{rel} of p_{opt}. It is not difficult to prove that this relaxation is not worse (and often better) than the convex hull relaxation of Section 4.3.

In order to improve the quality of this relaxation, let us choose a non-negative integer d and observe that $(\lambda_1 + \cdots + \lambda_q)^d$ has value one on the standard simplex. Therefore, (59) is obviously equivalent to

$$(\lambda_1 + \cdots + \lambda_q)^d \, E\left(\sum_{\nu=1}^{q} \lambda_\nu \delta^\nu, P\right) \succeq 0 \ \text{ for all } \ \lambda_\nu \geq 0 \ \text{ with } \ \sum_{\nu=1}^{q} \lambda_\nu = 1. \tag{61}$$

As explained for $d = 0$, this is true if the matrix-coefficients of the involved polynomial of degree $d + 2$ are positive semi-definite, which can again be compactly expressed as $G_d(P) \preceq 0$ for some real-linear map $G_d(.)$. Note that $G_d(.)$ can be easily constructed recursively for $d = 0, 1, 2, \ldots$. If we choose again $H(P) = P$, we arrive at a whole family of relaxations, parameterized by $d = 0, 1, 2, \ldots$, for computing upper bounds $p^d_{\rm rel}$ on $p_{\rm opt}$ as in Section 6.3. Note that the number of variables of this relaxation is fixed, while the number of LMI constraints in $G_d(P) \preceq 0$ grows exponentially with d. In practice, it often happens for small values of d that $p^d_{\rm rel}$ is (approximately) equal to $p_{\rm opt}$, which can be detected with the procedure as described in Section 6.5. We stress that the rank-one principle has been shown to be applicable as well [63].

As the most essential benefit of the suggested construction, let us now clarify why the upper bound relaxation family is asymptotically exact:

$$\lim_{d\to\infty} p^d_{\rm rel} = p_{\rm opt}.$$

The key for convergence is the following matrix-extension of a classical result due to Pólya [22, 56, 63]: If (59) holds for $\succeq$ replaced by $\succ$, there exists a non-negative integer d such that all coefficients of the polynomial in (61) are positive definite. Note that an alternative relaxation with guaranteed asymptotic exactness for boxes $\boldsymbol{\delta}$ (based on an iterated application of the KYP Lemma) has been suggested in [10].

7.2 Matrix sum-of-squares decompositions

In order to prepare the construction of relaxations for implicitly described uncertainty sets, let us concentrate in this section on the question of how to verify whether a given Hermitian-valued function $S(\delta)$ is globally positive semi-definite. This is certainly true if it can be represented as

$$S(\delta) = T(\delta)' T(\delta) \tag{62}$$

with some (not necessarily square and typically tall) matrix-valued function $T(\delta)$. We then say that $S(\delta)$ is a sum-of-squares (SOS). This terminology is clearly motivated if $S(\delta)$ is scalar; then $T(\delta)$ is a column, and $S(\delta)$ equals the sum of the (absolute) squares of the components of $T(\delta)$. The precise relation between positivity and the existence of sum-of-squares is often discussed for real polynomial and rational functions, and motivated Hilbert to formulate his famous seventeenth problem [59]. In control, an SOS representation (with additional properties) is called spectral factorization, and a multitude of computational algorithms have been developed if $\delta \in \mathbb{R}$ or $\delta \in i\mathbb{R}$, and if $S(\delta)$ and $T(\delta)$ are polynomial or rational in δ. Only rather recently it has been observed that the SOS property can be reduced to an LMI problem even if δ is multivariable, with a large variety of applications in control [52, 15, 55, 36, 35].

A computational procedure for verifying whether $S(\delta)$ is SOS proceeds as follows: With scalar-valued basis functions $u_1(\delta), \ldots, u_r(\delta)$, one just searches for the coefficient matrices $X_1, \ldots, X_r$ in the expansion

$$T(\delta) = X_1 u_1(\delta) + \cdots + X_r u_r(\delta) = XU(\delta), \quad X = \begin{pmatrix} X_1 & \cdots & X_r \end{pmatrix}, \quad U(\delta) := \begin{pmatrix} Iu_1(\delta) \\ \vdots \\ Iu_r(\delta) \end{pmatrix}.$$

We say that $S(\delta)$ is SOS with respect to $U(\delta)$, if there exists a coefficient matrix X with

$$S(\delta) = [XU(\delta)]'[XU(\delta)] = U(\delta)'(X'X)U(\delta).$$

It is easy to see that the change of variables $Y = X'X$ convexifies.

Lemma 21 *$S(\delta)$ is SOS with respect to $U(\delta)$ iff there exists some Y with*

$$S(\delta) = U(\delta)'YU(\delta) \quad \textit{and} \quad Y \succeq 0. \tag{63}$$

Proof. We have shown that the existence of Y with (63) is necessary. If such a Y exists, we factorize it as $Y = X'X$ and reverse the arguments to prove sufficiency. ■

Note that (63) just involves one affine equation and one semi-definite inequality constraint, and is hence a standard LMI problem. In the literature, various techniques have been suggested how to explicitly parameterize all solutions of the linear equation in (63), which leads to seemingly different but conceptually equivalent LMI problems, possibly just related by Lagrange duality [52, 15, 53, 45, 44, 39].

Remark 22

- *We stress that these and the subsequent insights are in no way restricted to polynomial or rational functions in δ. For example, efficient numerical techniques concerning one-variable trigonometric polynomials are discussed in [60]. The suggested schemes actually apply to arbitrary so-called C^*-function algebras. Interesting results for polynomials in non-commutative variables can be found in [32, 34, 33].*
- *Since Lemma 21 holds for arbitrary $U(\delta)$, one can reflect a particular problem structure in the choice of this basis matrix $U(\delta)$ in order to reduce the size of the LMI problem (63). For scalar polynomials, nice algebraic results are provided in [29], while systematic techniques for matrix-valued SOS decompositions that exploit control theoretic structure remain an important topic for future research.*

Remark 23 *Since $\boldsymbol{\delta}$ is a subset of some finite-dimensional matrix space, it can be viewed (in a standard fashion via a parametrization using some finite basis) as a subset of $\mathbb{R}^p$ for some p. If $S(\delta)$ and $T(\delta)$ are polynomial matrices which satisfy (62), it is elementary to see that*

$$d_j(S) = 2d_j(T) \quad \textit{for all} \quad j = 1, \ldots, p.$$

Here we denote by $d_j(S)$ or $d_j(T)$ the largest exponent of δ_j in all monomials which are required to represent $S(\delta)$ or $T(\delta)$ respectively. Therefore, in the numerical search for $T(\delta)$, a generic choice for $u_1(\delta), \ldots, u_r(\delta)$ is the list of all monomials which satisfy $2d_j(u_\nu) \leq d_j(S)$ for $j = 1, \ldots, p$, $\nu = 1, \ldots, r$. Newton-polytope techniques allow a priori reductions for improved computational complexity [59].

7.3 Implicitly described polytopes and semi-algebraic sets

Sum-of-squares decompositions guarantee global positivity, while we are rather interested in positivity on subsets $\boldsymbol{\delta}$. In contrast to the Pólya relaxation, let us now discuss how to guarantee (58) if $\boldsymbol{\delta}$ is implicitly described as

$$\boldsymbol{\delta} = \{\delta : \ g_1(\delta) \leq 0, \ldots, g_r(\delta) \leq 0\}$$

where $g_1(\delta), \ldots, g_r(\delta)$ are real-valued and real-linear. From now on we assume that $\boldsymbol{\delta} \subset \mathbb{R}^p$ (Remark 23), and that all polynomials have real coefficients.

It is a straightforward consequence of weak Lagrange duality that (58) holds, if there exist positive semi-definite Lagrange multiplier matrices $S_1, \ldots, S_r$ such that

$$E(\delta, P) + S_1 g_1(\delta) + \cdots + S_r g_r(\delta) \text{ is SOS.}$$

Indeed, $E(\delta, P) \succeq -S_1 g_1(\delta) - \cdots - S_r g_r(\delta)$ since SOS matrices are globally positive semi-definite; if $\delta \in \boldsymbol{\delta}$ we can exploit $g_1(\delta) \leq 0, \ldots, g_r(\delta) \leq 0$ to infer $E(\delta, P) \succeq 0$. Due to non-convexity, it cannot be expected that this sufficient condition is necessary as well. This motivates to improve the sufficient condition by allowing for more freedom in the Lagrange multipliers, and take them as *polynomial* functions in δ. In order to make sure that they are globally positive semi-definite, we take multiplier polynomials $S_1(\delta), \ldots, S_r(\delta)$ which are SOS. Again by weak Lagrange duality, (58) holds if there exist SOS matrices $S_1(\delta), \ldots, S_r(\delta)$ such that

$$E(\delta, P) + S_1(\delta) g_1(\delta) + \cdots + S_r(\delta) g_r(\delta) \text{ is SOS.} \tag{64}$$

If $\boldsymbol{\delta}$ is a compact polytope, this latter condition turns out to be necessary for the strict version of (58) [64]. With a suitable constraint qualification, this is even true for polynomial constraint functions which means that $\boldsymbol{\delta}$ is semi-algebraic [65].

Theorem 24 *Suppose that (58) holds for $\succ$ replacing $\succeq$. Moreover, let either one of the following constraint qualifications be satisfied:*

- $g_1(\delta), \ldots, g_r(\delta)$ *are real-affine and* $\boldsymbol{\delta}$ *is compact.*
- $g_1(\delta), \ldots, g_r(\delta)$ *are polynomials for which there exist some* ρ *and SOS polynomials* $s_1(\delta), \ldots, s_r(\delta)$ *such that* $\rho^2 - \|\delta\|^2 + s_1(\delta) g_1(\delta) + \cdots + s_r(\delta) g_r(\delta)$ *is SOS.*

Then there exist SOS matrices $S_1(\delta), \ldots, S_r(\delta)$ *with (64).*

Remark 25 *This result extends a fundamental theorem of Putinar [57] from scalar polynomials to polynomial matrices. If the semi-algebraic set* $\boldsymbol{\delta}$ *is known to be contained in a ball of radius* ρ *around zero, we can add one redundant constraint with* $g_{r+1}(\delta) = \|\delta\|^2 - \rho^2$ *in order to enforce the validity of the constraint qualification.*

Let us now reveal how to construct computable relaxations. For this purpose, we just restrict all $S_\nu(\delta)$ to be SOS with respect to some a priori fixed basis matrices $U_\nu(\delta)$ (Lemma 21). Clearly, (58) holds if there exist $Y_0 \succeq 0,\ Y_1 \succeq 0, \ldots, Y_r \succeq 0$ such that

$$E(\delta, P) + [U_1(\delta)' Y_1 U_1(\delta)]\, g_1(\delta) + \cdots + [U_r(\delta)' Y_r U_r(\delta)]\, g_r(\delta) = U_0(\delta)' Y_0 U_0(\delta). \tag{65}$$

If we denote by $\mathcal{Y}$ the linear space of all $y = (P, Y_0, Y_1, \ldots, Y_r)$ which satisfy the linear equation (65), and if we define the linear maps

$$G(y) = -\operatorname{diag}(Y_0, Y_1, \ldots, Y_r) \quad \text{and} \quad H(y) = P,$$

we have constructed a relaxation that indeed fits with the general scheme of Section 6.3. For reliable numerical implementations, let us stress the importance to choose the basis matrices in order to make sure that there exists at least one $y_0 \in \mathcal{Y}$ with $G(y_0) \prec 0$.

In this fashion, we have again constructed a whole family of relaxations parameterized by $U_0(\delta), U_1(\delta), \ldots, U_r(\delta)$. A generic choice can be made as follows. Take some sequence

$$u_1(\delta) = 1,\ u_2(\delta),\ u_3(\delta),\ u_4(\delta),\ \ldots$$

of pairwise different monomials (or linearly independent polynomials) which span the space of all scalar polynomials in δ. Fix

$$d_0, d_1, \ldots, d_r \in \mathbb{N} \ \text{ and define } \ U_\nu(\delta) := \mathrm{col}(I u_1(\delta), \ldots, I u_{d_\nu}(\delta)) \ \text{ for } \ \nu = 0, 1, \ldots, r.$$

(In practice d_0 will depend on $d_1, \ldots, d_r$ as observed in Remark 23.) The resulting relaxation defines an upper bound $p_{\mathrm{rel}}(d_0, d_1, \ldots, d_r)$ on p_{opt}. Under constraint qualifications, Theorem 24 allows to easily show that the relaxation family is asymptotically exact:

$$\lim_{d_0 \to \infty, \ldots, d_r \to \infty} p_{\mathrm{rel}}(d_0, d_1, \ldots, d_r) = p_{\mathrm{opt}}.$$

Example 26 *Consider the system (35) with*

$$\left(\begin{array}{cc} A & B(\delta_1, \delta_2) \\ C & D \end{array} \right) = \left(\begin{array}{cc|c} -2 & 1 & -3\delta_2 \frac{(2a^2+a)\delta_1^2\delta_2 + 2a\delta_1^2 + 2a\delta_2 + \delta_1^2 - 2}{2 - 2a^2\delta_2^2 - \delta_1^2 + a^2\delta_1^2\delta_2^2} \\ 0 & -2 & 1 \\ \hline -1 & 0 & 0 \end{array} \right), \quad a \in [0, 1],$$

and $\boldsymbol{\delta} = [-0.7, 0.8] \times [-0.7, 0.8]$*, while the rate-of-variation is not constrained. An upper bound on the* L_2*-gain of* $w \to z$ *is computed by using relaxations to solve (39) with a parameter-independent* X*. Moreover, we extract some* δ_* *as described in Section 6.5 and compute* $\|C(sI - A)B(\delta_*) + D\|_\infty$*, which is a lower bound on the worst-case* L_2*-gain for time-invariant parameters.*

Let us apply the convex-hull relaxation (Section 4.3), the Pólya relaxation with degrees $d = 0, 2, 4$ *(Section 7.1), and the SOS relaxation of the present section; for the latter we describe the uncertainty box by two quadratic inequalities and choose the corresponding SOS multiplier matrices of degree two. Figure 5 depicts the computed upper and lower bounds plotted versus the parameter* $a \in [0, 1]$*. For* $a \in [0, 0.25]$*, the bounds based on the convex hull relaxation coincide. Therefore, the computed value is guaranteed to be not conservative and hence exact (for both arbitrary fast and constant uncertainties). On the other hand, for* $a \in [0.4, 1]$*, the relaxation gap is very large, while it remains unclear from this computation whether it is caused by conservatism of the relaxation or by the intrinsic gap between arbitrary fast and constant uncertainties. Already the Pólya relaxation for* $d = 0$ *drastically reduces the gap and even reveals exactness up to* $a \in [0.3, 0.77]$*, with further improvements for* $d = 2, 4$*. The SOS relaxation reveals exactness up to* $a \in [0.34, 0.4]$*, which underpins that there is no intrinsic gap between fast and slow parametric uncertainties for most values of* a*.*

7.4 Uncertainty sets with LMI representations

For example full block uncertainties in robustness analysis require the inclusion of uncertainty sets with an LMI description. This just means that

$$\boldsymbol{\delta} = \{\delta \in \mathbb{R}^p : \ G_1(\delta) \preceq 0, \ldots, G_r(\delta) \preceq 0\},$$

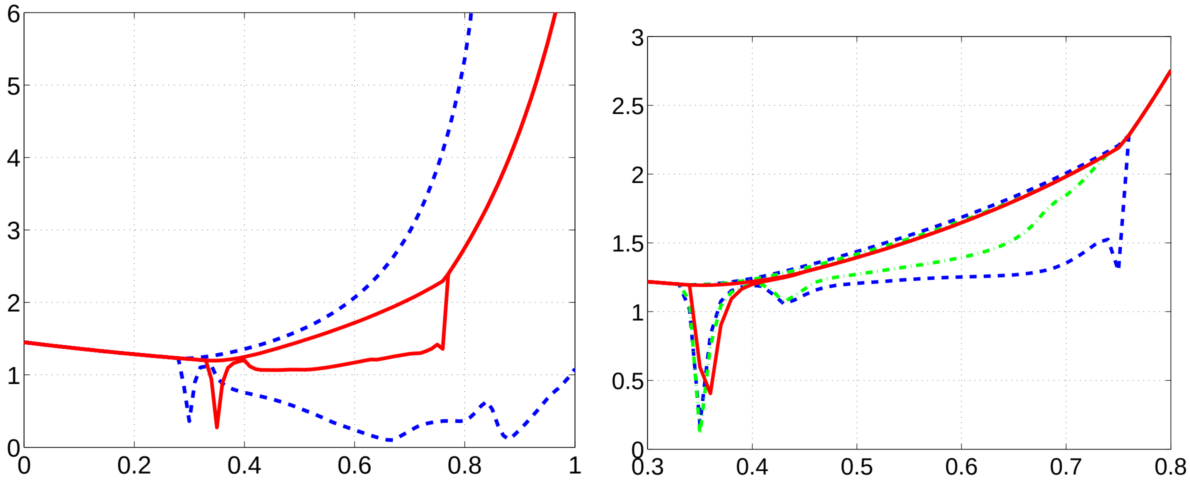

Figure 5: L_2-gain upper and lower bounds. Left: Convex-hull relaxation (dashed) and Pólya relaxation with $d = 0$ (full). Right: Pólya relaxation for $d = 2$ (dashed), $d = 4$ (dash-dotted) and SOS relaxation (full).

where $G_1(\delta), \ldots, G_r(\delta)$ are linear symmetric-valued mappings. In Section 3.7, we have already provided the key tool for extending Lagrange relaxations from scalar to matrix-valued constraints.

Suppose $E(\delta, P)$ has dimension $e \times e$. Due to Lemma 9, property (58) is guaranteed if there exist SOS matrices $S_0(\delta), S_1(\delta), \ldots, S_r(\delta)$ with

$$E(\delta, P) + (S_1(\delta), G_1(\delta))_e + \cdots + (S_r(\delta), G_r(\delta))_e = S_0(\delta). \tag{66}$$

This condition can be used to systematically construct relaxation families in literally the same way as for scalar-valued constraints in the previous section. Since all these relaxations can be subsumed to the framework of Section 6, it is in particular possible to verify exactness numerically. Moreover, due to the following extension of Theorem 24, also asymptotic exactness is guaranteed, even if $G_1(\delta), \ldots, G_r(\delta)$ are polynomial in δ [65].

Theorem 27 *Suppose that the symmetric-valued polynomial matrices $G_1(\delta), \ldots, G_r(\delta)$ satisfy the following constraint qualification: There exist some real ρ and SOS matrices $S_1(\delta), \ldots, S_r(\delta)$ such that*

$$\rho^2 - \|\delta\|^2 + \langle S_1(\delta), G_1(\delta)\rangle + \cdots + \langle S_r(\delta), G_r(\delta)\rangle \quad \text{is SOS.}$$

Then the strict version of (58) implies that there exist SOS matrices $S_0(\delta), S_1(\delta), \ldots, S_r(\delta)$ which satisfy (66).

7.5 General uncertain polynomial programs

Let us finally remark that both Theorems 24 and 27 are in no way depending on the particular structure of the polynomial matrix that is involved in (58). Therefore, the suggested relaxation strategy easily extends to the following general robust polynomial semi-definite program [65]:

$$\begin{array}{rl} \text{infimize} & \langle c, x\rangle \\ \text{subject to} & F(x,\delta) \succ 0 \ \text{ for all } \ \delta \ \text{ with } \ G_1(\delta) \preceq 0, \ldots, G_r(\delta) \preceq 0. \end{array} \tag{67}$$

Here $F(x,\delta)$ and $G_1(\delta),\ldots,G_r(\delta)$ are symmetric-valued, with affine dependence on $x \in \mathcal{X}$ and polynomial dependence on δ. If $f(\delta)$ is a scalar polynomial, this formulation comprises polynomial semi-definite programs [44, 38]

$$\begin{array}{rl} \text{supremize} & f(\delta) \\ \text{subject to} & G_1(\delta) \preceq 0, \ldots, G_r(\delta) \preceq 0 \end{array} \tag{68}$$

as a particular case: Just choose $x \in \mathbb{R}$, $F(x,\delta) = x - f(\delta)$ and $c = 1$. If $G_1(\delta),\ldots,G_r(\delta)$ are scalar-valued, we recover the problems addressed in [45, 53, 66].

Let $F(x,\delta)$ have dimension $e \times e$ and consider

$$\begin{array}{rl} \text{infimize} & \langle c, x\rangle \\ \text{subject to} & \epsilon > 0, \ F(x,\delta) + \sum_{\nu=1}^{r} (S_\nu(\delta), G_\nu(\delta))_e - \epsilon I = S_0(\delta), \end{array} \tag{69}$$

where $S_0(\delta), S_1(\delta), \ldots, S_r(\delta)$ vary over all SOS matrices. In view of our preparations, it is now routine how to construct relaxations families by just constraining to matrices that are SOS with respect to given basis matrices, and it is as well elementary to see that the value of all these relaxations provide upper bounds on (67). Under the constraint qualification of Theorem 27, the value of (67) actually equals (69), which can be used to show asymptotic exactness for suitable chosen sequences of basis matrices as in Section 7.3.

8 Conclusions

Starting from classical frequency domain inequalities, we have highlighted the prominent role of robust linear matrix inequalities in robust control. Particular emphasis has been laid on the systematic construction of relaxations within a unifying framework that allows to numerically check exactness. Under rather generic circumstances, we were able to even design relaxation families that can be shown to be asymptotically exact. As a major topic for future research, it remains to understand how to reflect system-theoretic problem structure in these relaxations families in order to keep the computational complexity manageable, even if addressing large-sized problems of industrial relevance.

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