# Multi-Objective Control without Youla Parameterization

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**Abstract.** It is rather well-understood how to systematically design controllers that achieve multiple norm-bound specifications imposed on different channels of a control system. However, all known approaches to such problems are based on the Youla parametrization of all stabilizing controllers. This involves a transformation of the model description on the basis of a fixed stabilizing controller, and a wrong choice of this controller might require to unduly increase the controller order to closely approach optimality.

In this paper we suggest a novel procedure to multi-objective controller design which avoids the Youla parametrization and which directly applies to the generalized plant framework. In addition, we discuss various theoretical and practical numerical benefits of this new approach.

## 1 Introduction

In this paper we confine our attention to discrete-time linear time-invariant systems which admit a finite-dimensional state-space description. We use the standard notation to denote by $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ the input-output operator or the transfer matrix which is defined by the system $x(t+1) = Ax(t) + Bu(t),\ y(t) = Cx(t) + Du(t)$.

Consider a generalized plant with two performance channels and one control channel described as

$$\begin{pmatrix} z_1 \\ z_2 \\ y \end{pmatrix} = \left[\begin{array}{c|ccc} A & B_1 & B_2 & B \\ \hline C_1 & D_1 & D_{12} & E_1 \\ C_2 & D_{21} & D_2 & E_2 \\ C & F_1 & F_2 & 0 \end{array}\right] \begin{pmatrix} w_1 \\ w_2 \\ u \end{pmatrix}. \tag{1}$$

The inter-connection of (1) with a controller

$$u = \left[\begin{array}{c|c} A_K & B_K \\ \hline B_K & D_K \end{array}\right] y = Ky \tag{2}$$

is denoted as

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \left[\begin{array}{c|cc} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \hline \mathcal{C}_1 & \mathcal{D}_1 & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_2 \end{array}\right] \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \mathcal{T}_1(K) & \mathcal{T}_{12}(K) \\ \mathcal{T}_{21}(K) & \mathcal{T}_2(K) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{3}$$

**Problem Formulation**. We consider the following multi-objective or structured $H_\infty$-control problem: Minimize $\gamma$ such that there exists a stabilizing controller (2) (a controller for which all eigenvalues of $\mathcal{A}$ are located in the open unit disk) which renders the following $H_\infty$-norm constraints on the diagonal blocks of the controlled closed-loop system satisfied:

$$\|\mathcal{T}_1(K)\|_\infty < \gamma \quad \text{and} \quad \|\mathcal{T}_2(K)\|_\infty < \gamma. \tag{4}$$

Let us denote the optimal achievable bound by $\gamma_*$.

Apart from being a mathematically challenging extension of standard single-objective $H_\infty$-control [3, 5, 7], our main practical motivation for such controller design techniques is as follows: They allow to enforce loop-shaping requirements with independent weights on unrelated subsystems of arbitrary closed-loop inter-connection without having to artificially include transfer matrix blocks of no interest [18].

In contrast to multi-objective control problems that involve different norm constraints on the diagonal blocks of the closed-loop system [1, 2, 19, 13, 6, 16], this more specific problem has found attention in [12, 11, 10, 14]. Without any exception, all existing approaches to approximately solve the genuine multiple norm controller design problems are based on the Youla parameterization. On the basis of a fixed stabilizing controller, one can determine a stable transfer matrix $T$ such that the set of all closed-loop transfer matrices $\mathcal{T}(K)$ that result from stabilizing controllers $K$ is given by all

$$\begin{pmatrix} T_1 & T_{12} \\ T_{21} & T_2 \end{pmatrix} + \begin{pmatrix} T_{13} \\ T_{23} \end{pmatrix} Q \begin{pmatrix} T_{31} & T_{32} \end{pmatrix} \tag{5}$$

if the Youla parameter $Q$ varies in the set $RH_\infty$ of all stable transfer matrices of appropriate dimension.

If (1) corresponds to a one-block problem [4] where $\begin{pmatrix} T_{13} \\ T_{23} \end{pmatrix}$ and $\begin{pmatrix} T_{21} & T_{31} \end{pmatrix}$ have full row and column rank on the whole unit circle respectively, rational interpolation theory allows to equivalently translate the multi-objective control problem into an LMI-problem [11, 10]. This makes it possible to compute the optimal value and close-to-optimal controllers by solving a fixed-sized finite-dimensional optimization problem.

If the problem does not have a one-block nature, it has been suggested to perform a relaxation by designing a controller which minimizes an upper bound of the actual cost. The solution of these so-called mixed design problems [1, 8, 17, 9] leads to controllers of the same oder as the underlying plant, but it is generally hard to estimate in how far the upper bound relates to the exact optimal value.

All suggested solutions to solve the genuine multi-objective control problem proceed along the following lines. Choose a sequence of scalar stable transfer functions $q_0$, $q_1$, ... which span a dense subspace in $RH_\infty$, where we recall that the FIR basis $q_j(z) = z^{-j}$ is a simple standard choice. Now consider the problem of minimizing $\gamma$ over all coefficient matrices $X_0, X_1, ..., X_k$ that satisfy

$$\|T_1 + T_{13}\left(\sum_{j=0}^{k} X_j q_j\right) T_{31}\|_\infty < \gamma \quad \text{and} \quad \|T_2 + T_{23}\left(\sum_{j=0}^{k} X_j q_j\right) T_{32}\|_\infty < \gamma$$

and denote the optimal value by $\gamma_k$. Since the coefficient matrices $X_j$ enter the transfer matrices affinely, it is possible to determine $\gamma_k$ by solving a standard LMI problem [13, 6, 16]. Moreover, it is not difficult to prove that $\gamma_k$ converges to $\gamma_*$ for $k \to \infty$, just by the density property of the sequence $q_j$. (Of course, this slightly specialized structure of parameterizing a dense sequence of subspaces in the matrix space $RH_\infty$ can be easily generalized to using different scalar basis functions for each matrix entry.) Let us briefly sketch the fundamental disadvantages of this approach that motivated the alternatives suggested in this paper:

- It is rather unclear how to choose the initial stabilizing controller in order to guarantee a fast convergence rate in $\gamma_k \to \gamma_*$. If convergence is slow, $\gamma_k$ is close to $\gamma_*$ only for large $k$ such that the McMillan degree of $\sum_{j=0}^{k} X_j q_j$ and hence that of the corresponding multi-objective controller (2) will be large, even if the order of (1) is small.

- Varying $X_j$ in the parameterization $\sum_{j=0}^{k} X_j q_j$ amounts to adjusting the residues of the Youla parameter while leaving its poles fixed. As in single-objective $H_\infty$-control, it would be highly desirable to somehow transform the problem of optimizing simultaneously over both the residues and the poles of the Youla parameter into a convex optimization problem. Unfortunately, such a procedure is unknown. Therefore, it would at least be desirable to favorably adjust the poles of $q_j$ in an iterative procedure in order to improve the approximation quality for a *fixed* number of $k$ basis functions. However, in the sketched framework it seems unclear how to gather information about improved pole locations of the Youla parameter from residue optimization. Recall that we have suggested in [13] a technique which allows to reconstruct optimal pole locations of the Youla parameter, at least asymptotically if $k \to \infty$.

As the main idea of this paper we observe that the multi-objective control problem is equivalent to minimizing

$$\left\| \begin{pmatrix} \mathcal{T}_1(K) & \mathcal{T}_{12}(K) + Q_{12} \\ \mathcal{T}_{21}(K) + Q_{21} & \mathcal{T}_2(K) \end{pmatrix} \right\|_\infty \tag{6}$$

over all stabilizing controllers (2) and over all $Q_1$, $Q_2$ in $RH_\infty$ of appropriate dimension. This just follows, for a fixed stabilizing controller $K$, from

$$\min_{Q_{12},\, Q_{21} \in RH_\infty} \left\| \begin{pmatrix} \mathcal{T}_1(K) & \mathcal{T}_{12}(K) + Q_{12} \\ \mathcal{T}_{21}(K) + Q_{21} & \mathcal{T}_2(K) \end{pmatrix} \right\|_\infty = \max\{\|\mathcal{T}_1(K)\|_\infty, \|\mathcal{T}_2(K)\|_\infty\}$$

which is an obvious consequence of Parrot's theorem. Hence, the blocks $Q_{12}$ and $Q_{21}$ serve to annihilate the contribution of the off-diagonal blocks in computing the $H_\infty$-norm of the closed-loop transfer matrix. Although this problem can be viewed as a structured controller design problem as discussed in [14], it is again only known under a specific one-block hypothesis how to reduce it to an LMI problem. Instead, we pursue in this paper the idea to approximately annihilate the influence of the off-diagonal blocks by confining $Q_{12}$, $Q_{21}$ to admit an expansion

$$\begin{pmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{pmatrix} = \sum_{j=0}^{k} \begin{pmatrix} 0 & V_j \\ U_j & 0 \end{pmatrix} \begin{pmatrix} u_j & 0 \\ 0 & v_j \end{pmatrix} \tag{7}$$

with scalar $RH_\infty$ functions $u_j$, $v_j$ and matrix valued coefficients $U_j$, $V_j$ with the same dimension as $\mathcal{T}_{21}(K)$, $\mathcal{T}_{12}(K)$ respectively. We arrive at the problem of minimizing

$$\left\| \begin{pmatrix} \mathcal{T}_1(K) & \mathcal{T}_{12}(K) \\ \mathcal{T}_{21}(K) & \mathcal{T}_2(K) \end{pmatrix} + \sum_{j=0}^{k} \begin{pmatrix} 0 & V_j \\ U_j & 0 \end{pmatrix} \begin{pmatrix} u_j & 0 \\ 0 & v_j \end{pmatrix} \right\|_\infty \tag{8}$$

over $U_j$, $V_j$ and over all stabilizing controller $K$. Again, it is straightforward to show that the optimal value of this problem converges to $\gamma_*$ if $u_0, u_1, \ldots$ and $v_0, v_1, \ldots$ both span a dense subspace of $RH_\infty$ and if $k \to \infty$.

We will show in this paper that this latter optimization problem admits a direct solution in terms of the description (1) and in terms of state-space realizations of $u_j$, $v_j$ without invoking any Youla parameterization. We provide a proof which is based on the elimination of the parameters that defined the controller $K$ and which is a non-trivial alternative to the procedure suggested in [14]. This leads to a more efficiently solvable LMI problem, and the construction of a suitable controller $K$ can be based on any desired $H_\infty$ controller design algorithm (Section 2). In addition we reveal how this alternative scenario allows to iteratively adapt the poles of the chosen basis functions $u_j$, $v_j$ by an intermediate model-reduction step in order to improve the approximation quality for fixed expansion length $k$ (Section 3). A numerical example serves to reveal the benefits of our novel approach (Section 4).

## 2 Parametric Dynamic Optimization

Let us first determine an LFT description of the transfer matrix which is involved in (8). For that purpose we collect the real parameters into the matrix

$$P = \left(\begin{array}{cc|c|cc} 0 & V_0 & \cdots & 0 & V_k \\ U_0 & 0 & \cdots & U_k & 0 \end{array}\right) \tag{9}$$

and we determine a minimal realization

$$\left[\begin{array}{c|cc} \widehat{A} & \widehat{B}_1 & \widehat{B}_2 \\ \hline \widehat{C} & \widehat{D}_1 & \widehat{D}_2 \end{array}\right] = \begin{pmatrix} u_0 I & 0 \\ 0 & v_0 I \\ \vdots & \vdots \\ u_k I & 0 \\ 0 & v_k I \end{pmatrix}. \tag{10}$$

Now it is simple to verify that the transfer matrix in (8) is given by

$$\begin{pmatrix} z_1 \\ z_2 \\ y \\ \widehat{y} \end{pmatrix} = \left[\begin{array}{cc|cccc} A & 0 & B_1 & B_2 & B & 0 \\ 0 & \widehat{A} & \widehat{B}_1 & \widehat{B}_2 & 0 & 0 \\ \hline C_1 & 0 & D_1 & D_{12} & E_1 & I \\ C_2 & 0 & D_{21} & D_2 & E_2 & 0 \\ C & 0 & F_1 & F_2 & 0 & 0 \\ 0 & \widehat{C} & \widehat{D}_1 & \widehat{D}_2 & 0 & 0 \end{array}\right] \begin{pmatrix} w_1 \\ w_2 \\ u \\ \widehat{u} \end{pmatrix}, \quad u = Ky, \quad \widehat{u} = P\widehat{y}. \tag{11}$$

Recall that the problem is to minimize the $H_\infty$-norm of the channel $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \to \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ of (11) where $K$ varies in the set of all stabilizing controller and $P$ in the set of all real parameters with the structure (9). It is our main intention to reveal how this parametric dynamic optimization problem can be converted into a finite-dimensional LMI problem.

It turns out that such problems can be solved for systems which admit the (even more general) LFT description

$$\begin{pmatrix} z \\ y_1 \\ y_2 \end{pmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{13} & B_{11} & B_{12} & B_{13} \\ 0 & A_{22} & A_{23} & B_{21} & B_{22} & 0 \\ 0 & 0 & A_{33} & B_{31} & 0 & 0 \\ \hline C_{11} & C_{12} & C_{13} & D_{11} & D_{12} & D_{13} \\ 0 & C_{22} & C_{23} & D_{21} & 0 & 0 \\ 0 & 0 & C_{33} & D_{31} & 0 & 0 \end{array}\right] \begin{pmatrix} w \\ u_1 \\ u_2 \end{pmatrix}, \quad u_1 = Ky_1, \quad u_2 = Py_2. \tag{12}$$

Such a such state-space description is characteristic for transfer matrices in which $u_1 \to y_1$ is strictly proper and the channels $u_1 \to y_2$, $u_2 \to y_1$, $u_2 \to y_2$ vanish identically. If re-connecting the static gain $u_2 = Py_2$ we arrive at

$$\left[\begin{array}{ccc|cc} A_{11} & A_{12} & A_{13} + B_{13}PC_{33} & B_{11} + B_{13}PD_{31} & B_{12} \\ 0 & A_{22} & A_{23} & B_{21} & B_{22} \\ 0 & 0 & A_{33} & B_{31} & 0 \\ \hline C_{11} & C_{12} & C_{13} + D_{13}PC_{33} & D_{11} + D_{13}PD_{31} & D_{12} \\ 0 & C_{22} & C_{23} & D_{21} & 0 \end{array}\right] =: \left[\begin{array}{c|cc} A(P) & B(P) & B_2 \\ \hline C(P) & D(P) & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]. \tag{13}$$

We conclude that (12) is equivalently described as

$$\begin{pmatrix} z \\ y_1 \end{pmatrix} = \left[\begin{array}{c|cc} A(P) & B(P) & B_2 \\ \hline C(P) & D(P) & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \begin{pmatrix} w \\ u_1 \end{pmatrix}, \quad u_1 = Ky_1. \tag{14}$$

Now it is simple to characterize whether there is a stabilizing controller $K$ which renders the $H_\infty$-norm of $w \to z$ smaller than $\gamma$ [5, 7]: There have to exist real symmetric matrices $X$ and $Y$ with

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} > 0, \tag{15}$$

$$\Psi^T \left(\begin{array}{cc} I & 0 \\ A(P) & B(P) \\ \hline 0 & I \\ C(P) & D(P) \end{array}\right)^T \left(\begin{array}{cc|cc} -X & 0 & 0 & 0 \\ 0 & X & 0 & 0 \\ \hline 0 & 0 & -\gamma I & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma} I \end{array}\right) \left(\begin{array}{cc} I & 0 \\ A(P) & B(P) \\ \hline 0 & I \\ C(P) & D(P) \end{array}\right) \Psi < 0, \tag{16}$$

$$\Phi^T \left(\begin{array}{cc} -A(P)^T & -C(P)^T \\ I & 0 \\ \hline -B(P)^T & -D(P)^T \\ 0 & I \end{array}\right)^T \left(\begin{array}{cc|cc} -Y & 0 & 0 & 0 \\ 0 & Y & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{\gamma}I & 0 \\ 0 & 0 & 0 & \gamma I \end{array}\right) \left(\begin{array}{cc} -A(P)^T & -C(P)^T \\ I & 0 \\ \hline -B(P)^T & -D(P)^T \\ 0 & I \end{array}\right) \Phi > 0. \quad (17)$$

Here $\Phi$ and $\Psi$ are basis matrices of $\ker(B_2^T \ D_{12}^T)$ and $\ker(C_2 \ D_{21})$ respectively. For technical reasons we require a specific structure of these annihilators. Indeed, since the first and the last block of $C_2$ and $B_2$ vanish according to their definition in (13), we can assume without loss of generality that they admit the structure

$$\Phi = \left(\begin{array}{cc} 0 & \Phi_{11} \\ 0 & \Phi_{21} \\ I & 0 \\ \hline 0 & \Phi_{41} \end{array}\right) \quad \text{and} \quad \Psi = \left(\begin{array}{cc} I & 0 \\ 0 & \Psi_{21} \\ 0 & \Psi_{31} \\ \hline 0 & \Psi_{41} \end{array}\right). \quad (18)$$

We have thus reduced the original problem to one of determining a structured matrix $P$ and $X$, $Y$ which satisfy the matrix inequalities (15)-(17). As such these inequalities are not convex in all variables, and thy cannot be rendered convex by a simple Schur complement argument.

For a specific parametric dynamic optimization problem which results from robust controller design against uncertain stochastic signals, we have discussed in [15] how one can exploit the special structure of the system's state-space realization in order to convexify these inequalities. In this paper we present a non-trivial novel modification in order to achieve the same goal without any further structural properties of the describing matrices in (12).

In extending [16] let us introduce the transformations

$$X \to R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{12}^T & R_{22} & R_{23} \\ R_{13}^T & R_{23}^T & R_{33} \end{pmatrix} \quad \text{and} \quad Y \to S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12}^T & S_{22} & S_{23} \\ S_{13}^T & S_{23}^T & S_{33} \end{pmatrix} \quad (19)$$

such that the blocks of $R$ and $S$ in a partition corresponding to that of $A(P)$ satisfy the relations

$$\underbrace{\begin{pmatrix} R_{11} & 0 & 0 \\ R_{12}^T & I & 0 \\ R_{13}^T & 0 & I \end{pmatrix}}_{R_1} X = \underbrace{\begin{pmatrix} I & -R_{12} & -R_{13} \\ 0 & R_{22} & R_{23} \\ 0 & R_{23}^T & R_{33} \end{pmatrix}}_{R_2}$$

and

$$Y \underbrace{\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ S_{13}^T & S_{23}^T & S_{33} \end{pmatrix}}_{S_1} = \underbrace{\begin{pmatrix} S_{11} & S_{12} & -S_{13} \\ S_{12}^T & S_{22} & -S_{23} \\ 0 & 0 & I \end{pmatrix}}_{S_2}.$$

It is easy to verify that (19) are bijective transformation from the set of positive definite matrices onto the the set of $R$, $S$ with the properties

$$\boldsymbol{X}(R) := \begin{pmatrix} R_{11} & 0 & 0 \\ 0 & R_{22} & R_{23} \\ 0 & R_{23}^T & R_{33} \end{pmatrix} > 0 \text{ and } \boldsymbol{Y}(S) := \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{12}^T & S_{22} & 0 \\ 0 & 0 & S_{33} \end{pmatrix} > 0. \tag{20}$$

It turns out that one can transform the non-convex matrix inequalities (15)-(17) in $P$, $X$, $Y$ into convex linear matrix inequalities in the new variables $P$, $R$, $S$. For that purpose we need to introduce, in addition to $\boldsymbol{X}(R)$ and $\boldsymbol{Y}(S)$, the functions

$$\boldsymbol{Z}(R,S) := (R_1 S_1)^T = \begin{pmatrix} R_{11} & R_{12} & R_{13} + S_{13} \\ 0 & I & S_{23} \\ 0 & 0 & S_{33} \end{pmatrix},$$

$$\boldsymbol{A_1}(P,R) := R_2 A(P) R_1^T = \begin{pmatrix} A_{11}R_{11} & A_{12} + A_{11}R_{12} - R_{12}A_{12} & \begin{matrix} A_{13} + B_{13}PC_{33} + A_{11}R_{13} - \\ -R_{12}A_{23} - R_{13}A_{33} \end{matrix} \\ 0 & R_{22}A_{22} & R_{22}A_{23} + R_{23}A_{33} \\ 0 & R_{23}^T A_{22} & R_{23}^T A_{23} + R_{33}A_{33} \end{pmatrix},$$

$$\boldsymbol{A_2}(P,S) := S_1^T A(P) S_2 = \begin{pmatrix} A_{11}S_{11} + A_{12}S_{21} & A_{11}S_{12} + A_{12}S_{22} & \begin{matrix} A_{13} + B_{13}PC_{33} + S_{13}A_{33} - \\ -A_{11}S_{13} - A_{12}S_{23} \end{matrix} \\ A_{22}S_{12}^T & A_{22}S_{22} & A_{23} + S_{23}A_{33} - A_{22}S_{23} \\ 0 & 0 & S_{33}A_{33} \end{pmatrix},$$

$$\boldsymbol{B_1}(P,R) := R_2 B(P) = \begin{pmatrix} B_{11} + B_{13}PD_{31} - R_{12}B_{21} - R_{13}B_{31} \\ R_{22}B_{21} + R_{23}B_{31} \\ R_{23}^T B_{21} + R_{33}B_{31} \end{pmatrix},$$

$$\boldsymbol{B_2}(P,S) := S_1^T B(P) = \begin{pmatrix} B_{11} + B_{13}PD_{31} + S_{13}B_{31} \\ B_{21} + S_{23}B_{31} \\ S_{33}B_{31} \end{pmatrix},$$

$$\boldsymbol{C_1}(P,R) := C(P)R_1^T = \begin{pmatrix} C_{11}R_{11} & C_{11}R_{12}+C_{12} & C_{13}+D_{13}PC_{33}+C_{11}R_{13} \end{pmatrix},$$

$$\boldsymbol{C_2}(P,S) := C(P)S_2 =$$
$$= \begin{pmatrix} C_{11}S_{11}+C_{12}S_{12}^T & C_{11}S_{12}+C_{12}S_{22} & C_{13}+D_{13}PC_{33}-C_{11}S_{13}-C_{12}S_{23} \end{pmatrix}.$$

We observe that all the above functions are affine in the variables $P$, $R$, $S$.

Now we are ready to formulate the main results of this paper, a full solution of the parametric dynamic optimization problem in terms of nicely structured linear matrix inequalities.

**Theorem 1** *There exists a stabilizing dynamic controller $K$ and a parameter $P$ which renders the $H_\infty$-norm of $w \to z$ for the system (12) smaller than $\gamma$ if and only if there exist $P$ and symmetric $R$, $S$ that satisfy the matrix inequalities*

$$\begin{pmatrix} \boldsymbol{Y}(S) & \boldsymbol{Z}(R,S) \\ \boldsymbol{Z}(R,S)^T & \boldsymbol{X}(R) \end{pmatrix} > 0, \tag{21}$$

$$\begin{pmatrix} \Psi^T \begin{pmatrix} \boldsymbol{X}(R) & 0 \\ 0 & \gamma I \end{pmatrix} \Psi & \Psi^T \begin{pmatrix} \boldsymbol{A_1}(P,R) & \boldsymbol{B_1}(P,R) \\ \boldsymbol{C_1}(P,R) & D(P) \end{pmatrix}^T \\ \begin{pmatrix} \boldsymbol{A_1}(P,R) & \boldsymbol{B_1}(P,R) \\ \boldsymbol{C_1}(P,R) & D(P) \end{pmatrix} \Psi & \begin{pmatrix} \boldsymbol{X}(R) & 0 \\ 0 & \gamma I \end{pmatrix} \end{pmatrix} > 0, \tag{22}$$

$$\begin{pmatrix} \Phi^T \begin{pmatrix} \boldsymbol{Y}(S) & 0 \\ 0 & \gamma I \end{pmatrix} \Phi & \Phi^T \begin{pmatrix} \boldsymbol{A_2}(P,R) & \boldsymbol{B_2}(P,R) \\ \boldsymbol{C_2}(P,R) & D(P) \end{pmatrix} \\ \begin{pmatrix} \boldsymbol{A_2}(P,R) & \boldsymbol{B_2}(P,R) \\ \boldsymbol{C_2}(P,R) & D(P) \end{pmatrix}^T \Phi & \begin{pmatrix} \boldsymbol{Y}(S) & 0 \\ 0 & \gamma I \end{pmatrix} \end{pmatrix} > 0. \tag{23}$$

**Proof.** Due to our preparations the proof is short. Since $R$ and $S$ vary in the set of matrices with (20), we conclude that $R_1$ and $S_1$ are non-singular. Hence we can transform (15) by congruence into

$$\begin{pmatrix} S_1^T Y S_1 & S_1^T R_1^T \\ R_1 S_1 & R_1 X R_1^T \end{pmatrix} > 0.$$

Due to $R_1 X R_1^T = R_2 R_1^T = \boldsymbol{X}(R)$ and $S_1^T Y S_1 = S_1^T S_2 = \boldsymbol{Y}(S)$ this is, by definition of the functions $\boldsymbol{X}(R)$, $\boldsymbol{Y}(S)$, $\boldsymbol{Z}(R,S)$, equivalent to (21).

By direct calculation one verifies that

$$\left(\begin{array}{cc} I & 0 \\ 0 & \Psi_{21} \\ 0 & \Psi_{31} \\ \hline 0 & \Psi_{41} \end{array}\right) \underbrace{\begin{pmatrix} R_{11} & R_{12}\Psi_{22} + R_{13}\Psi_{32} \\ 0 & I \end{pmatrix}}_{T} = \left(\begin{array}{ccc|c} R_{11} & R_{12} & R_{13} & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & I \end{array}\right) \left(\begin{array}{cc} I & 0 \\ 0 & \Psi_{21} \\ 0 & \Psi_{31} \\ \hline 0 & \Psi_{41} \end{array}\right).$$

Since $T$ is non-singular, we can transform (16) by congruence with $T$ into the equivalent inequality

$$\Psi^T \left(\begin{array}{c} * \\ * \\ \hline * \\ * \end{array}\right)^T \left(\begin{array}{cc|cc} -X & 0 & 0 & 0 \\ 0 & X & 0 & 0 \\ \hline 0 & 0 & -\gamma I & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma} I \end{array}\right) \left(\begin{array}{cc} R_1^T & 0 \\ A(P)R_1^T & B(P) \\ \hline 0 & I \\ C(P)R_1^T & D(P) \end{array}\right) \Psi < 0.$$

This is obviously equivalent to

$$\Psi^T \left(\begin{array}{c} * \\ * \\ \hline * \\ * \end{array}\right)^T \left(\begin{array}{cc|cc} -R_1 X R_1^T & 0 & 0 & 0 \\ 0 & (R_2^{-1})^T X R_2^{-1} & 0 & 0 \\ \hline 0 & 0 & -\gamma I & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma} I \end{array}\right) \left(\begin{array}{cc} I & 0 \\ R_2 A(P)R_1^T & R_2 B(P) \\ \hline 0 & I \\ C(P)R_1^T & D(P) \end{array}\right) \Psi < 0.$$

Again due to $R_1 X R_1^T = R_2 R_1^T = \boldsymbol{X}(R)$ and $(R_2^{-1})^T X R_2^{-1} = (R_2^{-1})^T R_1^{-1} = (R_1 R_2^T)^{-1} = (R_2 R_1^T)^{-1} = \boldsymbol{X}(R)^{-1}$, we arrive at

$$\Psi^T \left(\begin{array}{c} * \\ * \\ \hline * \\ * \end{array}\right)^T \left(\begin{array}{cc|cc} -\boldsymbol{X}(R) & 0 & 0 & 0 \\ 0 & \boldsymbol{X}(R)^{-1} & 0 & 0 \\ \hline 0 & 0 & -\gamma I & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma} I \end{array}\right) \left(\begin{array}{cc} I & 0 \\ \boldsymbol{A_1}(P,R) & \boldsymbol{B_1}(P,R) \\ \hline 0 & I \\ \boldsymbol{C_1}(P,R) & D(P) \end{array}\right) \Psi < 0.$$

Due to (21), a simple Schur complement argument allows to equivalently rearrange this inequality into (22).

Finally, similar arguments show how to transform (17) into

$$\Phi^T \left(\begin{array}{c} * \\ * \\ \hline * \\ * \end{array}\right)^T \left(\begin{array}{cc|cc} -\boldsymbol{Y}(S)^{-1} & 0 & 0 & 0 \\ 0 & \boldsymbol{Y}(S) & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{\gamma} I & 0 \\ 0 & 0 & 0 & \gamma I \end{array}\right) \left(\begin{array}{cc} -\boldsymbol{A_2}(P,S)^T & -\boldsymbol{C_2}(P,S)^T \\ I & 0 \\ \hline -\boldsymbol{B_2}(P,S)^T & -D(P)^T \\ 0 & I \end{array}\right) \Phi > 0$$

which is, in turn, equivalent to (23).

■

It is now possible to apply standard SDP-solvers in order to compute the smallest $\gamma_{\rm opt}$ for which $P$, $R$, $S$, $\gamma$ satisfy (21)-(23). Let us assume that one has determined a parameter $P$ for some $\gamma > \gamma_{\rm opt}$ such that (21)-(23) are feasible. This assures that, for this specific parameter $P$, there does exist a stabilizing controller $K$ for (14) such that the $H_\infty$-norm of $w \to z$ is smaller than $\gamma$. Since $P$ is fixed, the construction of $K$ amounts to solving a standard $H_\infty$-control which can be done by any out of the multitude of Riccati- or LMI-based algorithms. The McMillan degree of $K$ will (generically and typically) equal the degree of the generalized plan (12).

# 3 A Heuristic Iteration

In the previous section we have seen how to minimize (8) over all stabilizing controllers $K$ and over all parameters $U_j$, $V_j$ for fixed scalar stable transfer functions $u_j$, $v_j$. Moreover, we have stressed that the choice of basis functions which span a dense subset of $RH_\infty$ allows to approximate the optimal value of the genuine multi-objective control problem (6) up to an arbitrary accuracy. However, increasing the number of basis functions will increase the McMillan degree of (10) and hence of (11) which leads to the same growth of the degree of $K$. Even worse, the sizes of the variables $R$, $S$ in (21)-(22) grow quadratically in the McMillan degree of (11). In view of the restrictions on the number of variables of current SDP algorithms for reasonable computation times, it is thus highly desirable to be able to improve the approximation without increasing the number of basis functions but, instead, by trying to adjust their poles.

In this section we propose a heuristic iterative algorithm which involves a model reduction step by balanced truncation. Although one cannot easily provide rigorous arguments on the achievable improvements, it will be demonstrated by means of an example in the next section that this technique can be very beneficial for restraining the controller degree.

The iteration starts with arbitrary stable basis functions $u_j$, $v_j$. If no a priori knowledge is available, one would typically take $u_j(z) = z^{j-1}$ and $v_j(z) = z^{j-1}$. With this choice of basis functions one minimizes (8) to determine (close to) optimal $K$ and $U_j$, $V_j$. Recall that it is actually desired to have $\|\mathcal{T}_{21}(K) + \sum_j U_j u_j\|_\infty$ and $\|\mathcal{T}_{12}(K) + \sum_j V_j v_j\|_\infty$ as small as possible such that (8) is close to $\max\{\|\mathcal{T}_1(K)\|_\infty, \|\mathcal{T}_2(K)\|_\infty\}$. Hence the poles of $\mathcal{T}_{21}(K)$ and $\mathcal{T}_{12}(K)$ are good candidates to build the new basis functions $\tilde{u}_j$, $\tilde{v}_j$ to proceed with the iteration. At this point it is important to recognize that these poles are typically different from those of $u_j$, $v_j$ due to the construction of the dynamic controller $K$ by an

$H_\infty$-design step which performs an optimization over both residues *and* poles together! Note that the McMillan degrees of $\mathcal{T}_{21}(K)$, $\mathcal{T}_{12}(K)$ typically equal $2n + \widehat{n}$ if $n$, $\widehat{n}$ denote the sizes of $A$, $\widehat{A}$ in the realization (11). We conclude that the McMillan degree of the dynamic controller in the next iteration step is guaranteed to be not larger than that of $K$ if the new poles are chosen such that the McMillan degree of (10) is at most $\widehat{n}$. This suggests to perform model reduction for $\mathcal{T}_{21}(K)$, $\mathcal{T}_{12}(K)$ in order to determine a reduced set of poles for continuing the iteration.

In this schematic algorithm there are many concrete choices to be made and we refer to the next section for one concrete implementation on a simple academic example to reveal the benefit of such schemes.

# 4 Numerical Example

Consider a discrete-time system with two SISO performance channels, one control input and two control outputs defined according to (1) with

$$\left(\begin{array}{c|ccc} A & B_1 & B_2 & B \\ \hline C_1 & D_1 & D_{12} & E_1 \\ C_2 & D_{21} & D_2 & E_2 \\ C & F_1 & F_2 & 0 \end{array}\right) = \left(\begin{array}{ccc|ccc} 8/15 & 13/15 & 4/15 & 1/10 & 0 & 0 \\ 2/3 & 0 & -2/5 & 0 & 1/10 & 0 \\ -4/15 & 4/15 & 2/15 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right).$$

Consequently, the closed-loop transform matrix has dimension $2 \times 2$ whereas all Youla parameters are of dimension $1 \times 2$.

For this unstable system the following experiments have been performed:

a) **Youla approach with FIR basis.** Determine a Youla parameterization with an observer-based controller which places the closed-loop poles into $\pm 0.1$, $\pm 0.2$, $\pm 0.3$. With the basis functions $z^{-j}$, $j = 1, \ldots, N$, for both Youla parameters, determine the best achievable approximation for $\gamma_*$ where $N$ is increased from 1 to 10. The computed values are plotted versus the generic controller order as the top dashed curve in Figure 1.

b) **Novel technique starting from FIR basis.** Apply the procedure as sketched in Section 3 where choosing, for each $N = 1, \ldots, 10$, basis functions $u_j$, $v_j$, $j =$

$1,\ldots,N$, such that (10) has McMillan degree $2N$. Start with $u_j(z) = z^{-j}$, $v_j(z) = z^{-j}$ and determine $\max\{\|\mathcal{T}_1(K)\|_\infty, \|\mathcal{T}_2(K)\|_\infty\}$ for the controller which results from minimizing (8). This leads to the second from top dashed curve in Figure 1. Finally, the iteration as in Section 3 proceeds by performing a balanced truncation of both $\mathcal{T}_{21}(K)$ and $\mathcal{T}_{12}(K)$ (typically of McMillan degree $3+2N$) which leads to $N$ new poles to define $u_j$ and $v_j$ respectively such that (10) has, again, McMillan degree $2N$. After five steps of this iteration the values of $\max\{\|\mathcal{T}_1(K)\|_\infty, \|\mathcal{T}_2(K)\|_\infty\}$ are plotted in lowest dashed curve in Figure 1. Figure 3 also depicts the corresponding value curves for the three intermediate iterations.

c) **Mixed design based Youla approach with FIR basis.** With the only difference of basing the Youla parameterization on a problem-related mixed controller, perform the same computations as in a) which leads to the top full-line curve in Figure 1.

d) **Novel technique starting with poles from mixed design.** With the only difference of initializing the basis functions $u_j$, $v_j$ with poles of $\mathcal{T}_{21}(K)$, $\mathcal{T}_{12}(K)$ for a mixed controller $K$, perform the same iteration as in b). The first step results in the value indicated by the circle in Figure 1 and five iterations lead to the lowest full-line curve in Figure 1. Figure 4 depicts the value curves for the intermediate iteration steps.

This example serves to demonstrate the considerable benefit of the novel iteration suggested in this paper if compared to standard Youla-parameterization based approaches. Without using a priori knowledge from a mixed controller as in a) and b) we observe that our technique leads to only slight improvements over the Youla-parameterization-technique with the specifically chosen pole-placing controllers. However, even for this generic initial set-up, the benefit of systematically adjusting the poles leads to a striking improvement of the approximation as can be read off from the dashed lines in Figure 1. If starting from a problem-oriented mixed controller our novel technique is noticeable improved over the those based on the corresponding Youla-parameterization, and the additional iteration leads to results that are pretty close to those without any a priori knowledge, as can be read off from the full lines and the circle in Figure 1. The actual benefit is quantified in Figure 2 where we plot the percentage deviation from the best (lowest) upper bound on the exact optimal value for each controller order.

# 5 Conclusions

We have shown how to solve control problems with multiple LMI-objectives directly in terms of the original system description without performing a Youla parameterization of all stabilizing controllers. As demonstrated by means of an example, iterative schemes allow to considerably improve the approximation quality without having to unduly increase the controller order. All the techniques in this paper admit immediate extensions to multiple $H_\infty$-norm specifications on more than two channels of the controlled system.

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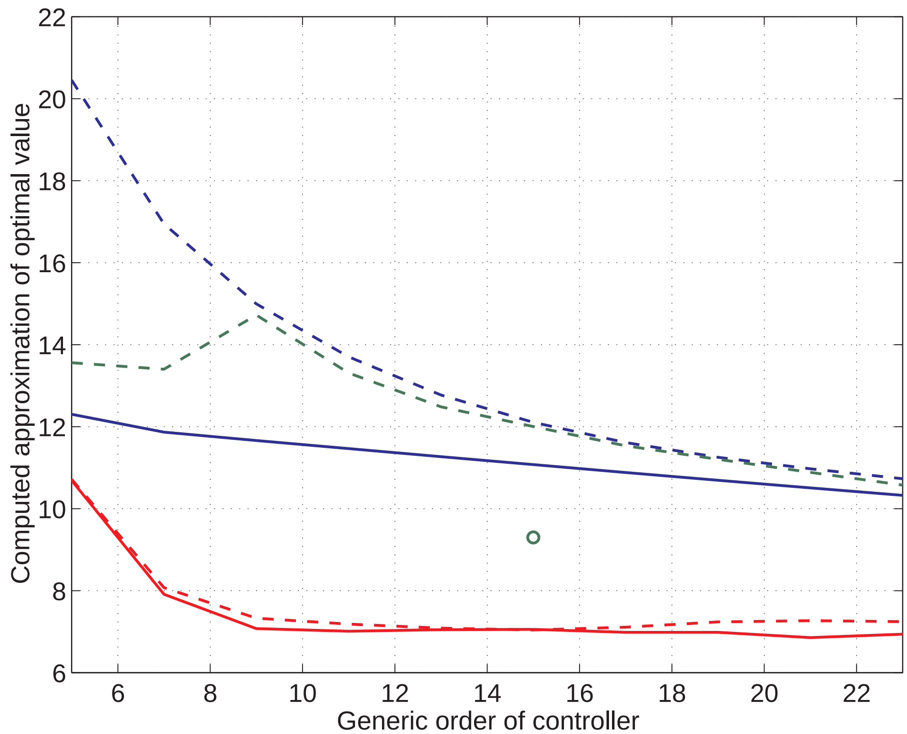


Figure 1: Optimal values for different approximation schemes

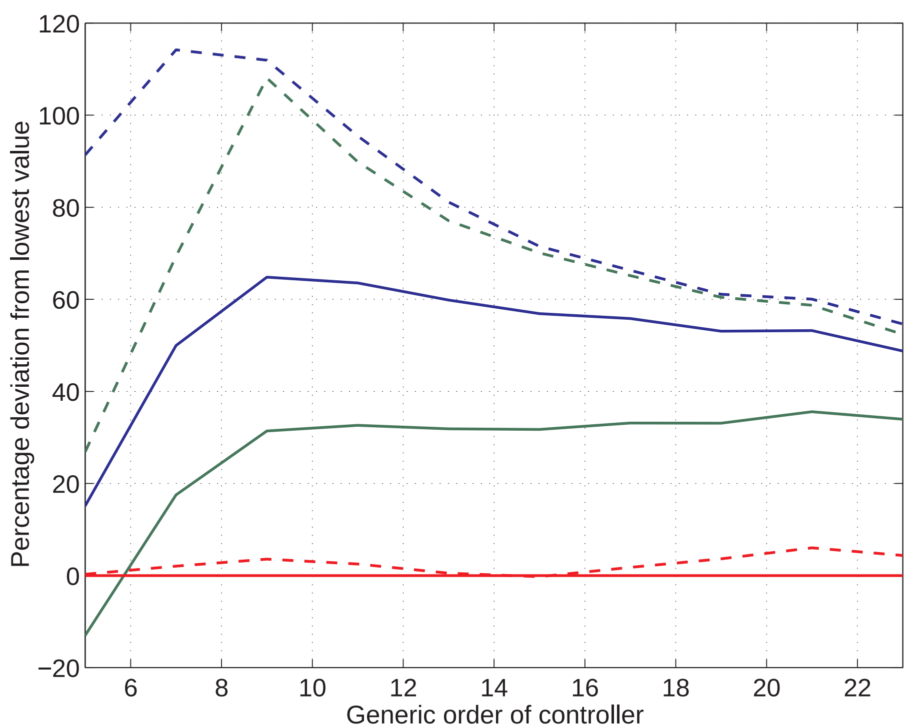


Figure 2: Degradation of different algorithms in percent

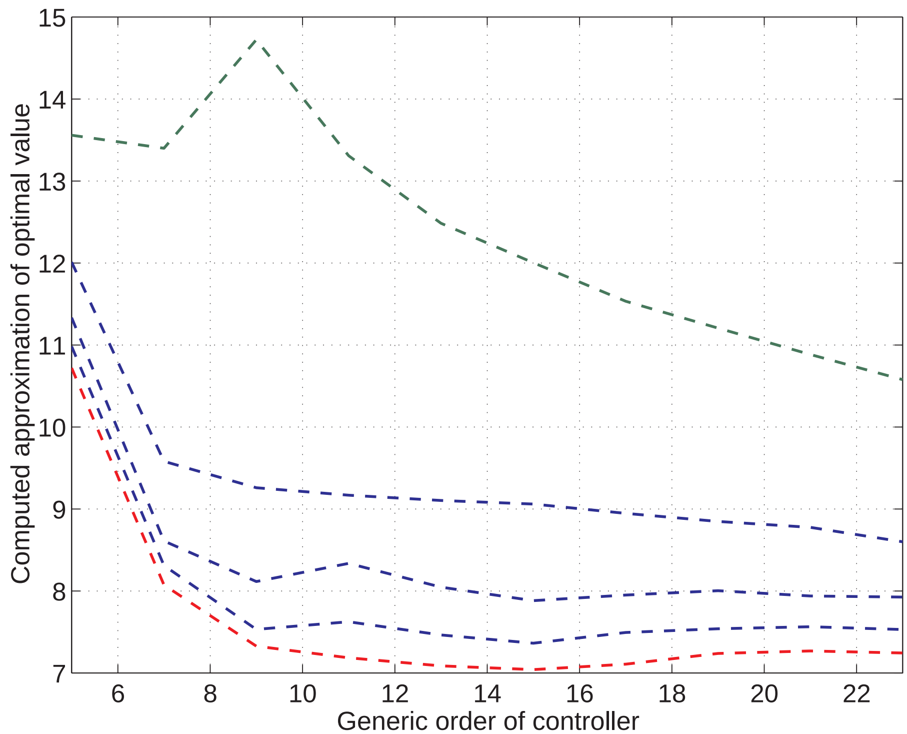


Figure 3: Progress of iteration in b)

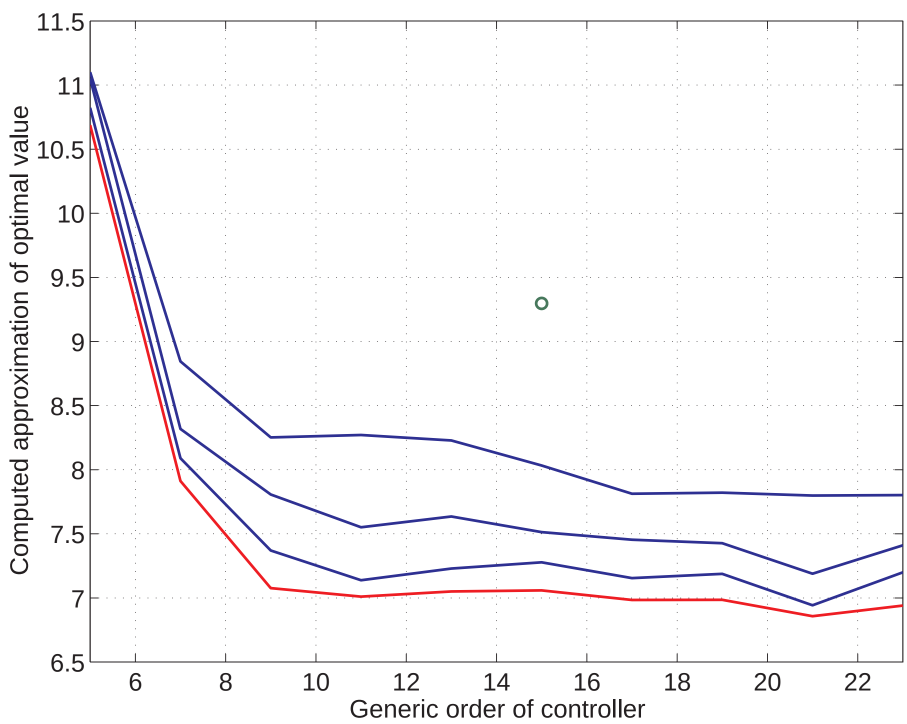


Figure 4: Progress of iteration in d)