A sum-of-squares approach to fixed-order $\boldsymbol{H}_\infty$-synthesis

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1 Introduction

Recent improvements of semi-definite programming solvers and developments on polynomial optimization have resulted in a large increase of the research activity on the application of the so-called sum-of-squares (SOS) technique in control. In this approach non-convex polynomial optimization programs are approximated by a family of convex problems that are relaxations of the original program [4, 22]. These relaxations are based on decompositions of certain polynomials into a sum of squares. Using a theorem of Putinar [28] it can be shown (under suitable constraint qualifications) that the optimal values of these relaxed problems converge to the optimal value of the original problem. These relaxation schemes have recently been applied to various non-convex problems in control such as Lyapunov stability of nonlinear dynamic systems [25, 5] and robust stability analysis [15].

In this work we apply these techniques to the fixed order or structured $\mathcal{H}_\infty$-synthesis problem. $\mathcal{H}_\infty$-controller synthesis is an attractive model-based control design tool which allows incorporation of modeling uncertainties in control design. We concentrate on $\mathcal{H}_\infty$-synthesis although the method can be applied to other performance specifications that admit a representation in terms of Linear Matrix Inequalities (LMI's). It is well-known that an $\mathcal{H}_\infty$-optimal full order controller can be computed by solving two algebraic Riccati equations [7]. However, the fixed order $\mathcal{H}_\infty$-synthesis problem is much more difficult. In fact it is one of the most important open problems in control engineering, in the sense that until now there do not yet exist fast and reliable methods to compute optimal fixed order controllers. As the basic setup we consider the closed-loop interconnection as shown below, where the linear system P is the generalized plant and K is a linear controller.

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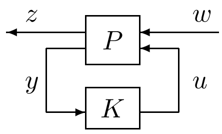

Given P, we want to find a controller K of a given order n_c (independent of that of P) such that the closed-loop interconnection is internally (asymptotically) stable and such that the $\mathcal{H}_\infty$-norm of the closed-loop transfer function from w to z is minimized.

The resulting optimization problem is non-convex and difficult to solve. Various approaches have been presented in the literature based on sequential solution of LMI's [9, 2, 8, 20], nonlinear Semi-Definite Programs (SDP's) [23, 1, 18], Branch and Bound methods [35] and, more recently, polynomial optimization using SOS [14]. In this latter method a so-called central polynomial is a priori chosen such that a sufficient condition for the $\mathcal{H}_\infty$-performance can be formulated in terms of a positivity test of two matrix polynomials with coefficients that depend linearly on the controller variables. This linear dependence allows to find the variables satisfying these positivity constraints by LMI optimization. The method however depends crucially on the choice of this central polynomial which is in general difficult. Furthermore this technique cannot be straightforwardly extended to MIMO (Multiple Input Multiple Output) $\mathcal{H}_\infty$-optimal controller synthesis. In contrast, the method to be discussed below can directly be applied to MIMO synthesis.

The main result of this paper is the construction of a sequence of SOS polynomial relaxations

- that require the solution of LMI problems whose size grows only quadratically in the number of states and
- whose optimal value converge from below to the fixed-order $\mathcal{H}_\infty$ performance.

The computation of *lower bounds* allows to add a stopping criterion to the algorithms mentioned above with a guaranteed bound on the difference of the performance of the computed controller and the optimal fixed order $\mathcal{H}_\infty$ performance. This is important since, except for the branch and bound method, these algorithms can in general not guarantee convergence to the globally optimal solution.

A trivial lower bound on the fixed order performance is, of course, the optimal performance achievable with a controller of the same order as the plant. Boyd and Vandenberghe [3] proposed lower bounds based on convex relaxations of the fixed order synthesis problem. These lower bounds cannot be straightforwardly improved to reduce the gap to the optimal fixed order performance. For our sequence of relaxations this gap is systematically reduced to zero.

After the problem formulation in Section 2, we show in Section 3 how a suitable matrix sum-of-squares relaxation technique can be directly applied to the non-convex semi-definite optimization problem resulting from the bounded real lemma. We will prove that the values of these relaxations converge to the optimal value. Although this convergence property is well-known for polynomial problems with scalar constraints [22], to the best of our knowledge this result is new for matrix-valued inequalities. (During the writing of this paper we became aware of the independent recent work of Kojima [24], that presents the same result with a different proof). This convergence result is of value for a variety of matrix-valued optimization problems. Examples in control are input-output selection, where the integer constraints of type $p \in \{0, 1\}$ are replaced by a quadratic constraint $p(p-1) = 0$, and spectral factorization of multidimensional transfer functions to asses dissipativity of linear shift-invariant distributed systems [26]. Here, our goal is to apply it to the fixed order $\mathcal{H}_\infty$ synthesis problem. Unfortunately, for plants with high state-dimension this direct technique leads to an unacceptable complexity. As the main reason, the resulting relaxations involve the search for an SOS polynomial in *all* variables, including the Lyapunov matrix in the bounded real lemma inequality constraint. Therefore, the size of the LMI relaxations grows exponentially in the number of state-variables.

In Section 4 we describe how to overcome this deficiency by constructing a relaxation scheme without the exponential growth in the state dimension through two-fold sequential dualization. First we dualize in the variables that grow with the state dimension, which leads to the re-formulation of the fixed order synthesis problem as a robust analysis problem with the controller variables as parametric uncertainty. On the one hand, this allows to apply the wide spectrum of robust analysis techniques to the fixed order controller design problem. On the other hand, robustness has to be verified only with respect to the small number of controller parameters which is the essence of keeping the growth of the relaxation size in the state-dimension polynomial.

The purpose of Section 5 is to discuss a novel approach to solve robust LMI problems based on SOS matrices, including a direct and compact description of the resulting linear SDP's with full flexibility in the choice of the underlying monomial basis. This leads to an asymptotically exact family of LMI relaxations for computing lower bounds on the optimal fixed-order $\mathcal{H}_\infty$-norm whose size only grows quadratically in the dimension of the system state. We will reveal as well that existing results based on straightforward scalarization techniques fail to guarantee these growth properties.

Our technique is appropriate for the design of controllers with a few decision variables (such as Proportional Integral Derivative (PID) controllers) for plants with moderate Mc-Millan degree. In Section 6 we apply the method to the fixed order $\mathcal{H}_\infty$-synthesis problem on two systems: an academic model with Mc-Millan degree 4 and a model of an active suspension system which has a generalized plant of a Mc-Millan degree 27.

2 Fixed-order H_∞ controller synthesis

Consider the $\mathcal{H}_\infty$-reduced order synthesis problem with a closed-loop system described by $A(p)$, $B(p)$, $C(p)$ and $D(p)$, where p parameterizes the to-be-constructed controller and varies in the compact set $\mathcal{P}$. Compactness can, for instance, be realized by restricting the controller variables to a Euclidean ball

$$\mathcal{P} := \{p \in \mathbb{R}^{n_p} \mid \|p\| \leq M\}. \tag{1}$$

In practice, most structured control problems have large state dimension and few controller variables that enter affinely in the closed-loop state space description. Suppose the generalized plant of order n admits the state space description

$$\begin{pmatrix} \dot{x} \\ z \\ y \end{pmatrix} = \left(\begin{array}{c|cc} A^{\mathrm{ol}} & B_1^{\mathrm{ol}} & B_2^{\mathrm{ol}} \\ \hline C_1^{\mathrm{ol}} & D_{11}^{\mathrm{ol}} & D_{12}^{\mathrm{ol}} \\ C_2^{\mathrm{ol}} & D_{21}^{\mathrm{ol}} & 0 \end{array}\right) \begin{pmatrix} x \\ w \\ u \end{pmatrix},$$

where $(\cdot)^{\mathrm{ol}}$ stands for 'open loop' and $A^{\mathrm{ol}} \in \mathbb{R}^{n\times n}$, $B_1^{\mathrm{ol}} \in \mathbb{R}^{n\times m_1}$, $B_2^{\mathrm{ol}} \in \mathbb{R}^{n\times m_2}$, $C_1^{\mathrm{ol}} \in \mathbb{R}^{p_1\times n}$, $C_2^{\mathrm{ol}} \in \mathbb{R}^{p_2\times n}$, $D_{11}^{\mathrm{ol}} \in \mathbb{R}^{p_1\times m_1}$, $D_{12}^{\mathrm{ol}} \in \mathbb{R}^{p_1\times m_2}$ and $D_{21}^{\mathrm{ol}} \in \mathbb{R}^{p_2\times m_1}$. For simplicity we assume the direct-feedthrough term D_{22}^{ol} to be zero. In Remark 7 we will discuss how our method can be applied to control problems with nonzero D_{22}^{ol}. Now consider, for instance, a PID-controller described by

$$k(p) = p_1 + p_2\frac{1}{s} + p_3\frac{s}{\tau s+1},$$

which admits the state space realization

$$\left(\begin{array}{c|c} A_K(p) & B_K(p) \\ \hline C_K(p) & D_K(p) \end{array}\right) := \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & -\frac{1}{\tau} & \frac{1}{\tau} \\ \hline p_2 & -\frac{p_3}{\tau} & p_1 + \frac{p_3}{\tau} \end{array}\right)$$

and suppose that we want to find the optimal proportional, integral and derivative gains p_1, p_2 and p_3 respectively. This structure has been used by Ibaraki and Tomizuka [19] for $\mathcal{H}_\infty$-optimal PID tuning of a hard-disk drive using the cone complementarity method [9]. See also Grassi and Tsakalis [11, 12] and Grimble and Johnson [13] for $\mathcal{H}_\infty$ and LQG-optimal PID-tuning respectively. Interconnecting the plant with the PID-controller yields a closed-loop state-space representation with matrices

$$\left(\begin{array}{c|c} A(p) & B(p) \\ \hline C(p) & D(p) \end{array}\right) = \left(\begin{array}{cc|c} A^{\mathrm{ol}} + B_2^{\mathrm{ol}} D_K(p) C_2^{\mathrm{ol}} & B_2^{\mathrm{ol}} C_K(p) & B_1^{\mathrm{ol}} + B_2^{\mathrm{ol}} D_K(p) D_{21}^{\mathrm{ol}} \\ B_K(p) C_2^{\mathrm{ol}} & A_K(p) & B_K(p) D_{21}^{\mathrm{ol}} \\ \hline C_1^{\mathrm{ol}} + D_{12}^{\mathrm{ol}} D_K(p) C_2^{\mathrm{ol}} & D_{12}^{\mathrm{ol}} C_K(p) & D_{11}^{\mathrm{ol}} + D_{12}^{\mathrm{ol}} D_K(p) D_{21}^{\mathrm{ol}} \end{array}\right)$$

which depend affinely on p. We intend to solve the fixed order $\mathcal{H}_\infty$-synthesis problem

$$\inf_{p\in\mathcal{P},\, A(p)\ \mathrm{stable}} \|D(p) + C(p)(sI - A(p))^{-1}B(p)\|_\infty^2.$$

Due to the bounded real lemma we can solve instead the equivalent problem

$$\begin{array}{rl} \text{infimize} & \gamma \\ \text{subject to} & p \in \mathcal{P}, \quad X \in \mathcal{S}^n, \quad X \succ 0 \\ & B_\infty(X,p,\gamma) \prec 0 \end{array} \tag{2}$$

where $\mathcal{S}^n$ denotes the set of real symmetric $n \times n$ matrices and

$$B_\infty(X,p,\gamma) := \begin{pmatrix} A(p)^T X + XA(p) & XB(p) \\ B(p)^T X & -\gamma I \end{pmatrix} + \begin{pmatrix} C(p)^T \\ D(p)^T \end{pmatrix} \begin{pmatrix} C(p) & D(p) \end{pmatrix}.$$

Let t^p_{opt} denote the optimal value of (primal) problem (2). Due to the bilinear coupling of X and p, this problem is a non-convex Bilinear Matrix Inequality problem (BMI). The number of bilinearly coupled variables is $\frac{1}{2}(n+n_c)(n+n_c+1)+n_p$ (with n_p denoting the number of free controller variables) which grows quadratically with n. However it has a linear objective and matrix valued polynomial constraints. It will be shown in the next section that we can compute a sequence of SOS relaxations of this problem that converges from below to the optimal value.

3 A direct polynomial SDP-approach

In this section we present an extension of the scalar polynomial optimization by SOS decompositions [22] to optimization problems with scalar polynomial objective and nonlinear polynomial semi-definite constraints. We formulate the relaxations in terms of Lagrange duality with SOS polynomials as multipliers which seems a bit more straightforward than the corresponding dual formulation based on the problem of moments [22].

3.1 Polynomial semi-definite programming

For $x \in \mathbb{R}^n$ let $f(x)$ and $G(x)$ denote scalar and symmetric-matrix-valued polynomials in x, and consider the following polynomial semi-definite optimization problem with optimal value d_{opt}:

$$\begin{array}{rl} \text{infimize} & f(x) \\ \text{subject to} & G(x) \preceq 0 \end{array} \tag{3}$$

Since multiple SDP-constraints can be easily collected into one single SDP-constraint by diagonal augmentation, it is clear that (2) is captured by this general formulation.

With any matrix $S \succeq 0$, the value $\inf_{x\in\mathbb{R}^{n_x}} f(x) + \langle S, G(x)\rangle$ is a lower bound for d_{opt} by standard weak duality. However, not even the maximization of this lower bound over $S \succeq 0$ allows to close the duality gap due to non-convexity of the problem. This is the reason for considering, instead, Lagrange multiplier matrices $S(x) \succeq 0$ which are polynomial functions of x. Still

$\inf_{x\in\mathbb{R}^{n_x}} f(x) + \langle S(x), G(x)\rangle$ defines a lower bound of d_{opt}, and the best lower bound that is achievable in this fashion is given by the supremal t for which there exists a polynomial matrix $S(x) \succeq 0$ such that

$$f(x) + \langle S(x), G(x)\rangle - t > 0 \quad \text{for all} \quad x \in \mathbb{R}^{n_x}.$$

In order to render the determination of this lower bound computational we introduce the following concept. A symmetric matrix-valued $n_G \times n_G$-polynomial matrix $S(x)$ is said to be a (matrix) sum-of-squares (SOS) if there exists a (not necessarily square and typically tall) polynomial matrix $T(x)$ such that

$$S(x) = T(x)^T T(x).$$

If $T_j(x)$, $j = 1, \dots, q$ denote the rows of $T(x)$, we infer

$$S(x) = \sum_{j=1}^{q} T_j(x)^T T_j(x).$$

If $S(x)$ is a scalar then $T_j(x)$ are scalars which implies $S(x) = \sum_{j=1}^{q} T_j(x)^2$. This motivates our terminology since we are dealing with a generalization of classical scalar SOS representations. Very similar to the scalar case, every SOS matrix is globally positive semi-definite, but the converse is not necessarily true.

Let us now just replace all inequalities in the above derived program for the lower bound computations by the requirement that the corresponding polynomials or polynomial matrices are SOS. This leads to the following optimization problem:

$$\begin{array}{ll} \text{supremize} & t \\ \text{subject to} & S(x) \text{ and } f(x) + \langle S(x), G(x)\rangle - t \text{ are SOS} \end{array} \tag{4}$$

If fixing upper bounds on the degree of the SOS matrix $S(x)$, the value of this problem can be computed by solving a standard linear SDP as will be seen in Section 5. In this fashion one can construct a family of LMI relaxations for computing increasingly improving lower bounds. Under a suitable constraint qualification, due to Putinar for scalar problems, it is possible to prove that the value of (4) actually equals d_{opt}. To the best of our knowledge, the generalization to matrix valued problems as formulated in the following result has, except for the recent independent work of Kojima [24], not been presented anywhere else in the literature.

Theorem 1. *Let d_{opt} be the optimal solution of (3) and suppose the following constraint qualification holds true: There exists some $r > 0$ and some SOS matrix $R(x)$ such that*

$$r - \|x\|^2 + \langle R(x), G(x)\rangle \quad \textit{is SOS.} \tag{5}$$

Then the optimal value of (4) equals d_{opt}.

Proof. The value of (4) is not larger than $d_{\rm opt}$. Since trivial for $d_{\rm opt}=\infty$, we assume that $G(x)\preceq 0$ is feasible. Choose any $\epsilon>0$ and some $\hat{x}$ with $G(\hat{x})\preceq 0$ and $f(\hat{x})\leq d_{\rm opt}+\epsilon$. Let us now suppose that $S(x)$ and $f(x)+\langle S(x),G(x)\rangle-t$ are SOS. Then

$$d_{\rm opt}+\epsilon-t\geq f(\hat{x})-t\geq f(\hat{x})+\langle S(\hat{x}),G(\hat{x})\rangle-t\geq 0$$

and thus $d_{\rm opt}+\epsilon\geq t$. Since ϵ was arbitrary we infer $d_{\rm opt}\geq t$.

To prove the converse we first reveal that, due to the constraint qualification, we can replace $G(x)$ by $\hat{G}(x)=\operatorname{diag}(G(x),\|x\|^2-r)$ in both (3) and (4) without changing their values. Indeed if $G(x)\preceq 0$ we infer from (5) that $r-\|x\|^2\geq r-\|x\|^2+\langle R(x),G(x)\rangle\geq 0$. Therefore the extra constraint $\|x\|^2-r\leq 0$ is redundant for (3). We show redundancy for (4) in two steps. If $S(x)$ and $f(x)-t+\langle S(x),G(x)\rangle$ are SOS we can define the SOS matrix $\hat{S}(x)=\operatorname{diag}(S(x),0)$ to conclude that $f(x)-t+\left\langle\hat{S}(x),\hat{G}(x)\right\rangle$ is SOS (since it just equals $f(x)-t+\langle S(x),G(x)\rangle$). Conversely suppose that $\hat{S}(x)=\hat{T}(x)^T\hat{T}(x)$ and $\hat{t}(x)^T\hat{t}(x)=f(x)-t+\left\langle\hat{S}(x),\hat{G}(x)\right\rangle$ are SOS. Partition $\hat{T}(x)=(T(x)\ u(x))$ according to the columns of $\hat{G}(x)$. With the SOS polynomial $v(x)^Tv(x)=r-\|x\|^2+\langle R(x),G(x)\rangle$ we infer

$$\begin{aligned}&\hat{t}(x)^T\hat{t}(x)=f(x)-t+\left\langle T(x)^TT(x),G(x)\right\rangle+u(x)^Tu(x)(\|x\|^2-r)=\\&=f(x)-t+\left\langle T(x)^TT(x),G(x)\right\rangle+u(x)^Tu(x)(\langle R(x),G(x)\rangle-v(x)^Tv(x))=\\&=f(x)-t+\left\langle T(x)^TT(x)+u(x)^Tu(x)R(x),G(x)\right\rangle-u(x)^Tu(x)v(x)^Tv(x).\end{aligned}$$

With $R(x)=R_f(x)^TR_f(x)$ we now observe that

$$S(x):=T(x)^TT(x)+u(x)^Tu(x)R(x)=\begin{pmatrix}T(x)\\u(x)\otimes R_f(x)\end{pmatrix}^T\begin{pmatrix}T(x)\\u(x)\otimes R_f(x)\end{pmatrix}$$

and

$$s(x):=\hat{t}(x)^T\hat{t}(x)+u(x)^Tu(x)v(x)^Tv(x)=\begin{pmatrix}\hat{t}(x)\\u(x)\otimes v(x)\end{pmatrix}^T\begin{pmatrix}\hat{t}(x)\\u(x)\otimes v(x)\end{pmatrix}$$

are SOS. Due to $f(x)-t+\langle S(x),G(x)\rangle=s(x)$ the claim is proved.

Hence from now one we can assume without loss of generality that there exists a standard unit vector v_1 with

$$v_1^TG(x)v_1=\|x\|^2-r. \tag{6}$$

Let us now choose a sequence of unit vectors $v_2,v_3,\ldots$ such that v_i, $i=1,2,\ldots$ is dense in the Euclidean unit sphere, and consider the family of scalar polynomial optimization problems

$$\begin{array}{rl}\text{infimize} & f(x)\\ \text{subject to} & v_i^TG(x)v_i\leq 0,\quad i=1,\ldots,N\end{array} \tag{7}$$

with optimal values d_N. Since any x with $G(x) \preceq 0$ is feasible for (7), we infer $d_N \leq d_{\rm opt}$. Moreover it is clear that $d_N \leq d_{N+1}$ which implies $d_N \to d_0 \leq d_{\rm opt}$ for $N \to \infty$. Let us prove that $d_0 = d_{\rm opt}$. Due to (6) the feasible set of (7) is contained in $\{x \in \mathbb{R}^{n_x} \mid \|x\|^2 \leq r\}$ and hence compact. Therefore there exists an optimal solution x_N of (7), and we can choose a subsequence N_ν with $x_{N_\nu} \to x_0$. Hence $d_0 = \lim_{\nu\to\infty} d_{N_\nu} = \lim_{\nu\to\infty} f(x_{N_\nu}) = f(x_0)$. Then $d_0 = d_{\rm opt}$ follows if we can show that $G(x_0) \preceq 0$. Otherwise there exists a unit vector v with $\epsilon := v^T G(x_0) v > 0$. By convergence there exists some K with $\|G(x_{N_\nu})\| \leq K$ for all ν. By density there exists a sufficiently large ν such that $K\|v_i - v\|^2 + 2K\|v_i - v\| < \epsilon/2$ for some $i \in \{1, \ldots, N_\nu\}$. We can increase ν to guarantee $v^T G(x_{N_\nu}) v \geq \epsilon/2$ and we arrive at

$$
\begin{aligned}
0 \geq v_i^T G(x_{N_\nu}) v_i = \\
&= (v_i - v)^T G(x_{N_\nu})(v_i - v) + 2v^T G(x_{N_\nu})(v_i - v) + v^T G(x_{N_\nu}) v \geq \\
&\qquad \geq -K\|v_i - v\|^2 - 2K\|v_i - v\| + \epsilon/2 > 0,
\end{aligned}
$$

a contradiction.

Let us finally fix any $\epsilon > 0$ and choose N with $d_N \geq d_{\rm opt} - \epsilon/2$. This implies $f(x) - d_{\rm opt} + \epsilon > 0$ for all x with $v_i^T G(x) v_i \leq 0$ for $i = 1, \ldots, N$. Due to (6) we can apply Putinar's scalar representation result [28] to infer that there exist polynomials $t_i(x)$ for which

$$
f(x) - d_{\rm opt} + \epsilon + \sum_{i=1}^{N} t_i(x)^T t_i(x) v_i^T G(x) v_i \quad \text{is SOS.} \tag{8}
$$

With the SOS matrix

$$
S_N(x) := \sum_{i=1}^{N} v_i t_i(x)^T t_i(x) v_i^T = \begin{pmatrix} t_1(x) v_1^T \\ \vdots \\ t_N(x) v_N^T \end{pmatrix}^T \begin{pmatrix} t_1(x) v_1^T \\ \vdots \\ t_N(x) v_N^T \end{pmatrix}
$$

we conclude that $f(x) - d_{\rm opt} + \epsilon + \langle S_N(x), G(x) \rangle$ equals (8) and is thus SOS. This implies that the optimal value of (4) is at least $d_{\rm opt} - \epsilon$, and since $\epsilon > 0$ was arbitrary the proof is finished.

■

Theorem 1 is a natural extension of a theorem of Putinar [28] for scalar polynomial problems to polynomial SDP's. Indeed, Lasserre's approach [22] for minimizing $f(x)$ over scalar polynomial constraints $g_i(x) \leq 0$, $i = 1, \ldots, m$, is recovered with $G(x) = \mathrm{diag}(g_1(x), \ldots, g_m(x))$. Moreover the constraint qualification in Theorem 1 is a natural generalization of that used by Schweighofer [33].

Remark 1. It is a direct consequence of Theorem 1 that, as in the scalar case [32], the constraint qualification (5) can be equivalently formulated as follows: there exist an SOS matrix $R(x)$ and an SOS polynomial $s(x)$ such that

$$
\{x \in \mathbb{R}^{n_x} \mid \langle R(x), G(x) \rangle - s(x) \leq 0\} \quad \text{is compact.}
$$

3.2 Application to the $\mathcal{H}_\infty$ fixed order control problem

The technique described above can directly be applied to (2), except that the constraint qualification is not necessarily satisfied. This can be resolved by appending a bounding inequality $X \preceq M_X I$ for some large value $M_X > 0$. An SOS relaxation of the resulting BMI problem is formulated with

$$G_1(X,p,\gamma) := -X, \quad G_2(X,p,\gamma) := B_\infty(X,p,\gamma),$$
$$G_3(X,p,\gamma) := X - M_X I, \quad G_4(X,p,\gamma) := \|p\|^2 - M$$

as follows

$$\begin{aligned} \text{supremize:} \quad & \gamma \\ \text{subject to:} \quad & \gamma + \sum_{i=1}^{4} \langle S_i(X,p,\gamma), G_i(X,p,\gamma) \rangle \quad \text{is SOS} \\ & S_i(X,p,\gamma) \ \text{ is SOS}, \quad i = 1, \ldots, 4. \end{aligned} \tag{9}$$

It has already been indicated that this problem is easily translated into a linear SDP if we impose a priori bounds on the degrees of all SOS matrix polynomials. However, as the main trouble, this technique suffers from large number of variables for higher order relaxations, especially when the order of the underlying system state is large. Since S_1, S_2 and S_3 are polynomials in X, the size of the relaxation grows *exponentially* with the order of the system. In our numerical experience the size of the LMI problems of the SOS relaxations grows so fast that good lower bounds can be only computed for systems with state dimension up to about 4. Therefore it is crucial to avoid the need for constructing SOS polynomials in the Lyapunov variable X. As the second main contribution of this paper, we reveal in the next section how to overcome this deficiency.

4 Conversion to robustness analysis

For translating the nonlinear synthesis problem in (2) to an equivalent robustness analysis problem, the key idea is to apply *partial* Lagrange dualization [17]: Fix the controller variables p and dualize with respect to the Lyapunov variable X. We will show that one is required to determine parameter-dependent dual variables, in full analogy to computing parameter-dependent Lyapunov function for LPV systems. As the main advantage, this reformulation allows us to suggest novel SOS relaxations that grow exponentially only in the number of controller (or uncertain) parameters p and that can be shown to grow only *quadratically* in the number of the system states, in stark contrast to the relaxations of Section 3.2.

4.1 Partial dualization

In this section we need the additional assumption that $(A(p), B(p))$ is controllable for every $p \in \mathcal{P}$. For fixed $p = p_0 \in \mathcal{P}$, (2) is an LMI problem in X and γ:

$$\begin{array}{rl} \text{infimize} & \gamma \\ \text{subject to} & X \in \mathcal{S}^n, \qquad X \succ 0, \qquad B_\infty(X, p_0, \gamma) \prec 0. \end{array} \tag{10}$$

Let us partition the dual variable Z for the constraint $B_\infty(X, p, \gamma) \prec 0$ in (2) as

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{pmatrix}. \tag{11}$$

Then the Langrange dual reads as follows:

$$\begin{array}{rl} \text{supremize} & \text{Tr}\left(\left[\, C(p_0)\ D(p_0) \,\right] Z \left[\, C(p_0)\ D(p_0) \,\right]^T\right) \\ \text{subject to} & A(p_0)Z_{11} + Z_{11}A(p_0)^T + B(p_0)Z_{12}^T + Z_{12}B(p_0)^T \succeq 0 \\ & \text{Tr}(Z_{22}) \le 1, \qquad Z \succeq 0. \end{array} \tag{12}$$

Let $t_{\text{opt}}^d(p_0)$ denote the dual optimal value of (12). Note that (12) is strictly feasible for all $p_0 \in \mathcal{P}$ as is shown in Appendix A. This implies $t_{\text{opt}}^d(p) = t_{\text{opt}}^p(p)$ and, as a consequence, it allows to draw the following conclusion. Given any $t \in \mathbb{R}$ suppose that the function $Z(p)$ satisfies

$$\text{Tr}\left(\left[\, C(p)\ D(p) \,\right] Z(p) \left[\, C(p)\ D(p) \,\right]^T\right) > t, \tag{13}$$

$$A(p)Z_{11}(p) + Z_{11}(p)A(p)^T + B(p)Z_{12}(p)^T + Z_{12}(p)B^T(p) \succ 0, \tag{14}$$

$$\text{Tr}(Z_{22}(p)) < 1, \qquad Z(p) \succ 0, \tag{15}$$

for all $p \in \mathcal{P}$. Then it is clear that $t_{\text{opt}}^d(p) \ge t$ and hence $t_{\text{opt}}^p(p) \ge t$ hold for all $p \in \mathcal{P}$. Therefore t is a lower bound on the best achievable controller performance. It is thus natural to maximize t over some class of functions $Z(.)$ in order to determine tight lower bounds on the value of (2). Our construction allows to show that this lower bound is actually tight if optimizing over matrix polynomials $Z(.)$.

Theorem 2. *Let γ_{opt} be the optimal solution to* (2)*. Let t_{opt} be the supremal t for which there exists a polynomial matrix $Z(p) \in \mathcal{S}^{n+m_1}$ satisfying* (13)-(15) *for all $p \in \mathcal{P}$. Then $\gamma_{opt} = t_{opt}$.*

Proof. We have already seen that $\gamma_{\text{opt}} \ge t_{\text{opt}}$. Now suppose $\gamma_{\text{opt}} \ge t_{\text{opt}} + \epsilon$ for some $\epsilon > 0$. For any fixed $p_0 \in P$, the optimal value of (10) and hence that of (12) are not smaller than γ_{opt}. Since (12) is strictly feasible there exists Y^0 (partitioned as (11)) with

$$\begin{aligned}\mathrm{Tr}\left(\left[\,C(p_0)\ D(p_0)\,\right] Y^0 \left[\,C(p_0)\ D(p_0)\,\right]^T\right) - \epsilon/2 > t_{\mathrm{opt}},\\ A(p_0)Y^0_{11} + Y^0_{11}A(p_0)^T + B(p_0){Y^0_{12}}^T + Y^0_{12}B^T(p_0) \succ 0,\\ \mathrm{Tr}(Y^0_{22}) < 1, \qquad Y^0 \succ 0.\end{aligned}$$

Since the inequalities are strict and $\mathcal{P}$ is compact, we can use a partition of unity argument [29] to show that there actually exists a *continuous* function $Y(p)$ such that

$$\mathrm{Tr}\left(\left[\,C(p)\ D(p)\,\right] Y(p) \left[\,C(p)\ D(p)\,\right]^T\right) - \epsilon/4 > t_{\mathrm{opt}}, \quad (16)$$

$$A(p)Y_{11}(p) + Y_{11}(p)A(p)^T + B(p)Y_{12}(p)^T + Y_{12}(p)B^T(p) \succ 0, \quad (17)$$

$$\mathrm{Tr}(Y_{22}(p)) < 1, \qquad Y(p) \succ 0, \quad (18)$$

for all $p \in \mathcal{P}$. Due to the Stone-Weisterstrass theorem about the approximation of continuous functions by polynomials on compacta, we can even choose $Y(p)$ to be a matrix polynomial. This allows to conclude $t_{\mathrm{opt}} \geq t_{\mathrm{opt}} + \epsilon/4$, a contradiction which finishes the proof.

In actual computations we optimize over functions $Z(\cdot)$ belonging to an increasing sequence of finite-dimensional subspaces of matrix-valued polynomials. Then the difference of the computed lower bound to the actual optimal $\mathcal{H}_\infty$ performance is non-decreasing. If we restrict the search to a subspace of degree bounded matrix polynomials, and if we let the bound on the degree grow to infinity, Theorem 2 guarantees that the corresponding lower bounds converge from below to the globally optimal $\mathcal{H}_\infty$ performance.

We have thus reduced the $\mathcal{H}_\infty$ synthesis problem to a robust analysis problem with complicating variables p and polynomial robustness certificates $Z(p)$. In Section 5 we will discuss how (13)-(15) can be relaxed to standard LMI constraints via suitable SOS tests.

Remark 2. The proposed partial dualization technique is not at all restricted to fixed-order $\mathcal{H}_\infty$ optimal control. Straightforward variations do apply to a whole variety of other interesting problems, such as designing structured controllers for any performance criterion that admits an analysis LMI representation (as e.g. general quadratic performance, H_2-performance or placement of closed-loop poles in LMI regions [6]).

Remark 3. We require the controller parameters to lie in a compact set in order to be able to apply the theorem of Stone-Weierstrass. From a practical point of view this is actually not restrictive since the controller parameters have to be restricted in size for digital implementation. Moreover one can exploit the flexibility in choosing the set $\mathcal{P}$ in order to incorporate the suggested lower bound computations in branch-and-bound techniques.

Remark 4. The controllability assumption is needed to prove that the dual (12) is strictly feasible for all $p \in \mathcal{P}$. Controllability can be verified by a Hautus test: $(A(p), B(p))$ is controllable for all $p \in \mathcal{P}$ if and only if

$$P_{\rm H}(\lambda, p) := \begin{pmatrix} A(p) - \lambda I & B(p) \end{pmatrix} \text{ has full row rank for all } \lambda \in \mathbb{C},\ p \in \mathcal{P}. \quad (19)$$

This property can be verified by the method described in Section 3. Indeed suppose $K \in \mathbb{R}$ is chosen with $\|A(p)\| \leq K$ for all $p \in \mathcal{P}$ (with $\|\cdot\|$ denoting the spectral norm). Then (19) holds true if and only if the real-valued polynomial

$$\begin{aligned} F_{\rm H}(a, b, p) :&= |\det(P_{\rm H}(a + bi, p)P_{\rm H}(a + bi, p)^*)|^2 \\ &= \det(P_{\rm H}(a + bi, p)P_{\rm H}(a + bi, p)^*)^* \det(P_{\rm H}(a + bi, p)P_{\rm H}(a + bi, p)^*) \end{aligned}$$

is strictly positive on $[-K, K] \times [-K, K] \times \mathcal{P}$. This can be tested with SOS decompositions, provided that $\mathcal{P}$ has a representation that satisfies (5). The upper bound K on the spectral norm of A on $\mathcal{P}$ can also be computed with SOS techniques.

Remark 5. The utmost right constraint in (14) ($Z(p) \succ 0$ for all $p \in \mathcal{P}$) can be replaced by the (generally much) stronger condition

$$Z(p) \text{ is SOS in } p.$$

As we will see in Section 5.6 this may reduce the complexity of our relaxation problems. Theorem 2 is still true after the replacement, since for any matrix-valued polynomial $Y(p)$ that satisfies (13)-(15), we can find a unique matrix valued function $R(p)$ on $\mathcal{P}$ that is the Cholesky factor of $Z(p)$ for all $p \in \mathcal{P}$. Furthermore $R(p)$ is continuous on $\mathcal{P}$ if $Z(p)$ is, because the Cholesky factor of a matrix can be computed by a sequence of continuity preserving operations on the coefficients of the matrix [10]. Again by Weierstrass' theorem there exists an approximation of the continuous $R(p)$ on $\mathcal{P}$ by a polynomial $\tilde{R}(p)$ such $Z(p) := \tilde{R}(p)^T \tilde{R}(p)$ satisfies (13)-(15). The constructed matrix-valued polynomial $Z(p)$ is indeed SOS.

4.2 Finite-dimensional approximation

Suppose that $Z_j : \mathbb{R}^{n_p} \mapsto \mathcal{S}^{n+m_1}$, $j = 1, 2, \ldots, N$, is a set of linearly independent symmetric-valued polynomial functions in p (such as a basis for the real vector space of all symmetric matrix polynomials of degree at most d). Let us now restrict the search of $Z(.)$ in Theorem 2 to the subspace

$$\mathcal{Z}_N := \left\{ Z(., z) \mid Z(., z) := \sum_{j=1}^{N} z_j Z_j(.),\ z = (z_1, \ldots, z_N) \in \mathbb{R}^N \right\}. \quad (20)$$

Then (13)-(15) are polynomial inequalities in p that are affine in the coefficients z for the indicated parameterization of the elements $Z(., z)$ in $\mathcal{Z}_N$. With $y := \text{col}(t, z)$, $c := \text{col}(1, 0_{n_z})$ and

$$F(p, y) := \text{diag}(F_{11}(p, z) - t, F_{22}(p, z), Z(p, z), 1 - \text{Tr}(Z_{22}(p, z))) \quad (21)$$

where

$$F_{11}(p,z) := \mathrm{Tr}\left(\left[\,C(p)\ D(p)\,\right] Z(p,z) \left[\,C(p)\ D(p)\,\right]^T\right),$$
$$F_{22}(p,z) := Z_{11}(p,z)A(p)^T + A(p)Z_{11}(p,z) + Z_{12}(p,z)B(p)^T + B(p)Z_{12}^T(p,z)^T$$

the problem to be solved can be compactly written as follows:

$$\begin{array}{ll}\text{supremize} & c^T y \\ \text{subject to} & F(p,y) \succ 0 \quad \text{for all} \quad p \in \mathcal{P}.\end{array} \tag{22}$$

This problem involves a semi-infinite semi-definite constraint on a matrix polynomial in p *only*, i.e. not on the state variables. This allows to construct relaxations that rely on SOS-decomposition for polynomials in p, which is the key to keep the size of the resulting LMI-problem *quadratic* in the number of system states, as opposed to the exponential growth of the size of the LMI-problem for the direct approach discussed in Section 3.

5 Robust analysis by SOS-decompositions

Let us now consider the generic robust LMI problem

$$\begin{array}{ll}\text{supremize} & c^T y \\ \text{subject to} & F(x,y) \succ 0 \text{ for all } x \in \mathbb{R}^{n_x} \text{ with } g_i(x) \le 0,\ i = 1,\ldots,n_g\end{array} \tag{23}$$

where $F(x,y) \in \mathcal{S}^r$ and $g_i(x) \in \mathbb{R}$ are polynomial in x and affine in y respectively. The problems (3) and (23) differ in two essential structural properties. First, (3) is just a semi-definite polynomial minimization problem, whereas (23) has a linear objective with a semi-infinite linear SDP constraint, where the dependence on the parameter x is polynomial. Second, in the relaxation for (3) we had to guarantee positivity for *scalar* polynomials, whereas (23) involves a *matrix-valued* positivity constraint. Despite these structural differences the relaxations suggested in this section are similar to those in Section 3.

5.1 Scalar constraints

Let us first consider scalar-valued semi-infinite constraints which corresponds to $F(x,y)$ being of dimension 1×1 in (23). If for some y there exist $\epsilon > 0$ and SOS polynomials $s_i(x)$, $i = 1,\ldots,n_g$, such that

$$F(x,y) + \sum_{i=1}^{n_g} s_i(x) g_i(x) + \epsilon \ \text{ is SOS in } \ x, \tag{24}$$

then it is very simply to verify that $c^T y$ is a lower bound on the optimal value of (23). The best possible lower bound is achieved with the supremal $c^T y$ over all $\epsilon > 0$ and all SOS polynomials $s_i(x)$ satisfying (24), and we have equality if $G(x) = \mathrm{diag}(g_1(x),\ldots,g_{n_g}(x))$ satisfies the constraint qualification (5). The proof is a a variant of that of Theorem 1 and is hence omitted.

5.2 Scalarization of matrix-valued constraints

Let us now assume that $F(x,y)$ is indeed *matrix-valued.* Our intention is to illustrate why a straightforward scalarization technique fails to lead to the desired properties of the corresponding LMI relaxations. Indeed define $f(v,x,y) := v^T F(x,y)v$ and

$$h_i(v,x) = g_i(x) \;\; i = 1,\ldots,n_g, \;\; h_{n_g+1}(v,x) = 1 - v^T v, \;\; h_{n_g+2}(v,x) = v^T v - 2.$$

Then $F(x,y) \succ 0$ for all x with $g_i(x) \leq 0$, $i = 1,\ldots,n_g$, is equivalent to $f(v,x,y) > 0$ for all (x,v) with $h_i(v,x) \leq 0$, $i = 1,\ldots,n_g+2$. As in Section 5.1 this condition can be relaxed as follows: there exists $\epsilon > 0$ and SOS polynomials $s_i(v,x)$, $i = 1,\ldots,n_g+2$, such that

$$f(v,x,y) + \sum_{i=1}^{n_g+2} s_i(v,x)h_i(v,x) + \epsilon \;\text{ is SOS}$$

(with exactness of the relaxation under constraint qualifications). Unfortunately, despite $f(v,x,y)$ is quadratic in v, no available result allows to guarantee that the SOS polynomials $s_i(v,x)$, $i = 1,\ldots,n_g+2$, can be chosen quadratic in v without loosing the relaxation's exactness. Hence one has to actually rely on higher order SOS polynomials in v to guarantee that the relaxation gap vanishes. In our specific problem, v has $2n+m_1+1$ components such that the relaxation size grows exponentially in the system state-dimension, and we fail to achieve the desired polynomial growth in n.

5.3 Matrix-valued constraints

This motivates to fully avoid scalarization as follows. We replace (22) by requiring the existence of $\epsilon > 0$ and SOS matrices $S_i(x)$ of the same dimension as $F(x,y)$ such that

$$F(x,y) + \sum_{i=1}^{n_g} S_i(x)g_i(x) + \epsilon I \;\text{ is an SOS matrix in }\; x. \tag{25}$$

It is easy to see that the optimal value of (23) is bounded from below by the largest achievable $c^T y$ for which there exist $\epsilon > 0$ and SOS matrices $S_i(x)$ with (25). It is less trivial to show that this relaxation is *exact* which has been proved in our paper [31]. This is the key step to see that one can construct a family of LMI relaxations for computing arbitrarily tight lower bounds on the optimal of (23) whose sizes grow exponentially only in the number of components of x.

Remark 6. We have performed partial dualization (in the high dimensional variables) in order to arrive at the formulation of (23). It is interesting to interpret the replacement of the semi-infinite constraint in (23) by (25) as a second Lagrange relaxation step in the low dimensional variable x. In this sense the suggested relaxation can be viewed as a full SOS Lagrange dualization of the original nonlinear semi-definite program, and exponential growth is avoided by the indicated split into two steps.

5.4 Verification of matrix SOS property

Let us now discuss how to construct a linear SDP representation of (25) if restricting the search of the SOS matrices $S_i(x)$, $i = 1, \ldots, n_g$, to an arbitrary subspace of polynomials matrices. The suggested procedure allows for complete flexibility in the choice of the corresponding monomial basis with a direct and compact description of the resulting linear SDP, even for problems that involve SOS matrices. Moreover it forms the basis for trying to reduce the relaxation sizes for specific problem instances.

For all these purposes let us choose a polynomial vector

$$u(x) = \mathrm{col}(u_1(x), \ldots, u_{n_u}(x))$$

whose components $u_j(x)$ are pairwise different x-monomials. Then $S(x)$ of dimension $r \times r$ is said to be SOS with respect to the monomial basis $u(x)$ if there exist real matrices T_j, $j = 1, \ldots, n_u$, such that

$$S(x) = T(x)^T T(x) \quad \text{with} \quad T(x) = \sum_{j=1}^{n_u} T_j u_j(x) = \sum_{j=1}^{n_u} T_j (u_j(x) \otimes I_r).$$

If $U = (T_1 \ \cdots \ T_{n_u})$ and if P denotes the permutation that guarantees

$$u(x) \otimes I_r = P[I_r \otimes u(x)],$$

we infer with $W = (UP)^T (UP) \succeq 0$ that

$$S(x) = [I_r \otimes u(x)]^T W [I_r \otimes u(x)]. \tag{26}$$

In order to render this relation more explicit let us continue with the following elementary concepts. If $M \in \mathbb{R}^{nr \times nr}$ is partitioned into $n \times n$ blocks as $(M_{jk})_{j,k=1,\ldots,r}$ define

$$\mathrm{Trace}_r(M) = \begin{pmatrix} \mathrm{Tr}(M_{11}) & \cdots & \mathrm{Tr}(M_{1r}) \\ \vdots & \ddots & \vdots \\ \mathrm{Tr}(M_{r1}) & \cdots & \mathrm{Tr}(M_{rr}) \end{pmatrix}$$

as well as the bilinear mapping $\langle .,. \rangle_r : \mathbb{R}^{mr \times nr} \times \mathbb{R}^{mr \times nr} \to \mathbb{R}^{r \times r}$ as

$$\langle A, B \rangle_r = \mathrm{Trace}_r(A^T B).$$

One then easily verifies that $[I_r \otimes u(x)]^T W [I_r \otimes u(x)] = \left\langle W, I_r \otimes u(x)u(x)^T \right\rangle_r$. If we denote the pairwise different monomials in $u(x)u(x)^T$ by $w_j(x)$, $j = 1, \ldots, n_w$, and if we determine the unique symmetric Q_j with

$$u(x)u(x)^T = \sum_{j=1}^{n_w} Q_j w_j(x),$$

we can conclude that

$$S(x) = \sum_{j=1}^{n_w} \langle W, I_r \otimes Q_j \rangle_r \, w_j(x). \tag{27}$$

This proves one direction of the complete characterization of $S(x)$ being SOS with respect to $u(x)$, to be considered as a flexible generalization of the Gram-matrix method [27] to polynomial matrices.

Lemma 1. *The matrix polynomial $S(x)$ is SOS with respect to the monomial basis $u(x)$ iff there exist necessarily unique symmetric S_j, $j = 1, \ldots, n_w$, such that $S(x) = \sum_{j=1}^{n_w} S_j w_j(x)$, and the linear system*

$$\langle W, I_r \otimes Q_j \rangle_r = S_j, \qquad j = 1, \ldots, n_w, \tag{28}$$

has a solution $W \succeq 0$.

Proof. If $W \succeq 0$ satisfies (28) we can determine a Cholesky factorization of PWP^T as $U^T U$ to obtain $W = (UP)^T(UP)$ and reverse the above arguments. ■

5.5 Construction of LMI relaxation families

With monomial vector $v_i(x)$ and some real vector b_i for each $i = 1, 2, \ldots, n_g$ let us represent the constraint functions as $g_i(x) = b_i^T v_i(x), i = 1, 2, \ldots, n_g$. Moreover, let us choose monomial vectors $u_i(x)$ of length n_i to parameterize the SOS matrices $S_i(x)$ with respect to $u_i(x)$ with $W_i \succeq 0$, $i = 0, 1, \ldots, n_g$, as in Section 5.4. With $v_0(x) = 1$ and $b_0 = -1$, we infer

$$\begin{aligned}
-S_0(x) \sum_{i=1}^{n_g} S_i(x) g_i(x) &= \sum_{i=0}^{n_g} \left\langle \; W_i, I_r \otimes \left[u_i(x)u_i(x)^T\right] \; \right\rangle_r \; \left[b_i^T v_i(x)\right] \\
&= \sum_{i=0}^{n_g} \left\langle \; W_i, \left(I_r \otimes \left[u_i(x)u_i(x)^T\right]\right) \left[b_i^T v_i(x)\right] \; \right\rangle_r \\
&= \sum_{i=0}^{n_g} \left\langle \; W_i, I_r \otimes \left(\left[u_i(x)u_i(x)^T\right] \otimes \left[b_i^T v_i(x)\right]\right) \; \right\rangle_r \\
&= \sum_{i=0}^{n_g} \left\langle \; W_i, I_r \otimes \left(\left[I_{n_i} \otimes b_i^T\right] \left[u_i(x)u_i(x)^T \otimes v_i(x)\right]\right) \; \right\rangle_r .
\end{aligned}$$

Let us now choose the pairwise different monomials

$$w_0(x) = 1, \qquad w_1(x), \dots, w_{n_w}(x)$$

to allow for the representations

$$u_i(x)u_i(x)^T \otimes v_i(x) = \sum_{j=0}^{n_w} P_{ij} w_j(x) \ \text{ and } \ F(x,y) = \sum_{j=0}^{n_w} A_j(y) w_j(x), \tag{29}$$

with symmetrically valued $A_j(y)$ that depend affinely on y. Then there exist $\epsilon > 0$ and SOS matrices $S_i(x)$ with respect to $u_i(x)$, $i = 0, \dots, n_g$, such that

$$F(x,y) + \sum_{i=1}^{n_g} S_i(x) g_i(x) + \epsilon I = S_0(x) \tag{30}$$

if and only if there exists a solution to the following LMI system:

$$\epsilon > 0, \quad W_i \succeq 0, \quad i = 0, \dots, n_g, \tag{31}$$

$$A_0(y) + \sum_{i=0}^{n_g} \left\langle \, W_i, I_r \otimes \left(\left[I_{n_i} \otimes b_i^T \right] P_{i0} \right) \, \right\rangle_r + \epsilon I = 0, \tag{32}$$

$$A_j(y) + \sum_{i=0}^{n_g} \left\langle W_i, I_r \otimes \left(\left[I_{n_i} \otimes b_i^T \right] P_{ij} \right) \, \right\rangle_r = 0, \quad j = 1, \dots, n_w. \tag{33}$$

We can hence easily maximize $c^T y$ over these LMI constraints to determine a lower bound on the optimal value of (23). Moreover these lower bounds are guaranteed to converge to the optimal value of (23) if we choose $u_i(x)$, $i = 0, \dots, n_g$, to comprise all monomials up to a certain degree, and if we let the degree bound grow to infinity.

5.6 Size of the relaxation of LMI problem

The size of the LMI relaxation for (23) is easily determined as follows. The constraints are (31), (32) and (33). The condition on the matrices W_i, $i = 0, 1, \dots, n_g$, to be nonnegative definite in (31) comprises for each $i = 0, 1, \dots, n_g$ one inequality in $\mathcal{S}^{rn_{u_i}}$, where (as mentioned earlier in the text) r and n_{u_i} denote the number of rows in $F(x,y)$ in (23) and the number of monomials for the i^{th} SOS matrix, $i = 0, 1, \dots, n_g$, respectively. On top of that (32) adds r^2 and (33) adds $n_w r^2$ scalar equation constraints to the LMI problem.

The decision variables in the LMI relaxation are the lower bound t, the matrices for the SOS representation $W_i \in \mathcal{S}^{rn_{u_i}}$, $i = 0, 1, \dots, n_g$, and the original optimization variables $y \in \mathbb{R}^{n_y}$. Since a symmetric matrix in $\mathcal{S}^{rn_{u_i}}$ can be parameterized by a vector in $\mathbb{R}^{\frac{1}{2} rn_{u_i}(rn_{u_i}+1)}$, we end up in total with

$$1 + n_y + \frac{1}{2}\sum_{i=0}^{n_g} rn_{u_i}(rn_{u_i} + 1) \tag{34}$$

scalar variables in our LMI problem.

This number may be further reduced by using an explicit representation of the degrees of freedom in W_i for each $i \in \{0, 1, \ldots, n_g\}$ by constructing a basis for the solution set of the linear constraints (32) and (33). Although this explicit parameterization may lead to a smaller number of variables, we consider for simplicity in the remainder of this text the implicit representation with (32) and (33).

We can further specify the size of the LMI problem for the fixed order $\mathcal{H}_\infty$ synthesis problem in terms of the sizes of dynamical system, which will show that the size of the LMI's depends *quadratically* on the number of states. If $\mathcal{P}$ is a ball of radius M described by (1), then it can be described by one polynomial inequality such that $n_g = 1$. The number of variables in y is equal to the dimension of the subspace $\mathcal{Z}_N$: $n_y = N = \dim(\mathcal{Z}_N)$. Since Z is a polynomial in p with matrix coefficients in $\mathcal{S}^{n+m_1}$, n_y grows at most quadratically in n. The number r of rows of each sub-block of $F(p, y)$ in (21) is

Block	$F_{11}(p, z) - t$	$F_{22}(p, z)$	$Z(p, z)$	$1 - \mathrm{Tr}(Z_{22}(p, z))$
Number of rows	1	n	$n + m_1$	1

which results in $r = 2 + 2n + m_1$ rows. This number of rows is reduced by about 50% to $r = 2 + n$ if we replace the utmost right constraint in (15) ($Z(p) \geq 0$ for all $p \in \mathcal{P}$) by requiring $Z(p)$ to be SOS, as we suggested in Remark 5.

The monomial vectors u_i, $i = 0, 1$, should be chosen such that $F(\cdot, y)$ can be expressed in terms of S_0 and $S_1 g_1$ as in (30) and they will be specified for the concrete examples in Section 6. Note that n_{u_i}, $i = 0, 1$, is independent of n since u_i, $i = 0, 1$, are polynomials in p.

Summarizing, n_y grows at most quadratically in n, r grows linearly in n and n_{u_i}, $i = 0, 1$, are independent of n. Equation (34) therefore implies that the number of LMI variables is indeed *quadratic* in n.

Remark 7. Although not limiting for unstructured controller synthesis [36], the assumption $D_{22} = 0$ on the direct-feedthrough term is usually restrictive if the controller is required to admit some structure. It is therefore useful to discuss how a nonzero D_{22} can be taken into account in our method. For $D_{22} \neq 0$ the closed-loop matrices are rational functions of p:

$$A(p) = \begin{pmatrix} A^{\mathrm{ol}} + B_2^{\mathrm{ol}} Q(p) D_K(p) C_2^{\mathrm{ol}} & B_2^{\mathrm{ol}} Q(p) C_K(p) \\ A_{21}(p) & A_K(p) + B_K(p) D_{22} Q(p) C_K(p) \end{pmatrix}$$

$$B(p) = \begin{pmatrix} B_1^{\mathrm{ol}} + B_2^{\mathrm{ol}} Q(p) D_K(p) D_{21}^{\mathrm{ol}} \\ B_K(p) D_{21} + B_K(p) D_{22} Q(p) D_K(p) D_{21}^{\mathrm{ol}} \end{pmatrix}$$

$$C(p) = \begin{pmatrix} C_1^{\mathrm{ol}} + D_{12} Q(p) D_K(p) C_2^{\mathrm{ol}} & D_{12} Q(p) C_K(p) \end{pmatrix}$$

$$D(p) = D_{11}^{\mathrm{ol}} + D_{12}^{\mathrm{ol}} Q(p) D_K(p) D_{21}^{\mathrm{ol}}$$

where $A_{21}(p) := B_K(p)C_2^{\mathrm{ol}} + B_K(p)D_{22}Q(p)D_K(p)C_2^{\mathrm{ol}}$ and

$$Q(p) := (I - D_K p D_{22})^{-1}.$$

This results in rational dependence of the left-hand sides of (13) and (14) on p, such that we can not directly apply the SOS test. Under the well-posedness condition that $I - D_K(p)D_{22}$ is nonsingular for all $p \in \mathcal{P}$, we can multiply (13) and (14) by $\det(I - D_K(p)D_{22})^2$. Since then $\det(I - D_K(p)D_{22})^2 > 0$ for all $p \in \mathcal{P}$, the resulting inequalities are equivalent to (13) and (14) and polynomial in p. This implies that our problem is a robust polynomial LMI problem (23) and we can apply the SOS technique. It is easy to see that testing well-posedness is a robust LMI problem as well.

Remark 8. In a similar fashion we can introduce a rational parameterization for $Z(p)$. Let $Z_j : \mathbb{R}^{n_p} \mapsto \mathcal{S}^{n+m_1}$, $j = 1, 2, \ldots, N$, be a set of linearly independent symmetric valued rational functions in p without poles in 0 (instead of polynomials as in Section 4.2). By multiplication of the inequalities (13)-(15) with the smallest common denominator of $Z(p)$ that is strictly positive for all $p \in \mathcal{P}$, their left-hand side becomes a polynomial of p. We expect such parameterizations of $Z(p)$ with rational functions to be much better than with polynomials, in the sense that they generate better lower bounds for the same size of the LMI problem. Comparison of these parameterizations is a topic for future research.

6 Application

We present lower bound computations for the fixed order $\mathcal{H}_\infty$ problem of a fourth order system and a 27$^{\mathrm{th}}$ order active suspension system. The results of this section have also been published in [16].

6.1 Fourth order system

We consider an academic example with

$$\left(\begin{array}{c|c|c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & D_{22} \end{array}\right) = \left(\begin{array}{cccc|c|cc} -7 & 4 & 0 & 0.2 & 0.9 & 0.2 & 0 \\ -0.5 & -2 & 0 & 0 & 2 & 0.2 & 0 \\ 3 & 4 & -0.5 & 0 & 0.1 & 0.1 & 0 \\ 3 & 4 & 2 & -1 & -4 & 0 & -0.2 \\ \hline 0 & -10 & -3 & 0 & 0 & 3 & -4 \\ \hline 0.8 & 0.1 & 0 & 0 & 0.3 & 0 & 0 \end{array}\right)$$

and computed lower bounds on the closed-loop $\mathcal{H}_\infty$-performance of all stabilising static controllers in a compact subset of $\mathbb{R}^{2\times 1}$. The open loop $\mathcal{H}_\infty$ norm is 47.6. We first computed an initial feedback law $K_{\mathrm{init}} = \begin{pmatrix} -38 & -28 \end{pmatrix}^T$, which gives a performance of 0.60. We computed bounds for the ball $\mathcal{P}_{\mathrm{ball}} =$

$\{p \in \mathbb{R}^2 | \; \|p\| \leq M\}$ with radius $M = 1.5$ around the initial controller, i.e. $K(p) := K_{\text{init}} + \begin{pmatrix} p_1 \; p_2 \end{pmatrix}^T$. Observe that $p \in \mathcal{P}$ is equivalent to $g_1(p) \leq 0$ where

$$g_1(p) := p_1^2 + p_2^2 - M^2, \tag{35}$$

which is the only constraint on p, such that $n_g = 1$ in (23). We therefore optimize over 2 SOS polynomials S_0 and S_1, both of dimension 10×10. The choice of the monomials u_0 and u_1 will be discussed below.

The resulting lower bound, the number of variables in the LMI and the size of the LFT are shown in Table 1 for various degrees in u_1 and Z. The monomial vector u_1 is represented in the table by (l_1, l_2), i.e. $u_1(p) = M^{l_1,l_2}(p)$ where M maps the maximal monomial orders l_1 and l_2 into a vector containing all mononials

$$p_1^i p_2^j, \qquad 0 \leq i \leq l_1, \qquad 0 \leq j \leq l_2.$$

In other words M^{l_1,l_2} is the monomial vector

$$M^{l_1,l_2}(p) := \begin{pmatrix} 1 \; p_1 \; p_1^2 \ldots p_1^{l_1} \; p_1p_2 \; p_1^2p_2 \ldots p_1^{l_1}p_2 \ldots p_1^{l_1}p_2^{l_2} \end{pmatrix}.$$

The vector (k_1, k_2) in the table denotes the maximal monomial degrees for Z in a similar fashion, e.g, $(k_1, k_2) = (2, 1)$ should be read as the parameterization

$$Z(p,z) = Z_0 + \textstyle\sum_{j=1}^N z_jE_j + z_{j+N}p_1E_j + z_{j+2N}p_1^2E_j + z_{j+3N}p_1p_2E_j + \\ + \textstyle\sum_{j=1}^N z_{j+4N}p_1^2p_2E_j + z_{j+5N}p_2E_j,$$

where $N = \dim(\mathcal{S}^{n+m_1}) = \frac{1}{2}(n + m_1 + 1)(n + m_1)$ and E_j, $j = 1, \ldots, N$, is a basis for $\mathcal{S}^{n+m_1}$. For this parameterization the number n_y of variables in y grows quadratically with n, which is the maximum growth rate of n_y as mentioned in Section 5.6. The monomial vector $u_0(p) = M^{q_1,q_2}(p)$ is chosen such that the monomials on the right-hand side of (30) match those on the left-hand side, i.e.

$$q_i = \left\lceil \frac{2 + \max\{k_i, 2l_i\}}{2} \right\rceil, \qquad i = 1, 2,$$

where $\lceil x \rceil$ is the smallest integer larger than or equal to x. For instance the lower bound in the right lower corner of Table 1 is computed with $q_1 = 3, q_2 = 3$ such that $u_0(p)$ is a monomial vector of length 16.

By a gridding technique we have found an optimal controller $p^{\text{opt}} = \begin{pmatrix} 1.33 \; 0.69 \end{pmatrix}^T \in \mathcal{P}$ with performance 0.254. From Table 1 it seems that the lower bound indeed approaches this value for increasing order in both u_1 and Z. The best lower bound is $t_{\text{opt}} = 0.251$, which is slightly smaller than the optimal performance 0.254. The number of variables in our implementation of the LMI relaxations is shown in Table 2. Each LMI problem has been solved with SeDuMi [34] in at most a few minutes.

Table 1. Lower bounds for 4th order system, various degrees of Z and S_1

(k_1,k_2), monomials in Z	(l_1,l_2), monomials in u_1: $(0,0)$	$(1,1)$	$(2,0)$	$(0,2)$
$(0,0)$	0.15584	0.16174	0.16174	0.16174
$(1,0)$	0.20001	0.20939	0.20959	0.20183
$(1,1)$	0.20319	0.21483	0.21331	0.20785
$(2,0)$	0.22259	0.2298	0.23097	0.22396
$(1,2)$	0.20642	0.22028	0.21886	0.21171
$(2,1)$	0.22669	0.23968	0.23936	0.22959
$(3,0)$	0.22361	0.24000	0.24212	0.22465
$(4,0)$	0.22465	0.24263	0.24373	0.22504
$(2,2)$	0.22737	0.24311	0.24298	0.23277
$(4,2)$	0.22889	0.25047	0.25069	0.23414

Table 2. Number of LMI variables for 4th order system, various degrees of Z and u_1

(k_1,k_2), monomials in Z	(l_1,l_2), monomials in u_1: $(0,0)$	$(1,1)$	$(2,0)$	$(0,2)$
$(0,0)$	41	1514	970	1960
$(1,0)$	220	1528	984	1974
$(1,1)$	413	1556	1012	2002
$(2,0)$	234	1542	998	1988
$(1,2)$	1211	1584	1370	2030
$(2,1)$	881	1584	1040	2030
$(3,0)$	578	1556	1012	2002
$(4,0)$	867	1570	1026	2016
$(2,2)$	1253	1626	1412	2072
$(4,2)$	2547	2920	2706	2981

6.2 Active suspension system

As a second example we consider the control of an active suspension system, which has been a benchmark system of a special issue of the European Journal of Control on fixed-order controller synthesis [21], see Figure 1. The goal is to compute a low-order discrete-time controller such that the closed-loop sensitivity and controller sensitivity satisfy certain frequency-dependent bounds. The system has 17 states and the weights of our 4-block $\mathcal{H}_\infty$ design contributed with 10 states, which add up to 27 states of the generalized plant. The full order design has closed-loop $\mathcal{H}_\infty$-norm 2.48. We computed a 5th order controller by closed-loop balanced residualization with performance 3.41. For more details on the fixed order $\mathcal{H}_\infty$-design the reader is referred to [18]. We computed lower bounds for changes in two diagonal elements of the state-space matrices of the controller

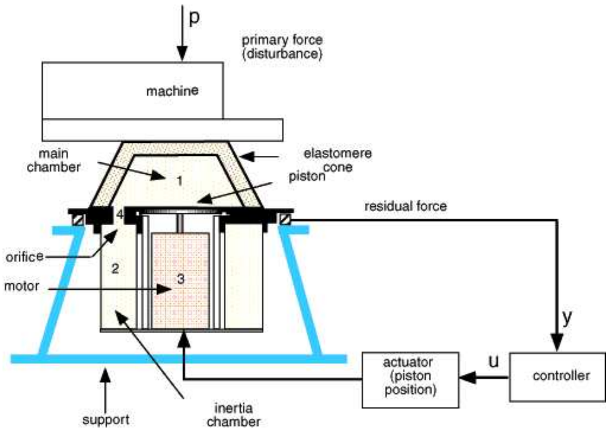

Fig. 1. Active suspension system

$$K(p) = \left(\begin{array}{c|c} A_K(p) & B_K \\ \hline C_K & D_K \end{array}\right)$$

$$= \left(\begin{array}{ccccc|c} -78.2 & 1129.2 & 173.24 & -97.751 & -130.36 & 6.6086 \\ -1240.9 & -78.2+p_1 & 111.45 & 125.12 & 76.16 & 21.445 \\ 0 & 0 & -6.0294 & 164.81+p_2 & 159 & -11.126 \\ 0 & 0 & 0 & -204.56 & 49.031 & -12.405 \\ 0 & 0 & 0 & -458.3 & -204.56 & -9.4469 \\ \hline -0.067565 & 0.19822 & -1.0047 & -0.069722 & 0.19324 & 0.0062862 \end{array}\right)$$

where p_1 and p_2 are free scalar controller variables. Table 3 shows computed lower bounds for various degrees in Z and u_1 and various controller sets $\mathcal{P}_{\text{ball}} := \{p \quad |\mathbb{R}^{n_p}, \|p\| \leq M\}$, $M \in \{5, 10, 50, 100\}$, together with the number of LMI variables. It is interesting to note that the lower bounds are clearly

Table 3. Lower bounds for suspension system, for various degrees of Z and S and $\mathcal{P}$ balls with various radii.

monomials		Radius M of $\mathcal{P} := \{p \vert p \in \mathbb{R}^{n_p}, \|p\| \leq M\}$				# LMI variables
Z	u_1	5	10	50	100	$M \in \{5, 10, 50, 100\}$
$(0,0)$	$(0,0)$	3.2271	3.0176	2.2445	1.9790	1719
$(1,0)$	$(0,0)$	3.2570	3.0732	2.3975	2.1585	2313
$(0,1)$	$(0,0)$	3.2412	3.0468	2.3725	2.2398	2313

better than the bounds computed with the S-procedure in [17]. It is a topic of our current research to investigate why and for which type of problems the relaxations based on SOS decompositions work better than those based on the S-procedure.

The example illustrates that the lower bound computation is feasible for larger order systems, although the size of the LMI problems still grows gradually.

7 Conclusions

We have shown that there exist sequences of SOS relaxations whose optimal value converge from below to the optimal closed-loop $\mathcal{H}_\infty$ performance for controllers of some a priori fixed order. In a first scheme we generalized a well-established SOS relaxation technique for scalar polynomial optimization to problems with matrix-valued semi-definite constraints. In order to avoid the resulting exponential growth of the relaxation size in both the number of controller parameters and system states, we proposed a second technique based on two sequential Lagrange dualizations. This translated the synthesis problem into a problem as known from robustness analysis for systems affected by time-varying uncertainties. We suggested a novel relaxation scheme based on SOS matrices that is guaranteed to be asymptotically exact, and that allows to show that the size of the relaxations only grow quadratically in the dimension of the system state.

We have applied the method to systems of McMillan degree 4 and 27 respectively. The first example illustrated that the lower bounds indeed converge to the optimal fixed-order $\mathcal{H}_\infty$-performance value. The second example showed the feasibility of the approach for plants with moderate McMillan degree, in the sense that we can compute nontrivial lower bounds by solving LMI problems with about 2300 variables.

A Proof of strict feasibility of the dual problem

Let us prove that (12) is strictly feasible for all $p_0 \in \mathcal{P}$. For an arbitrary $p_0 \in \mathcal{P}$, we need to show that there exists some W with

$$\begin{aligned} A(p_0)W_{11} + W_{11}A(p_0)^T + B(p_0)W_{12}^T + W_{12}B(p_0)^T \succ 0,\\ \mathrm{Tr}(W_{22}) < 1, \qquad W \succ 0. \end{aligned} \tag{36}$$

Since $(A(p_0), B(p_0))$ is controllable, one can construct an anti-stabilizing state-feedback gain, i.e., a matrix K such that $A(p_0) + B(p_0)K$ has all its eigenvalues in the open right-half plane. Hence there exists some $P \succ 0$ with

$$(A(p_0) + B(p_0)K)P + P(A(p_0) + B(p_0)K)^T \succ 0 \tag{37}$$

and rP also satisfies (37) for any $r > 0$. Then W defined by the blocks $W_{11} = rP$, $W_{12}^T = rKP$ and

$$W_{22} = W_{12}^T W_{11}^{-1} W_{12} + rI = r(K^T PK + I)$$

in the partition (11) satisfies (36) and $W \succ 0$ for arbitrary $r > 0$. The constructed W does the job if we choose in addition $r > 0$ sufficiently small to achieve $\mathrm{Tr}(W_{22}) = r\mathrm{Tr}(K^T PK + I) < 1$.

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