Mixed H_2/H_∞ Control[*]

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Abstract

In this article we provide a solution to the mixed H_2/H_∞ problem with reduced order controllers for time-varying systems in terms of the solvability of differential linear matrix inequalities and rank conditions, including a detailed discussion of how to construct a controller. Immediate specializations lead to a solution of the full order problem and the mixed H_2/H_∞ problem for linear systems whose description depends on unknown but in real-time measurable time-varying parameters. As done in the literature for the H_∞ problem, we resolve the quadratic mixed H_2/H_∞ problem by reducing it to the solution of a finite number of algebraic linear matrix inequalities. Moreover, we point out directions how to overcome the conservatism caused by assuming a particular parameter dependence or by using constant solutions of the differential matrix inequalities. For linear time-invariant systems, we reveal how to incorporate robust asymptotic tracking or disturbance rejection as an objective in the mixed H_2/H_∞ problem. Finally, we address the specializations to the fully general pure H_∞ or generalized H_2 problem, and provide quadratically convergent algorithms to compute optimal values. Our techniques do not only lead to insights into the structure of the solution sets of the corresponding linear matrix inequalities, but they also allow to explicitly describe the influence of various system zeros on the optimal values.

Notation

$\mathcal{C} = \mathcal{C}^- \cup \mathcal{C}^0 \cup \mathcal{C}^+$ is the complex plain partitioned into open half planes and the imaginary axis. $\mathcal{R}^n$ is equipped with the Euclidean norm, and $\mathcal{R}^{n\times m}$ with the corresponding induced

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norm, both denoted as $\|.\|$. $\rho(A)$ denotes the spectral radius of the matrix A, and $A^{-1/2}$ the unique positive semidefinite square root of A if $A \geq 0$. L_p denotes the signal space $L_p^n[0, \infty)$ (for an appropriate n) and is equipped with the standard norm $\|.\|_p$. Functions are tacitly assumed to be continuous and bounded, and smooth functions are, in addition, continuously differentiable. Time functions are functions defined on $[0, \infty)$. For a symmetric valued function X defined on S, X is said to be strictly positive ($X \gg 0$) if there exists an $\epsilon > 0$ with $X(s) \geq \epsilon I$ for all $s \in S$. For the system or input output mapping $\dot{x} = Ax + Bu,\ x(0) = 0,\ y = Cx + Du$, we use the notation $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. If $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a constant matrix, the system is called LTI, if it is a time function, the system is called LTV. The time function A is exponentially stable if the system $\dot{x} = Ax$ has this property.

1 Introduction

Recently, linear systems which depend on time-varying a priorily unknown but on-line measurable parameters have gained a lot of interest [50, 38],[3]-[8]. These so-called linear parametrically-varying (LPV) systems appear in robustness problems, in gain-scheduling techniques for nonlinear systems, or in synthesis problems for nonlinear systems that can be described by a differential inclusion [11]. For a detailed discussion we refer to the literature [50, 4].

Given an LPV system, the goal is to construct a controller which not only uses a measured output but, in addition, the on-line measured actual parameters as information in order to exponentially stabilize the system and to provide good performance properties. Until now, the performance objective was specified as an L_2 disturbance attenuation problem with the standard interpretation such as guaranteeing robust stability or tracking [36],[3]-[8]. This is the so-called H_∞ problem for LPV systems.

In principle, an LPV system can be viewed to be time-varying, and any design technique which is available for a specific choice of the performance measure can be used for controller construction. However, the actual parameter curve is not known a priorily and many existing synthesis techniques (as e.g. H_2 and H_∞ control for LTV systems) instantaneously require the knowledge of the parameter values over the whole time interval of interest. For LPV systems and the H_∞ problem, it is not difficult to propose a (pretty conservative) way out of this dilemma: Assume that the parameters are contained in an a priorily given set. Then replace the time-varying solutions of differential Riccati equations along the actual parameter curve by constant solutions of algebraic Riccati inequalities over the whole set of possible parameters. Only under additional hypotheses on the structure of the parameter set (polytopic) and on the dependence of the system on the parameters (affine and partly constant), the verification of the existence of a suboptimal controller and its construction can be reduced to solving a set of linear matrix inequalities [3]-[8].

The purpose of this article is to show that these ideas can be extended to the so-called mixed H_2/H_∞ problem (only whose LTI version has been addressed previously [9, 17, 25, 28, 29, 33, 42, 43, 51, 60]) such that not only robustness specifications (in terms of an H_∞ constraint) but also performance specifications (measured in H_2 norm like criteria or upper bounds thereof) can be taken into account. In fact, we provide a full proof for our central result, a solution of the reduced order mixed H_2/H_∞ problem for LTV systems in terms of the solvability of differential linear matrix inequalities and rank coupling conditions, and we give explicit formulas for a full order controller.

Then we address various specializations of our main result. We point out how to recover results on the pure H_∞ problem for LTV systems [39, 55]. More importantly, we solve the mixed H_2/H_∞ problem for LPV systems not only in the spirit of previous work but also including possible refinements to avoid conservatism. This encompasses a solution of the H_2 problem for LPV systems.

If the system is LTI, we obtain a new solution of the mixed H_2/H_∞ problem in terms of linear matrix inequalities where the underlying system is in no way restricted. Moreover, we show how to incorporate asymptotic tracking or disturbance rejection requirements by extending the system with a suitable internal model. This leads to a solution of the mixed H_2/H_∞ problem with robust regulation (for possibly large plant uncertainties) and extends [1, 2]. For the pure H_∞ problem, we generalize [47, 49] to systems having a nonzero direct feedthrough from the disturbances to the controlled outputs and provide a quadratically convergent algorithm to compute the optimal value. Finally, we reveal that such a computational scheme can be also obtained for the generalized H_2 problem [40], and we provide an explicit formula for the optimal value of the genuine H_2 problem, both for general LTI systems [52].

As auxiliary considerations, we investigate in detail the estimation inequality both for time-varying and for time-invariant data. In the latter case, we are not only able to gain insights into the structure of the solutions of this algebraic linear matrix inequality, but we can also explicitly display the influence of various system zeros on the solvability.

The article is organized as follows. In Section 2, we define the mixed H_2/H_∞ objective for time-varying systems and address the related analysis tests, including the role of scalings, for reasons of space mainly without proofs. Section 3 contains our main result, a solution to the reduced order controller mixed H_2/H_∞ control problem in terms of differential linear matrix inequalities, including explicit formulas for controllers. In Section 4, we show how the estimation differential inequality can be reduced to an initial value problem for a perturbed Riccati differential equation. For time-invariant data, we investigate the corresponding algebraic linear matrix inequality in full generality. Section 5 summarizes the consequences for our main result. In Section 6, we discuss linear parametrically-varying systems and demonstrate a controller construction by solving a finite number of algebraic linear matrix inequalities. In Section 7, we address the mixed H_2/H_∞ problem with robust regulation for LTI systems, and the Sections 8 and 9 are devoted to the pure H_∞ and (generalized) H_2 problems respectively.

Short proofs which provide insights into construction schemes are included in the text whereas more technical proofs are collected in the appendix.

2 Mixed H_2/H_∞ Performance Bounds

Consider the LTV system

$$z = Tw = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} w = \left[\begin{array}{c|c} A & G \\ \hline H_1 & F_1 \\ H_2 & F_2 \end{array}\right] w. \tag{1}$$

We interpret $w \to z_1$ as the robustness channel and $w \to z_2$ as the performance channel. To be more specific, we assume that the uncertainty of the system is described by $w = \Delta z$ where Δ comprises the set of (possibly nonlinear) operators $L_2 \to L_2$ with incremental gain [13] not larger than $1/\gamma$. If A is exponentially stable, the small-gain theorem implies that stability is preserved if $\|T_1\|_\infty < \gamma$, where

$$\|T_1\|_\infty := \sup_{w \in L_2, \|w\|_2 = 1} \|T_1 w\|_2$$

defines the operator norm of T_1 induced by signals in L_2. In the LTI case, there is a well-known test on the state space matrices which characterizes stability of A and $\|T_1\|_\infty < \gamma$, the so-called Bounded Real Lemma (BRL) [57, 22, 26]. It is not difficult to prove the following generalization to LTV systems. Recall that time functions are bounded, and that a symmetric valued time function X is strictly positive ($X \gg 0$) if there exists an $\epsilon > 0$ with $X(t) \geq \epsilon I$ for all $t \geq 0$.

Theorem 1 *(LTV Strict Bounded Real Lemma) The time function A is exponentially stable and* $\left\|\left[\begin{array}{c|c} A & G \\ \hline H_1 & F_1 \end{array}\right]\right\|_\infty < \gamma$ *iff there exists a smooth time function X such that*

$$X \gg 0, \quad \begin{pmatrix} \dot{X} + A^T X + XA & XG & H_1^T \\ G^T X & -\gamma I & F_1^T \\ H_1 & F_1 & -\gamma I \end{pmatrix} \ll 0. \tag{2}$$

Hence norm bounds can be characterized by the existence of strictly positive solutions to a strict differential linear matrix inequality (DLMI). Note that asking for the existence of a constant solution is equivalent to the popular concept of quadratic H_∞ performance for an LTV system [3]-[8]. If T_1 is LTI, it is no loss of generality to confine X in Theorem 1 to be *constant* and the inequalities (2) become algebraic.

The channel $w \to z_2$ of (1) is used for describing performance specifications. Indeed, we have in mind to generalize certain H_2 norm like criteria for LTI systems [40] to the LTV system T_2. Among the several possibilities, we pay special attention to the deterministic criterion of assessing performance by the largest amplitude of z_2 for all w of finite and bounded energy. This is particularly useful if (components of) z_2 are interpreted as tracking errors. To quantify the gain of T_2 mapping L_2 into L_∞, we define the induced norm

$$\|T_2\| := \sup_{w \in L_2, \|w\|_2 = 1} \|T_2 w\|_\infty.$$

If A is exponentially stable, let Y denote the (bounded) solution of the initial value problem

$$\dot{Y} = AY + YA^T + GG^T, \; Y(0) = 0. \tag{3}$$

If $F_2 = 0$, one has $\|T_2\|^2 = \sup_{t\geq 0} \|H_2 Y H_2^T(t)\|$. This allows to prove the following analogue of Theorem 1.

Theorem 2 *A is exponentially stable and* $\left\| \left[\begin{array}{c|c} A & G \\ \hline H_2 & F_2 \end{array} \right] \right\|^2 < \beta$ *iff* $F_2 = 0$ *and there exists a smooth* Z *with*

$$Z \gg 0, \quad \dot{Z} + A^T Z + ZA + ZGG^T Z \ll 0, \quad H_2 Z^{-1} H_2^T \ll \beta I. \tag{4}$$

If defining the size of the amplitude of z_2 with the spatial norm $\max_j |x_j|$, the squared gain of T_2 equals $\sup_{t\geq 0} \max_j d_j[H_2 Y H_2^T(t)]$, where $d_j(M)$ denotes the j-th diagonal element of the square matrix M, and Y solves (3). Theorem 2 remains valid for this norm after replacing $H_2 Z^{-1} H_2^T \ll \beta I$ by $\max_j d_j(H_2 Z^{-1} H_2^T) \ll \beta$.

An alternative measure arises with a stochastic interpretation. If w is white noise, we recall (due to $x_0 = 0$ and hence $E(x_0 x_0^T) = 0$) that $E(z_2 z_2^T) = H_2 Y H_2^T$ [30]. Then

$$\|T_2\|_2^2 := \sup_{t\geq 0} E(z_2^T z_2) = \sup_{t\geq 0} \text{trace}[H_2 Y H_2^T(t)]$$

defines the maximal output variance and is a generalization of the genuine H_2 norm to LTV systems. Theorem 2 persists to hold for the H_2 norm with $H_2 Z^{-1} H_2^T \ll \beta I$ replaced by $\text{trace}(H_2 Z^{-1} H_2^T) \ll \beta$.

Even for LTI systems, the synthesis problem of optimizing $\|.\|$ or $\|.\|_2$ of over all stabilizing controllers which keep a bound on the norm $\|.\|_\infty$ for a different channel seems very hard [28, 42]. This motivates to replace the objective functional by an upper bound. Let us define

$$J(T) := \inf\{\alpha \mid \exists \text{ smooth time function } X \text{ with (2) and } H_2 X^{-1} H_2 \ll \alpha I\} \tag{5}$$

(including, as usual, $J(T) = \infty$ if no solution of (2) exists.) Then $J(T) < \alpha$ implies $H_2 X^{-1} H_2 \ll \alpha I$ for some solution X of (2). Note that any such X satisfies $\dot{X} + A^T X + XA + \frac{1}{\gamma} XGG^T X \ll 0$. If $F_2 = 0$, Theorem 2 allows to infer $\|T_2\|^2 < \alpha\gamma$. We conclude

$$\|T_2\|^2 \leq \gamma J(T)$$

and $J(T)$ is indeed an upper bound of $\|T_2\|^2/\gamma$. If $\gamma = 1, \; H_1 = 0, \; F_1 = 0$, the solution sets of the DLMI in (2) and of the differential Riccati inequality in (4) are clearly identical, what implies $\|T_2\|^2 = J(T)$ and *recovers* $\|T_2\|^2$. Similar conclusions hold for the other norms of T_2.

One can clearly define $J(T)$ via (5) confining X to *constant* solutions of (2). This generalizes the so-called quadratic H_∞ performance specification to the quadratic mixed H_2/H_∞

specification and allows a specialization to the pure H_2 case. If T is LTI, it is well-known that this restriction to *constant* X causes no loss of generality, and we recover the definitions in [42, 40].

Let us finally comment on scalings. If one can take structural or other properties of the system uncertainty into account, the perturbation Δ is restricted to a certain subset $\boldsymbol{\Delta}_\gamma$ of all operators with incremental gain at most $1/\gamma$. It might then be possible to identify a class of scalings containing pairs of matrix valued time functions (S, S_1) with bounded inverses such that

$$S\Delta S_1 \text{ has incremental gain at most } 1/\gamma \text{ for all } \Delta \in \boldsymbol{\Delta}_\gamma.$$

The existence of a pair of scalings (S, S_1) in this class with

$$\|S_1^{-1}T_1S^{-1}\|_\infty < \gamma$$

is an obvious sufficient condition for stability robustness against uncertainties in $\boldsymbol{\Delta}_\gamma$, and it is weaker than $\|T_1\|_\infty < \gamma$. A most prominent specific example of this concept is the μ-upper bound with constant scalings. This gives a systematic tool to incorporate in this channel not only stability robustness requirements against structured uncertainties, but also robust performance specifications in the induced L_2-norm, as usually done for LTI systems [37]. One might as well specify a set of scalings S_2 for weighting the performance output and modeling alternative performance specifications [44]. The benefit of incorporating scalings S_2 for the channel z_2 remains to be explored.

Summarizing, if having fixed a class of scalings $\boldsymbol{S}$ consisting of triples (S_1, S_2, S) of time functions with bounded inverse, we can alternatively define

$$J_s(T) \text{ to be the infimal } \alpha$$

for which there exists an $(S_1, S_2, S) \in \boldsymbol{S}$ and a solution to the BRL inequality corresponding to $\|S_1^{-1}T_1S^{-1}\|_\infty < \gamma$ such that $S_2^{-1}H_2X^{-1}H_2^TS_2^{-T} \ll \alpha I$.

3 The Mixed H_2/H_∞ Synthesis Problem

Suppose a specific control task (including the specification of weightings) leads to the generalized LTV plant

$$\begin{pmatrix} y \\ z_1 \\ z_2 \end{pmatrix} = \left[\begin{array}{c|cc} A & B & G \\ \hline C & 0 & D \\ H_1 & E_1 & F_1 \\ H_2 & E_2 & F_2 \end{array}\right] \begin{pmatrix} u \\ w \end{pmatrix} \tag{6}$$

where A is of size $n \times n$. With the LTV controller

$$u = Ry = \left[\begin{array}{c|c} K & L \\ \hline M & N \end{array}\right] y, \tag{7}$$

the closed loop system is described as

$$z = T(R)w = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} T_1(R) \\ T_2(R) \end{pmatrix} w = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{G} \\ \hline \mathcal{H}_1 & \mathcal{F}_1 \\ \mathcal{H}_2 & \mathcal{F}_2 \end{array}\right] w$$

where

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{G} \\ \hline \mathcal{H}_1 & \mathcal{F}_1 \\ \mathcal{H}_2 & \mathcal{F}_2 \end{array}\right] = \left[\begin{array}{cc|c} A + BNC & BM & G + BND \\ LC & K & LD \\ \hline H_1 + E_1NC & E_1M & F_1 + E_1ND \\ H_2 + E_2NC & E_2M & F_2 + E_2ND \end{array}\right].$$

The controller R is called *stabilizing* if $\mathcal{A}$ is exponentially stable. We intend to minimize $J(T(R))$ over all LTV controllers R. Recall that $J(T(R)) < \infty$ automatically implies that R is stabilizing. It is standard to approach this problem via a suboptimality test: Characterize whether there exists an R with $J(T(R)) < \alpha$ or, equivalently, whether there exists an R and a smooth time function $\mathcal{X}$ such that

$$\mathcal{X} \gg 0, \quad \begin{pmatrix} \dot{\mathcal{X}} + \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{G} & \mathcal{H}_1^T \\ \mathcal{G}^T\mathcal{X} & -\gamma I & \mathcal{F}_1^T \\ \mathcal{H}_1 & \mathcal{F}_1 & -\gamma I \end{pmatrix} \ll 0, \quad \mathcal{H}_2\mathcal{X}^{-1}\mathcal{H}_2 \ll \alpha. \tag{8}$$

This not only guarantees robust stability (against perturbations with incremental gain at most $1/\gamma$) but also a performance level $\alpha\gamma$.

Recent approaches to the H_∞ problem for LTI systems are based on the following observations: The BRL inequality in (8) is, for a fixed $\mathcal{X}$, *linear in the controller parameters.* Hence one can eliminate these parameters using the so-called Projection Lemma (Lemma 7), what leads to a suboptimality test in terms of linear inequalities in (parts) of $\mathcal{X}$ and $\mathcal{X}^{-1}$ [22, 26, 49].

In the mixed problem (8), we have to fulfill *three* inequalities what makes it impossible to eliminate *all* controller parameters from the final characterization. Instead, we intend to keep as many (transformed) controller parameters as possible such that, still, matrix inequalities result which are linear in parts of $\mathcal{X}$ and $\mathcal{X}^{-1}$ *and* in the remaining (transformed) controller parameters. In fact, the key idea has its origins in [45, 49]: Eliminate K in (8) (which only affects the BRL inequality), transform L and M, and keep N to achieve the desired structure.

A central step in the proof is the following very simple explicit result for the solvability of a specially structured inequality. In fact, it will turn out in Section 4 that this is a version of the so-called 'completion of the squares argument' which is most suited for our purposes.

Lemma 3 *Let Q be a symmetric (partitioned) time function and consider the inequality*

$$\begin{pmatrix} Q_1 & Q_{21}^T & Q_{31}^T + X^T \\ Q_{21} & Q_2 & Q_{32}^T \\ Q_{31} + X & Q_{32} & Q_3 \end{pmatrix} \ll 0 \tag{9}$$

in the unstructured time function X. This inequality has a solution X iff

$$\begin{pmatrix} Q_1 & Q_{21}^T \\ Q_{21} & Q_2 \end{pmatrix} \ll 0 \quad \textit{and} \quad \begin{pmatrix} Q_2 & Q_{32}^T \\ Q_{32} & Q_3 \end{pmatrix} \ll 0. \tag{10}$$

If (9) is solvable, one particular solution is given by

$$X = Q_{32} Q_2^{-1} Q_{21} - Q_{31}. \tag{11}$$

Proof. If (9) has a solution then (10) just follow from (9) by canceling the first or third block row/column.

Now suppose that (10) holds what implies $Q_2 \ll 0$. The central trick is to cancel in (9) the block Q_{21} by a congruence transformation. To be specific, (9) is equivalent to

$$\begin{pmatrix} I & -Q_{21}^T Q_2^{-1} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} (9) \begin{pmatrix} I & 0 & 0 \\ -Q_2^{-1} Q_{21} & I & 0 \\ 0 & 0 & I \end{pmatrix} \ll 0$$

which rewrites to

$$\begin{pmatrix} Q_1 - Q_{21}^T Q_2^{-1} Q_{21} & 0 & Q_{31}^T - Q_{21}^T Q_2^{-1} Q_{32}^T + X^T \\ 0 & Q_2 & Q_{32}^T \\ Q_{31} - Q_{32} Q_2^{-1} Q_{21} + X & Q_{32} & Q_3 \end{pmatrix} \ll 0.$$

X defined in (11) is a solution since (10) implies $Q_1 - Q_{21}^T Q_2^{-1} Q_{21} \ll 0$. ■

Now we are ready to formulate and prove our main result.

Theorem 4 *There exists a controller* $R := \left[\begin{array}{c|c} K & L \\ \hline M & N \end{array}\right]$ *with K of size $k \le n$ which satisfies $J(T(R)) < \alpha$ iff there exist $\epsilon > 0$, time functions X, Y, Z with*

$$X \gg 0,\ Y \gg 0,\ X - Y^{-1} = ZZ^T,\ Z \text{ of size } n \times k,\ Z^T Z \gg 0, \tag{12}$$

and time functions F, J, N such that

$$\mathit{rank} \begin{pmatrix} F & NC \\ Y & I \\ I & X \end{pmatrix} = k + n,\ \mathit{rank} \begin{pmatrix} BN & Y & I \\ J & I & X \end{pmatrix} = k + n, \tag{13}$$

and such that the following DLMIs are satisfied:

$$\begin{pmatrix} \dot{X} + A^T X + XA + JC + (JC)^T & XG + JD & (H_1 + E_1 NC)^T \\ (XG + JD)^T & -\gamma I & (F_1 + E_1 ND)^T \\ H_1 + E_1 NC & F_1 + E_1 ND & -\gamma I \end{pmatrix} \ll 0, \tag{14}$$

$$\begin{pmatrix} -\dot{Y} + AY + YA^T + BF + (BF)^T & G + BND & (H_1 Y + E_1 F)^T \\ (G + BND)^T & -\gamma I & (F_1 + E_1 ND)^T \\ H_1 Y + E_1 F & F_1 + E_1 ND & -\gamma I \end{pmatrix} \ll 0, \tag{15}$$

$$\begin{pmatrix} \alpha - \epsilon I & H_2Y + E_2F & H_2 + E_2NC \\ (H_2Y + E_2F)^T & Y & I \\ (H_2 + E_2NC)^T & I & X \end{pmatrix} \geq 0. \tag{16}$$

Proof of necessity. Suppose, for some controller of size $k \leq n$, there exists a smooth time function $\mathcal{X}$ with (8). We fix $\mathcal{X}$ and view the BRL inequality in (8) as an inequality in K. To be specific, define $\mathcal{I} := \begin{pmatrix} 0 & I \end{pmatrix}$ to get

$$\mathcal{A} = \begin{pmatrix} A + BNC & BM \\ LC & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} K \begin{pmatrix} 0 & I \end{pmatrix} = \tilde{\mathcal{A}} + \mathcal{I}^T K \mathcal{I}.$$

Then the BRL inequality reads as

$$\begin{pmatrix} \dot{\mathcal{X}} + \tilde{\mathcal{A}}^T\mathcal{X} + \mathcal{X}\tilde{\mathcal{A}} + \mathcal{I}^T K^T \mathcal{I}\mathcal{X} + \mathcal{X}\mathcal{I}^T K \mathcal{I} & \mathcal{X}\mathcal{G} & \mathcal{H}_1^T \\ \mathcal{G}^T\mathcal{X} & -\gamma I & \mathcal{F}_1^T \\ \mathcal{H}_1 & \mathcal{F}_1 & -\gamma I \end{pmatrix} \ll 0. \tag{17}$$

The matrix $\mathcal{J} := \begin{pmatrix} I \\ 0 \end{pmatrix}$ clearly satisfies $\mathcal{I}\mathcal{J} = 0$. Hence (17) implies

$$\begin{pmatrix} \mathcal{J}^T[\dot{\mathcal{X}} + \tilde{\mathcal{A}}^T\mathcal{X} + \mathcal{X}\tilde{\mathcal{A}}]\mathcal{J} & \mathcal{J}^T\mathcal{X}\mathcal{G} & \mathcal{J}^T\mathcal{H}_1^T \\ \mathcal{G}^T\mathcal{X}\mathcal{J} & -\gamma I & \mathcal{F}_1^T \\ \mathcal{H}_1\mathcal{J} & \mathcal{F}_1 & -\gamma I \end{pmatrix} \ll 0. \tag{18}$$

Similarly, with

$$\mathcal{Y} := \mathcal{X}^{-1},$$

we infer $(\mathcal{I}\mathcal{X})\mathcal{Y}\mathcal{J} = 0$ and thus

$$\begin{pmatrix} \mathcal{J}^T[-\dot{\mathcal{Y}} + \mathcal{Y}\tilde{\mathcal{A}}^T + \tilde{\mathcal{A}}\mathcal{Y}]\mathcal{J} & \mathcal{J}^T\mathcal{G} & \mathcal{J}^T\mathcal{Y}\mathcal{H}_1^T \\ \mathcal{G}^T\mathcal{J} & -\gamma I & \mathcal{F}_1^T \\ \mathcal{H}_1\mathcal{Y}\mathcal{J} & \mathcal{F}_1 & -\gamma I \end{pmatrix} \ll 0. \tag{19}$$

Let us partition $\mathcal{X}$, $\mathcal{Y}$ according to $\mathcal{A}$ into n and k rows/columns as

$$\mathcal{X} = \begin{pmatrix} X & U \\ U^T & \hat{X} \end{pmatrix}, \quad \mathcal{Y} = \begin{pmatrix} Y & V \\ V^T & \hat{Y} \end{pmatrix}, \tag{20}$$

and let us recall

$$\mathcal{Y} = \begin{pmatrix} [X - U\hat{X}^{-1}U^T]^{-1} & -[X - U\hat{X}^{-1}U^T]^{-1}U\hat{X}^{-1} \\ -\hat{X}^{-1}U^T[X - U\hat{X}^{-1}U^T]^{-1} & [\hat{X} - U^T X^{-1} U]^{-1} \end{pmatrix}. \tag{21}$$

We can assume without loss of generality that

$$U \text{ (of dimension } n \times k\text{) satisfies } U^T U \gg 0;$$

if not true, just perturb $\mathcal{X}$ suitably without violating (8). The formula (21) reveals $Y^{-1} = X - U\hat{X}^{-1}U^T$. If Cholesky factorizing $\hat{X} = W^TW$ such that W is smooth, bounded, and has a bounded inverse, we arrive at (12) for $Z := UW^{-1}$.

A simple computation shows, with the identities

$$F = NCY + MV^T \ \text{ and } \ J = XBN + UL, \tag{22}$$

that the left-hand sides of (18), (14) and (19), (15) are *identical.* We remark that we define F and J via these equations for proving necessity, and we view (22) as equations in L and M for constructing a controller in the sufficiency proof. Under the hypothesis (12), it is now not difficult to see that (22) are solvable as equations in L and M iff the rank conditions (13) hold true. We clarify this for the first equation: It is solvable iff $\ker(V^T) \subset \ker(F - NCY)$. The (1,1) block of $\mathcal{X}\mathcal{Y} = I$ implies $XY + UV^T = I$ which reveals $\ker(I - XY) = \ker(V^T)$ since U has full column rank. Hence, solvability is equivalent to $\ker(I - XY) \subset \ker(F - NCY)$. By (12), $I - XY$ has rank k and, therefore, this inclusion is equivalent to the first condition in (13) since

$$\begin{pmatrix} F & NC \\ Y & I \\ I & X \end{pmatrix} \begin{pmatrix} I & 0 \\ -Y & I \end{pmatrix} = \begin{pmatrix} F - NCY & NC \\ 0 & I \\ I - XY & X \end{pmatrix}.$$

Finally, there is an $\epsilon > 0$ with $\mathcal{H}_2\mathcal{X}^{-1}\mathcal{H}_2^T \leq (\alpha - \epsilon)I$ which is equivalent to

$$\begin{pmatrix} (\alpha - \epsilon)I & \mathcal{H}_2 \\ \mathcal{H}_2^T & \mathcal{X} \end{pmatrix} \geq 0. \tag{23}$$

With the $(n + k) \times 2n$ function

$$\mathcal{Z} := \begin{pmatrix} Y & I \\ V^T & 0 \end{pmatrix}, \tag{24}$$

the inequality (23) implies

$$\begin{pmatrix} (\alpha - \epsilon)I & \mathcal{H}_2\mathcal{Z} \\ \mathcal{Z}^T\mathcal{H}_2^T & \mathcal{Z}^T\mathcal{X}\mathcal{Z} \end{pmatrix} \geq 0. \tag{25}$$

Recalling the definition of F and computing $\mathcal{Z}^T\mathcal{X}\mathcal{Z} = \mathcal{Z}^T \begin{pmatrix} I & X \\ 0 & U^T \end{pmatrix} = \begin{pmatrix} Y & I \\ I & X \end{pmatrix}$ leads to (16). This proves necessity. ■

Constructive proof of sufficiency. Define $U := Z$, $\hat{X} := I$, and $\mathcal{X} := \begin{pmatrix} X & U \\ U^T & I \end{pmatrix}$. Due to $X - UU^T \gg 0$, $\mathcal{X}$ is smooth, bounded, and strictly positive, and the same holds for $\mathcal{Y} := \mathcal{X}^{-1}$. Again because of (12) and (21), $\mathcal{Y}$ has Y as its left-upper block. If using the partitions (20), we have $U^TU \gg 0$ by hypothesis and, since $V = -YU$ from (21), $V^TV \gg 0$ as a consequence. Let us now define the time functions

$$M := (F - NCY)V(V^TV)^{-1} \text{ and } L := (U^TU)^{-1}U^T(J - XBN).$$

Since U and V have full column rank, these are the unique solutions of the equations (22) if they are solvable at all. The latter, however, is assured by (13) as clarified in the necessity proof.

With L, M, N given so far, we can define $\tilde{\mathcal{A}}$, $\mathcal{G}$, $\mathcal{H}_j$, $\mathcal{F}_j$, $j = 1, 2$. If we introduce $\mathcal{Z}$ as in (24), we infer that (16) is the same as (25). Since $\mathcal{Z}$ has full row rank and thus a right-inverse, we can get back to (23) which leads to the third inequality in (8). Moreover, (14) and (15) are identical to (18) and (19) respectively. Hence it remains to find a time function K which satisfies (17) and, therefore, leads to the BRL inequality in (8).

We use $\mathcal{X}\mathcal{I}^T K\mathcal{I} = \begin{pmatrix} Z \\ I_k \end{pmatrix} K \begin{pmatrix} 0 & I_k \end{pmatrix}$ where we display the size of the identity blocks by using the index. Due to $Z^T Z \gg 0$, there exists a smooth time function Z_e such that $\begin{pmatrix} Z & Z_e \end{pmatrix}$ has a smooth bounded inverse. With the first k rows S_1 and the last $n-k$ rows S_2 of this inverse, we conclude that

$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \text{ has a smooth bounded inverse and } \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} Z = \begin{pmatrix} I_k \\ 0 \end{pmatrix}.$$

With

$$\mathcal{S} := \begin{pmatrix} S_1 & 0 \\ S_2 & 0 \\ -S_1 & I_k \end{pmatrix} \text{ we get } \mathcal{S}\begin{pmatrix} Z \\ I_k \end{pmatrix} = \begin{pmatrix} I_k \\ 0 \\ 0 \end{pmatrix} \text{ and } \mathcal{S}\begin{pmatrix} 0 \\ I_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ I_k \end{pmatrix}.$$

Note that both $\mathcal{S}$ and $\mathcal{S}^{-1}$ are smooth time functions. We can left-multiply the first row of (17) with $\mathcal{S}$ and right-multiply the first column of (17) with $\mathcal{S}^T$. After this congruence transformation, (17) is equivalent to

$$\begin{pmatrix} Q_1 & Q_{12} + K & Q_{13} \\ Q_{21} + K^T & Q_2 & Q_{23} \\ Q_{31} & Q_{32} & Q_3 \end{pmatrix} \ll 0 \tag{26}$$

for some computable Q. Now we just note that (18), (19) are equivalent to $\begin{pmatrix} Q_1 & Q_{13} \\ Q_{31} & Q_3 \end{pmatrix} \ll 0$ and $\begin{pmatrix} Q_2 & Q_{23} \\ Q_{32} & Q_3 \end{pmatrix} \ll 0$. Lemma 3 then reveals that $K := Q_{13}Q_3^{-1}Q_{32} - Q_{12}$ is indeed a time function which leads to (26) and hence to (17). The controller construction is complete. ■

The following comments on Theorem 4 also apply, without explicitly mentioning, to all the other problems that will be considered in this article.

Remark on the dual problem. Consider the LTV system

$$\begin{pmatrix} y \\ z_1 \\ z_2 \end{pmatrix} = \left[\begin{array}{c|ccc} A & B & G & G_2 \\ \hline C & 0 & D & D_2 \\ H_1 & E_1 & F_1 & F_{12} \\ H_2 & E_2 & F_2 & F_{22} \end{array}\right] \begin{pmatrix} u \\ w \\ w_2 \end{pmatrix}.$$

The existence of a controller (7) such that the closed loop system described by $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \left[\begin{array}{c|cc} \mathcal{A} & \mathcal{G} & \mathcal{G}_2 \\ \hline \mathcal{H}_1 & \mathcal{F}_1 & \mathcal{F}_{12} \\ \mathcal{H}_2 & \mathcal{F}_2 & \mathcal{F}_{22} \end{array}\right] \begin{pmatrix} w \\ w_2 \end{pmatrix}$ admits a smooth $\mathcal{X}$ *with (8) and* $\mathcal{G}_2^T \mathcal{X} \mathcal{G}_2 \ll \alpha$ is characterized exactly as in Theorem 4 if just *adding*

$$\begin{pmatrix} (\alpha - \epsilon) I & (XG_2 + JD_2)^T & (G_2 + BND_2)^T \\ XG_2 + JD_2 & X & I \\ G_2 + BND_2 & I & Y \end{pmatrix} \geq 0.$$

Then the solution of the *dual* problem is obtained by canceling (16).

Remark on the controller construction. It is not difficult to see that any $\mathcal{X}$ satisfying (8) can be transformed by a coordinate change in the controller state (which amounts to a congruence transformation on $\mathcal{X}$) such that its right-lower block is identical to I. Hence this choice in the controller construction can be made without loss of generality.

Remark on scalings. Since scalings only change the data matrices, we can directly derive the corresponding characterization for $J_s(T(R)) < \alpha$ without new proof. In Theorem 4, we just need to require, in addition, the existence of scalings $(S_1, S_2, S) \in \boldsymbol{S}$ and replace the (2,2) identity blocks in (14) and (15) by $S^T S$, the (3,3) identity blocks by $S_1 S_1^T$, and the (1,1) identity block in (16) by $S_2 S_2^T$. Hence the DLMIs are *linear* in $(S^T S, S_1 S_1^T, S_2 S_2^T)$ for $(S_1, S_2, S) \in \boldsymbol{S}$. It might be possible to reparametrize $\{(S^T S, S_1 S_1^T, S_2 S_2^T) \mid (S_1, S_2, S) \in \boldsymbol{S}\}$ and to transform the inequalities (14)-(16) such that the new parameters enter linearly [37, 38, 36, 3]. Generally, however, one has to turn to heuristic ways out [41].

Remark on other performance measures. Let e_j denote the standard unit vector. Clearly, $\max_j d_j(\mathcal{H}_2 \mathcal{X}^{-1} \mathcal{H}_2^T) \ll \alpha$ is equivalent to $\begin{pmatrix} \alpha - \epsilon & e_j^T \mathcal{H}_2 \\ \mathcal{H}_2^T e_j & \mathcal{X} \end{pmatrix} \geq 0$ for all j and some $\epsilon > 0$. Hence, if we replace the performance inequality in (8) by this alternative one, we conclude without new proof that Theorem 4 remains valid if replacing (16) with

$$\forall j: \begin{pmatrix} \alpha - \epsilon & e_j^T (H_2 Y + E_2 F) & e_j^T (H_2 + E_2 NC) \\ (H_2 Y + E_2 F)^T e_j & Y & I \\ (H_2 + E_2 NC)^T e_j & I & X \end{pmatrix} \geq 0.$$

If H_2 has l rows, the same holds for $\operatorname{trace}(\mathcal{H}_2 \mathcal{X}^{-1} \mathcal{H}_2^T) \ll \alpha$ if replacing (16) with

$$\begin{pmatrix} \alpha - \epsilon & * & * & \cdots & * & * \\ (H_2 Y + E_2 F)^T e_1 & Y & I & & 0 & 0 \\ (H_2 + E_2 NC)^T e_1 & I & X & & 0 & 0 \\ \vdots & & & \ddots & & \\ (H_2 Y + E_2 F)^T e_l & 0 & 0 & & Y & I \\ (H_2 + E_2 NC)^T e_l & 0 & 0 & & I & X \end{pmatrix} \geq 0.$$

Note that one can easily extend all this to a system with more than one performance output and different performance specifications on each of these outputs; one just needs to add the corresponding inequalities in Theorem 4.

Remarks concerning direct feedthroughs. Just by setting $N = 0$, we can extract a suboptimality test for controllers of the form $\left[\begin{array}{c|c} K & L \\ \hline M & 0 \end{array}\right]$. Due to the H_2 nature of the performance specification, one might wish to include the requirement $F_2 + E_2ND = 0$ on N. This puts another *linear* restriction on N without destroying the structure of the DLMIs.

Remark on special problems. The state-feedback problem is obtained by setting $C = I$ and $D = 0$. As immediate necessary conditions for suboptimality, there exist time functions Y (smooth) and F with

$$\begin{pmatrix} -\dot{Y} + AY + YA^T + BF + (BF)^T & G & * \\ G^T & -\gamma I & F_1^T \\ H_1Y + E_1F & F_1 & -\gamma I \end{pmatrix} \ll 0, \ \begin{pmatrix} \alpha I & H_2Y + E_2F \\ * & Y \end{pmatrix} \gg 0.$$

If these two inequalities hold, it is easily seen directly that $N := FY^{-1}$ provides a *static* suboptimal feedback controller. An analoguous specialization tackles the full information problem $C = (I\ 0)^T$, $D = (0\ I)^T$. Similarly, we remark that the inequality (15) is related to an H_∞ estimation problem [34, 46].

Remark on quadratic performance. For solving the quadratic mixed H_2/H_∞ control problem, one needs to characterize the existence of a controller with (8) where $\mathcal{X}$ is restricted to be *constant.* Trivial simplifications of our proof lead to the conclusion that Theorem 4 fully applies if just asking for *constant* X, Y, and Z. Hence, one ends up having to solve *algebraic linear matrix inequalities (LMIs) in constant* X, Y, Z *and in time functions* F, J, N *along the parameter curve defined by the system data.*

Remark on LTI systems. If the system and the controllers are LTI, it causes no loss of generality to confine $\mathcal{X}$ in (8) to be constant. Hence Theorem 4 remains valid by specializing to *constant* X, Y, Z, and F, J, N. This is an LMI solution to the reduced order [9] or full order mixed H_2/H_∞ problem [17, 25, 28, 29, 33, 42, 43, 51, 60]. In contrast to all earlier papers, no technical assumptions on the system data are required. The design of full order controllers can be directly based on Theorem 5: Find the parameters X, Y and F, J, N by solving three coupled *linear matrix inequalities* [35, 24]. Then compute L, M, and K according to the formulas given after Theorem 5. Note that we do not require to include the a priori hypotheses that

$$(A, B) \text{ is stabilizable and } (A, C) \text{ is detectable} \tag{27}$$

since they are obvious *necessary conditions* for the existence of positive definite solutions of (14) and (15).

Remark on the pure H_∞ Problem. The whole discussion includes the pure H_∞ problem obtained with $H_2 = 0$, $E_2 = 0$, $F_2 = 0$. By (12), this just amounts to canceling

(16) in Theorem 4. In Theorem 5, one has to replace (28) by the coupling condition $\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \gg 0$.

Remark on the pure H_2 problem. As mentioned in Section 2, we can recover the pure generalized H_2 problem by setting $H_1 = 0$, $E_1 = 0$, $F_1 = 0$, $\gamma = 1$. In Theorem 4, this just amounts to canceling the third block rows/columns in (14) and (15). If allowing for nonproper controllers, one should include the linear constraint $F_2 + E_2ND = 0$ on N. To the best of our knowledge, this gives for the first time a solution of the H_2 problem in terms of DLMIs or, in the LTI case, in terms of LMIs.

Let us now specialize Theorem 4 to the case without any a priori restriction on the controller size. In fact, a slight modification of the above proof (if the controller has size larger than n) leads to the following result.

Theorem 5 *There exists a controller R with $J(T(R)) < \alpha$ iff there exist smooth time functions X, Y and time functions F, J, N such that the DLMIs (14) and (15) and*

$$\begin{pmatrix} \alpha I & H_2Y + E_2F & H_2 + E_2NC \\ (H_2Y + E_2F)^T & Y & I \\ (H_2 + E_2NC)^T & I & X \end{pmatrix} \gg 0 \tag{28}$$

are satisfied. If existing, the parameter K in R can be chosen of size n.

Hence, suboptimality is characterized in terms of the solvability of two differential inequalities and one algebraic inequality where all the unknowns X, Y, F, J, N *enter linearly*. This structural property will be essential for the discussions to follow.

Remark on the construction of a full order controller. We would like to provide explicit formulas for how to construct a full order controller. Given X and Y, Cholesky factorize $X - Y^{-1} = UU^T$ such that U and U^{-1} are smooth and bounded. Motivated by $XY + UV^T = I$, define $V = (I - YX)U^{-T}$ which is smooth and has a bounded inverse. Motivated by (22), define the controller parameters

$$M := (F - NCY)V^{-T} \quad \text{and} \quad L := U^{-1}(J - XBN). \tag{29}$$

Finally, with

$$Q_{21} = \dot{X}Y + \dot{U}V^T + (A + BNC)^T + X(A - BNC)Y + JCY + XBF,$$

$$\begin{pmatrix} Q_{31} & Q_{32} & \big| & Q_3 \end{pmatrix} = \left(\begin{array}{cc|cc} (G + BND)^T & (XG + JD)^T & -\gamma I & (F_1 + E_1ND)^T \\ H_1Y + E_1F & H_1 + E_1NC & F_1 + E_1ND & -\gamma I \end{array}\right),$$

a suitable K is given by

$$K = U^{-1}(Q_{32}^T Q_3^{-1} Q_{31} - Q_{21})V^{-T}.$$

(This is shown literally as in the sufficiency proof of Theorem 4 by choosing $\mathcal{S} := \mathcal{Z}^T$ with $\mathcal{Z}$ defined in (24).) Note that we could as well start with $U := X - Y^{-1}$ implying $V = -Y$ or, dually, $V := Y - X^{-1}$ implying $U = -X$.

4 Discussion of the Estimation DLMI

In this section we would like gain some insight into the DLMI (14) which is related to an H_∞ estimation problem. We separate the time-varying case in Section 4.1 from the time-invariant situation in Section 4.2 since the latter leads to an *algebraic* linear matrix inequality. Most of the proofs are found in the appendix.

4.1 Time-Varying Data

In the LTV case we proceed under the hypothesis $DD^T \gg 0$ and $E_1^T E_1 \gg 0$. If not true, these conditions can be enforced by (small) perturbations since the inequality (14) is strict. Then we can perform coordinate changes in the u- and y-space and orthogonal coordinate changes in the w- and z-space to obtain

$$\left(\begin{array}{cc|c} A & B & G \\ C & 0 & D \\ \hline H_1 & E_1 & F_1 \end{array}\right) = \left(\begin{array}{cc|cc} A & B & G_1 & G_2 \\ C & 0 & 0 & I \\ \hline H_1 & E_1 & F_{\bullet 1} & F_{\bullet 2} \end{array}\right) = \left(\begin{array}{cc|c} A & B & G \\ C & 0 & D \\ \hline H_{11} & 0 & F_{1\bullet} \\ H_{21} & I & F_{2\bullet} \end{array}\right) = \left(\begin{array}{cc|cc} A & B & G_1 & G_2 \\ C & 0 & 0 & I \\ \hline H_{11} & 0 & F_{11} & F_{12} \\ H_{21} & I & F_{21} & F_{22} \end{array}\right). \tag{30}$$

Under these hypotheses, the following result reduces the solvability of the DLMI to the test of whether a perturbed initial value problem has a bounded solution. We note that the proof provides, just by applying Lemma 3, a pretty quick derivation of the corresponding formulas.

Theorem 6 *If X satisfies the DLMI (14) then*

$$\|F_{\bullet 1}\| \ll \gamma, \quad \|F_{1\bullet}\| \ll \gamma, \tag{31}$$

and there exists an $\epsilon > 0$ such that the solution of the initial value problem

$$\begin{aligned} Z(0) = \epsilon I,\ \dot{Z} \ &= \ (A - G_2 C)Z + Z(A - G_2 C)^T - \gamma Z C^T C Z + \epsilon I \ - \\ &- \ \begin{pmatrix} G_1 & Z(H_1 - F_{\bullet 2} C)^T \end{pmatrix} \begin{pmatrix} -\gamma I & F_{\bullet 1}^T \\ F_{\bullet 1} & -\gamma I \end{pmatrix}^{-1} \begin{pmatrix} G_1^T \\ (H_1 - F_{\bullet 2} C) Z \end{pmatrix} \end{aligned} \tag{32}$$

exists on $[0, \infty)$ and remains bounded; for all small $\epsilon > 0$, $0 \ll Z \ll X^{-1}$. Conversely, (31) implies that

$$N := -F_{22} + \begin{pmatrix} F_{21} & 0 \end{pmatrix} \begin{pmatrix} -\gamma I & F_{11}^T \\ F_{11} & -\gamma I \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ F_{12} \end{pmatrix} \tag{33}$$

yields $\|F_1 + E_1 N D\| \ll \gamma$. If N is any time function with the latter property, and if the solution of (32) exists and is bounded on $[0, \infty)$ for some $\epsilon > 0$, then $Z \gg 0$, and (14) holds for $X := Z^{-1}$ and

$$J = -(XG_2 + \gamma C^T) + \begin{pmatrix} G_1^T X \\ H_1 - F_{\bullet 2} C \end{pmatrix}^T \begin{pmatrix} -\gamma I & F_{\bullet 1}^T \\ F_{\bullet 1} & -\gamma I \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ F_{\bullet 2} + E_1 N \end{pmatrix}. \tag{34}$$

Remark. The proof reveals that the set of all X satisfying the DLMI (14) (for some (J,N)) and the solution set of the differential Riccati inequality

$$\dot{X} + (A - G_2 C)^T X + X(A - G_2 C) - \gamma C^T C - \\ - \begin{pmatrix} X G_1 & H_1^T - C^T F_{\bullet 2}^T \end{pmatrix} \begin{pmatrix} -\gamma I & F_{\bullet 1}^T \\ F_{\bullet 1} & -\gamma I \end{pmatrix}^{-1} \begin{pmatrix} G_1^T X \\ H_1 - F_{\bullet 2} C \end{pmatrix} \ll 0 \tag{35}$$

are identical.

4.2 Time-Invariant Data

If the data matrices are constant, we characterize in this section whether there exist *constant* $X > 0, J, N$ which satisfy (14); hence we discuss an algebraic linear matrix inequality. As a necessary condition for solvability, we infer that (A, C) is detectable; this is assumed from now on.

For fixed (X, N), we can view (14) as an LMI in J only. Defining

$$\mathcal{L}(X,N) := \begin{pmatrix} A^T X + XA & XG & (H_1 + E_1 NC)^T \\ G^T X & -\gamma I & (F_1 + E_1 ND)^T \\ H_1 + E_1 NC & F_1 + E_1 ND & -\gamma I \end{pmatrix},$$

this LMI reads as

$$\mathcal{L}(X,N) + \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} J \begin{pmatrix} C & D & 0 \end{pmatrix} + \begin{pmatrix} C^T \\ D^T \\ 0 \end{pmatrix} J^T \begin{pmatrix} I & 0 & 0 \end{pmatrix} < 0.$$

If we introduce a matrix

$$K_y \text{ whose columns form a basis of the kernel of } \begin{pmatrix} C & D & 0 \end{pmatrix},$$

two obvious necessary conditions for the existence of a solution J are

$$\begin{pmatrix} -\gamma I & (F_1 + E_1 ND)^T \\ F_1 + E_1 ND & -\gamma I \end{pmatrix} < 0 \tag{36}$$

and $K_y^T \mathcal{L}(X,N) K_y < 0$. The latter inequality is in fact independent of N since we have $K_y^T \mathcal{L}(X,N) K_y = K_y^T \mathcal{L}(X,0) K_y$.

The so-called Projection Lemma [22, 26] reveals that these conditions are as well sufficient for the existence of J.

Lemma 7 *(Projection Lemma) For arbitrary A, B, $Q = Q^T$, the LMI*

$$A^T X B + B^T X^T A + Q < 0 \tag{37}$$

in the unstructered X has a solution iff

$$Ax = 0 \textit{ or } Bx = 0 \ (x \neq 0) \textit{ imply } x^T Q x < 0.$$

We reproduce a proof since it provides a construction scheme for J. Moreover, it also reveals that, in suitable coordinates, Lemma 7 reduces to Lemma 3 if the kernels of A and B together span the whole space.

Proof. Necessity is trivial. For proving sufficiency, let $S = (S_1\ S_2\ S_3\ S_4)$ be a nonsingular matrix such that the columns of S_2 span $\ker(A) \cap \ker(B)$, those of $(S_1\ S_2)$ span $\ker(A)$, and those of $(S_2\ S_3)$ span $\ker(B)$. With $(0\ 0\ A_3\ A_4) := AS$ and $(B_1\ 0\ 0\ B_4) := BS$, we observe that $(A_3\ A_4)$ and $(B_1\ B_4)$ have full column rank. Hence, the equation $\begin{pmatrix} A_3^T \\ A_4^T \end{pmatrix} X \begin{pmatrix} B_1 & B_4 \end{pmatrix} = \begin{pmatrix} Z_{31} & Z_{34} \\ Z_{41} & Z_{44} \end{pmatrix}$ has a solution X for each right-hand side. Therefore, $S^T(37)S$ is equivalent to

$$\left(\begin{array}{ccc|c} Q_1 & Q_{21}^T & Q_{31}^T + Z_{31}^T & Q_{41}^T + Z_{41}^T \\ Q_{21} & Q_2 & Q_{32}^T & Q_{42}^T \\ Q_{31} + Z_{31} & Q_{32} & Q_3 & Q_{43}^T + Z_{34} \\ \hline Q_{41} + Z_{41} & Q_{42} & Q_{43} + Z_{34}^T & Q_4 + Z_{44} + Z_{44}^T \end{array}\right) < 0$$

with *free* blocks Z_{ij}. The hypotheses now just amount to (10) and, hence, we can find a (constant) Z_{31} such that the marked 3×3 block is negative definite. Since Z_{44} is free, it can be chosen to render the whole matrix negative definite. ■

We have fully proved the following result.

Theorem 8 *For fixed X and N, the LMI (14) in J has a solution iff (36) and*

$$K_y^T \begin{pmatrix} A^T X + XA & XG & H_1^T \\ G^T X & -\gamma I & F_1^T \\ H_1 & F_1 & -\gamma I \end{pmatrix} K_y < 0 \tag{38}$$

hold.

We infer that N is only restricted by (36) and does not influence (38). Moreover, for a fixed N with (36), the set of all X satisfying (14) (for some J) is *identical* to the solution set of (38). This decouples the construction of X from the determination of (J, N) in (14). As for J, testing the existence of N with (36) and computing a solution just amounts to applying Lemma 7.

Hence it remains to discuss how to test the existence of a positive definite solution of (38). If D has full row rank, one can reduce this LMI in exactly the same manner as in Section 4.1 to an algebraic Riccati inequality. In the general case, we need to transform the system data. For this purpose we look at

$$\begin{pmatrix} A & G \\ C & D \\ H_1 & F_1 \end{pmatrix} \rightarrow \begin{pmatrix} S & S_{12} & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A & G \\ C & D \\ H_1 & F_1 \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & S_3 \end{pmatrix}$$

where S, S_2, S_3 are coordinate changes in the state-, y- and w-space (S_3 orthogonal) and S_{12} is an output-injection applied to (6). It is easily seen that (X, J, N) satisfies the LMI

(14) iff $(S^{-T}XS^{-1}, (S^{-T}J - S_{12})S_2^{-1}, N)$ satisfies the same LMI for the transformed data. Hence solutions of (38) transform as

$$X \to S^{-T}XS^{-1}.$$

As clarified in [46, 49], one can transform in this manner

$$\left(\begin{array}{c|c} A & G \\ \hline C & D \\ \hline H_1 & F_1 \end{array}\right) \text{ to } \left(\begin{array}{cc|ccc} A_r & 0 & G_r & 0 & 0 \\ G_sC_r & A_s & 0 & G_s & 0 \\ \hline 0 & C_s & 0 & 0 & 0 \\ D_r & D_s & 0 & 0 & I \\ \hline H_r & H_s & F_r & F_{r1} & F_{r2} \end{array}\right) \tag{39}$$

with the crucial property that

$$\left(\begin{array}{cc} A_s - \lambda I & G_s \\ C_s & 0 \end{array}\right) \text{ has full column rank for all } \lambda \in \mathcal{C}. \tag{40}$$

This transformation separates a

$$\text{regular subsystem} \left[\begin{array}{c|ccc} A & G_r & 0 & 0 \\ \hline C_r & 0 & I & 0 \\ D_r & 0 & 0 & I \\ \hline H_r & F_r & F_{r1} & F_{r2} \end{array}\right] \text{ from a singular part } \left[\begin{array}{c|c} A_s & G_s \\ \hline C_s & 0 \end{array}\right] \tag{41}$$

(what clarifies the indices s and r). For some more detailed explanations of this structure in the language of geometric control theory we refer to [49, 46]. It is important to observe that, on the basis of the structure algorithm, the regular subsystem can be computed in a numerically reliable manner [23]. For our purposes, the transformation is motivated by (40). It is well-known that left-invertible systems without any zeros have very nice properties. As an example, we mention a well-known fact from the theory of almost disturbance decoupling [58, 56]: if H_r and F_1 vanish, (38) has a positive definite solution *for all* $\gamma > 0$. This makes it plausible to expect that the singular part of the system can be factored out, and that the solvability of (38) can be characterized in terms of the regular subsystem only. The precise result reads as follows.

Lemma 9 *The inequality (38) has a (positive definite) solution iff*

$$\left(\begin{array}{ccc} A_r^TX_r + X_rA_r - \gamma(C_r^TC_r + D_r^TD_r) & X_rG_r & * \\ G_r^TX_r & -\gamma I & F_r^T \\ H_r - F_{r1}C_r - F_{r2}D_r & F_r & -\gamma I \end{array}\right) < 0 \tag{42}$$

has a (positive definite) solution X_r.

Let us introduce the abbreviations

$$\tilde{H}_r := H_r - F_{r1}C_r - F_{r2}D_r, \quad F_\gamma := \left(\begin{array}{cc} \gamma I & -F_r^T \\ -F_r & \gamma I \end{array}\right).$$

Then (42) is equivalent to $F_\gamma > 0$ and

$$R(X_r) := A_r^T X_r + X_r A_r - \gamma(C_r^T C_r + D_r^T D_r) + \begin{pmatrix} X_r G_r & \tilde{H}_r^T \end{pmatrix} F_\gamma^{-1} \begin{pmatrix} G_r^T X_r \\ \tilde{H}_r \end{pmatrix} < 0.$$

Note that this is just the ARI which corresponds to (32) for the regular subsystem (41). If $(-A_r, G_r)$ is stabilizable, a standard result implies that this strict ARI has a (positive definite) solution iff the largest or antistabilizing solution of the corresponding algebraic Riccati equation (ARE) $R(X_r) = 0$ exists (and is positive definite) [48, Theorem 2].

If (A_r, G_r) has uncontrollable modes in $\mathcal{C}^-$ or $\mathcal{C}^0$, the validation test is more involved. Let us display the critical uncontrollable modes by choosing S in (39) (w.l.o.g.) such that

$$\left(\begin{array}{c|c} A_r & G_r \end{array} \right) = \left(\begin{array}{ccc|c} A_1 & A_{12} & A_{13} & G_1 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \end{array} \right)$$

where

$$(-A_1, G_1) \text{ is stabilizable}, \quad \sigma(A_2) \subset \mathcal{C}^0, \quad \sigma(A_3) \subset \mathcal{C}^-.$$

This separates the uncontrollable modes of (A_r, G_r) in the open right-half plane (in (A_1, G_1)) from those on the imaginary axis (in A_2) and in the open left-half plane (in A_3). Let us partition the columns of

$$\begin{pmatrix} C_r \\ D_r \end{pmatrix} = \begin{pmatrix} C_1 & C_2 & C_3 \end{pmatrix}, \quad \tilde{H}_r = \begin{pmatrix} \tilde{H}_1 & \tilde{H}_2 & \tilde{H}_3 \end{pmatrix}$$

according to those of A_r. Finally, if A_2 has the eigenvalues $i\omega_j$, let E_j be a basis of the (complex) kernel of $A_2 - i\omega_j I$.

The following result provides a complete *verifiable* characterization of whether (38) has a positive definite solution.

Theorem 10 *The LMI (38) has a positive definite solution iff*

$$F_\gamma = \begin{pmatrix} \gamma I & -F_r^T \\ -F_r & \gamma I \end{pmatrix} > 0, \tag{43}$$

the unique solution X of

$$A_1^T X + X A_1 - \gamma C_1^T C_1 + \begin{pmatrix} X G_1 & \tilde{H}_1^T \end{pmatrix} F_\gamma^{-1} \begin{pmatrix} G_1^T X \\ \tilde{H}_1 \end{pmatrix} = 0 \tag{44}$$

with

$$\sigma\left(A_1 + \begin{pmatrix} G_1 & 0 \end{pmatrix} F_\gamma^{-1} \begin{pmatrix} G_1^T X \\ \tilde{H}_1 \end{pmatrix} \right) \subset \mathcal{C}^+ \tag{45}$$

exists and satisfies $X > 0$, and the unique solution Y of the linear equation

$$A_2^T Y + Y A_1 + A_{12}^T X - \gamma C_2^T C_1 + \begin{pmatrix} Y G_1 & \tilde{H}_2^T \end{pmatrix} F_\gamma^{-1} \begin{pmatrix} G_1^T X \\ \tilde{H}_1 \end{pmatrix} = 0 \tag{46}$$

satisfies

$$E_j^* \left[A_{12}^T Y^T + Y A_{12} - \gamma C_2^T C_2 + \begin{pmatrix} Y G_1 & \tilde{H}_2^T \end{pmatrix} F_\gamma^{-1} \begin{pmatrix} G_1^T Y^T \\ \tilde{H}_2 \end{pmatrix} \right] E_j < 0 \tag{47}$$

for all j.

Remark. Although these formulas look complicated, they have a very simple origin: After partitioning

$$X_r = \begin{pmatrix} X & Y^T & * \\ Y & Z & * \\ * & * & W \end{pmatrix}, \; R(X_r) = \begin{pmatrix} R_1(X_r) & R_{21}^T(X_r) & * \\ R_{21}(X_r) & R_2(X_r) & * \\ * & * & R_3(X_r) \end{pmatrix} \tag{48}$$

according to A_r, the block $R_1(X_r)$ is indentical to the left-hand side of (44), $R_{21}(X_r)$ is that of (46), and the left-hand side of (47) is nothing else than $E_j^* R_2(X_r) E_j$.

Let us now comment on how to test these properties. Clearly, (43) is just a matter of verification. Since $(-A_1, G_1)$ is stabilizable, to test the existence of a solution X of (44) with (45) is a standard problem. We can apply the results in [18, Section 7.2] to infer that X exists iff the Hamiltonian matrix

$$H := \begin{pmatrix} A_1 + \begin{pmatrix} G_1 & 0 \end{pmatrix} F_\gamma^{-1} \begin{pmatrix} 0 \\ \tilde{H}_1 \end{pmatrix} & \begin{pmatrix} G_1 & 0 \end{pmatrix} F_\gamma^{-1} \begin{pmatrix} G_1^T \\ 0 \end{pmatrix} \\ \gamma C_1^T C_1 - \begin{pmatrix} 0 & \tilde{H}_1^T \end{pmatrix} F_\gamma^{-1} \begin{pmatrix} 0 \\ \tilde{H}_1 \end{pmatrix} & -\left(A_1 - \begin{pmatrix} G_1 & 0 \end{pmatrix} F_\gamma^{-1} \begin{pmatrix} 0 \\ \tilde{H}_1 \end{pmatrix} \right)^T \end{pmatrix}$$

does not have eigenvalues on the imaginary axis. If true, and if the columns of $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ span the generalized eigenspace of H with respect to its eigenvalues in $\mathcal{C}^+$, then X_1 is square and nonsingular, and $X = X_2 X_1^{-1}$ satisfies (44)-(45). After all, one can easily verify $X > 0$. Due to (45) and $\sigma(A_2) \subset \mathcal{C}^0$, we can solve the linear equation (46) for a unique Y, and (47) is, again, only a matter of verification.

Remarks.

(a) This theorem provides insight in which parts of the system are relevant for the solvability of (38). First, the singular part (41) does not play any role. Due to (40), it is easily seen that the zeros of

$$\begin{pmatrix} A - sI & G \\ C & D \end{pmatrix} \tag{49}$$

are the uncontrollable modes of (A_r, G_r), which are separated according to the partition $\mathcal{C}^+ \cup \mathcal{C}^0 \cup \mathcal{C}^-$ into the uncontrollable modes of (A_1, G_1) and the eigenvalues of A_2, A_3 respectively. Hence, zeros in the open left-half plane are irrelevant (A_3 does not appear) and the influence of the zeros on the imaginary axis is explicitly displayed by (47). Similarly, the affect of the $\mathcal{C}^+$ zeros can be made explicit [46]. If $G_1 = 0$, we note that (44) is a *linear equation* in X and (45) trivially holds; the resulting simplifications for the following results are easy to extract [46, 49].

(b) The proofs of Theorems 9 and 10 provide explicit insights into the structure of the solution set of (38). In particular, one can extract which blocks of the solutions can be freely chosen or which ones can be made arbitrarily large.

(c) Suppose that D has full row rank and that $GD^T = 0$. Then we can assume w.l.o.g. $D = (0\ I)$ and conclude

$$\left(\begin{array}{c|c} A & G \\ \hline C & D \\ \hline H_1 & F_1 \end{array}\right) = \left(\begin{array}{c|cc} A_r & G_r & 0 \\ \hline D_r & 0 & I \\ \hline H_r & F_r & F_{r2} \end{array}\right). \tag{50}$$

Hence (39) does not have a singular part and $R(X_r) < 0$ is an algebraic Riccati inequality for the original matrices. Theorem 10 then characterizes the solvability of this general ARI without restrictions on the uncontrollable modes of (A, G). If $GD^T \neq 0$, one just needs to replace A by $A - GD^TC$ (see Section 4.1).

(d) If (49) has no zeros in $\mathcal{C}^0$, A_2 is empty (such that (46), (47) have to be cancelled) and one only needs to test the existence and positivity of X with (44)-(45). If (49) has no zeros in $\mathcal{C}^0 \cup \mathcal{C}^-$, an ARE in terms of the regular subsystem (41) results.

(e) Suppose $D = (0\ I)$ and $GD^T = 0$. If (49) has no zeros in $\mathcal{C}^- \cup \mathcal{C}^0$, then $X > 0$, (44)-(45) are conditions in the original data matrices. With $P = X^{-1}$, they can be rewritten as $P \geq 0$ and

$$AP + PA^T - \gamma PC^TCP + \begin{pmatrix} G_r & P(H_1^T - DF_1^T) \end{pmatrix} F_\gamma^{-1} \begin{pmatrix} G_r^T \\ (H_1 - F_1D^T)P \end{pmatrix} = 0,$$
$$\sigma\left(A - \gamma PC^TC + \begin{pmatrix} G_r & P(H_1^T - DF_1^T) \end{pmatrix} F_\gamma^{-1} \begin{pmatrix} 0 \\ H_1 - F_1D^T \end{pmatrix}\right) \subset \mathcal{C}^-.$$

If (49) has no $\mathcal{C}^0$ zeros, the same holds with $P = S^T \begin{pmatrix} X^{-1} & 0 \\ 0 & 0 \end{pmatrix} S$ (in the partition of A_r). This transforms the conditions (44)-(45) in special coordinates back to the original data matrices, and reveals the relation to the indefinite Riccati equations as appearing in [14, 15]. Again, if $GD^T \neq 0$, replace A by $A - GD^TC$.

Our main interest lies in the quick computation of the critical parameter

$$\gamma_c := \inf\{\gamma > 0 \mid \text{The LMI (38) has a positive definite solution.}\} \tag{51}$$

For this purpose we first clarify that one can give an explicit formula for the interval of those values γ for which (43) holds and a *symmetric* X with (44)-(45) exists. One needs to solve a standard LQ Riccati equation which is, due to the stabilizability of $(-A_1, G_1)$, always possible.

Theorem 11 *Let X_0 satisfy*

$$A_1^T X_0 + X_0 A_1 - C_1^T C_1 + X_0 G_1 G_1^T X_0 = 0, \quad \sigma(A_1 + G_1 G_1^T X_0) \subset \mathcal{C}^+.$$

Then (43) holds and there exists a symmetric X with (44) and (45) iff

$$\gamma > \gamma_e := \|F_r - (\tilde{H}_1 + F_r G_1^T X_0)(sI + A_1 + G_1 G_1^T X_0)^{-1} G_1\|_\infty.$$

Remark. Due to this formula, γ_e can be computed by quadratically convergent algorithms [10].

Here is now the central trick to determine γ_c: View the unique solutions of (44)-(46) as *functions X_γ and Y_γ of γ on the interval* (γ_e, ∞). Defining

$$U_\gamma = -\mathrm{diag}_j E_j^* \left[A_{12}^T Y_\gamma^T + Y_\gamma A_{12} - \gamma C_2^T C_2 + \begin{pmatrix} G_1^T Y_\gamma^T \\ \tilde{H}_2 \end{pmatrix}^T F_\gamma^{-1} \begin{pmatrix} G_1^T Y_\gamma^T \\ \tilde{H}_2 \end{pmatrix} \right] E_j, \qquad (52)$$

we can abbreviate Theorem 10 as follows.

Corollary 12 *The LMI (38) has a positive definite solution iff $\gamma > \gamma_c$ iff $\gamma > \gamma_e$, $X_\gamma > 0$, $U_\gamma > 0$.*

Due to (45), the implicit function theorem implies that X_γ and, hence, also Y_γ, U_γ are analytic functions on (γ_e, ∞).

By differentiating (44) with respect to γ, it is not difficult to see that X_γ is *nonincreasing* and *concave.* A perturbation trick allows to show that U_γ shares these properties with X_γ.

Lemma 13 *X_γ and U_γ are analytic functions on (γ_e, ∞) which are positive definite for some large γ, and which are nondecreasing (the first derivative is positive semidefinie) and concave (the second derivative is negative semidefinite).*

With

$$f_\gamma := \mathrm{diag}(X_\gamma, U_\gamma),$$

Corollary 12 clearly implies $\gamma_c = \inf\{\gamma > \gamma_e \mid f_\gamma > 0\}$. Since $f_\gamma > 0$ for some large γ, we infer $\gamma_c < \infty$. It might happen that f_γ, although nondecreasing, remains positive if γ decreases to γ_e, what implies $\gamma_c = \gamma_e$. In the other case, there exists some $\gamma_1 > \gamma_e$ with $f_{\gamma_1} \not> 0$. Due to the properties in Lemma 13, there is exactly one $\gamma \in (\gamma_1, \infty)$ for which f_γ is positive semidefinite and singular, and this parameter in fact coincides with γ_c [47]. (It cannot happen that f_γ is positive semidefinite and singular for more than one values of γ.) This implies $\gamma_1 \leq \gamma_c$ and γ_1 can be taken as the starting point for the following Newton algorithm.

Theorem 14 *Suppose f_γ has the properties as in Lemma 13. If $\gamma_e < \gamma_j \leq \gamma_c$, there exists a unique γ_{j+1} such that*

$$f_{\gamma_j} + f'_{\gamma_j}(\gamma_{j+1} - \gamma_j) \text{ is positive semidefinite and singular,}$$

and γ_{j+1} satisfies $\gamma_j \leq \gamma_{j+1} \leq \gamma_c$. The inductively defined sequence (γ_j) converges montonically and quadratically to γ_c.

Hence, given γ_j, one just needs to solve a symmetric generalized eigenvalue problem in order to find a better approximation γ_{j+1} of γ_c. For a proof we refer to [47].

As a final result, we provide an important property of

$$\boldsymbol{X}_\gamma := \{X^{-1} \mid X > 0 \text{ satisfies (38)}\}$$

which is identical to the set of X^{-1} if X varies over all positive definite solutions of (14) (for some (J, N)). We show that there exists a P_γ which is a *strict lower bound* and, at the same time, a *limit point* of $\boldsymbol{X}_\gamma$. For brevity, P_γ is called a *strict lower limit point.* As easily seen, any such strict lower limit point is necessarily unique. Hence, existence is the nontrivial and crucial point in the following result.

Theorem 15 *Let $\boldsymbol{X}_\gamma$ be nonempty. With $X_\gamma > 0$ satisfying (44)-(45), define $P_r(\gamma) := \begin{pmatrix} X_\gamma^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in the partition of A_r and $P_\gamma := S^T \begin{pmatrix} P_r(\gamma) & 0 \\ 0 & 0 \end{pmatrix} S$ in the partition of A. Then $P_\gamma \geq 0$ is the unique strict lower limit point of $\boldsymbol{X}_\gamma$. P_γ is analytic on (γ_c, ∞) and satisfies $P_\gamma' \leq 0$, $P_\gamma'' \geq 0$.*

Remark. If D has full row rank and (49) has no zeros in $\mathcal{C}^0$, P_γ just coincides with the solution of the estimation Riccati equation in [14, 15] (see Remark (e) after Theorem 10).

In this section we have generalized the results from [48, 49, 20] to arbitrary data and we considerably simplified some of the earlier proofs.

5 Consequences for the Mixed H_2/H_∞ Problem

We just observe that, for fixed (Y, F, N), the left-hand side of (28) is monotone in X. If combining Theorem 5 and 6, we arrive at the following corollary for LTV systems.

Corollary 16 *There exists a controller R with $J(T(R)) < \alpha$ iff there exists an $\epsilon > 0$ such that the solution of (32) exists on $[0, \infty)$ and remains bounded, and there exist time functions Y (smooth), F, N satisfying (15) and*

$$\begin{pmatrix} \alpha I & H_2 Y + E_2 F & H_2 + E_2 N C \\ (H_2 Y + E_2 F)^T & Y & I \\ (H_2 + E_2 N C)^T & I & Z^{-1} \end{pmatrix} \gg 0.$$

This decouples the construction of (J, X) from the construction of (Y, F, N) in Theorem 5. In addition, it is interesting to observe that Z can be *computed on-line* which will become important for linear parametrically-varying systems as discussed in Section 6.

If the system is LTI, we let all the unknowns in Theorem 5 be constant as well. Moreover, we note that (28) is equivalent to $X > 0$ and

$$\begin{pmatrix} \alpha I & H_2Y + E_2F \\ * & Y \end{pmatrix} - \begin{pmatrix} H_2 + E_2NC \\ I \end{pmatrix} X^{-1} \begin{pmatrix} H_2 + E_2NC \\ I \end{pmatrix}^T > 0. \tag{53}$$

Now we exploit Theorem 15 to infer that the set of all X^{-1} in Theorem 5 has the lower limit point P_γ. It is easily seen that we can replace X^{-1} in (53) by P_γ.

Corollary 17 *If the system is LTI, there exists an LTI controller R with $J(T(R)) < \alpha$ iff $\gamma > \gamma_c$ (as defined in (51)), and there exist a solution (Y, F, N) of (15) with*

$$\begin{pmatrix} \alpha I - (H_2 + E_2NC)P_\gamma(H_2 + E_2NC)^T & * \\ (H_2[Y - P_\gamma] + E_2[F - NCP_\gamma])^T & Y - P_\gamma \end{pmatrix} > 0. \tag{54}$$

Indeed, (53) leads to (54) by $-P_\gamma > -X^{-1}$, and (54) implies (53) since X can be chosen close to P_γ.

Note that (54) is equivalent (Schur complement) to the *linear* matrix inequality

$$\begin{pmatrix} I & W^T(H_2 + E_2NC)^T & 0 \\ (H_2 + E_2NC)W & \alpha I & * \\ 0 & (H_2[Y - P_\gamma] + E_2[F - NCP_\gamma])^T & Y - P_\gamma \end{pmatrix} > 0.$$

in (Y, F, N) if decomposing $P_\gamma = WW^T$. Hence, with the reformulation in Corollary 17, we have eliminated the variables (X, J) in the suboptimality test.

Note that [42] contains a separation result which converts, under technical hypotheses, the output feedback problem to a state-feedback problem. Corollary 17 is the full generalization to arbitrary LTI systems.

6 Linear Parametrically Varying Systems

A practically relevant specific time dependence in (6) arises if specifying (continuous bounded) functions

$$\begin{pmatrix} A(p) & B(p) & G(p) \\ C(p) & 0 & D(p) \\ H_1(p) & E_1(p) & F_1(p) \\ H_2(p) & E_2(p) & * \end{pmatrix}$$

defined on some set $\boldsymbol{P} \subset \mathcal{R}^m$, and letting the system be described by some unknown parameter curve $p(t) \in \boldsymbol{P}$. The available a priori information consists of the set $\boldsymbol{P}$ and the on-line information is the actual parameter value $p(t)$ at the time instant t. This structure comprises linear systems with time-varying measurable parameters and gain scheduling structures for possibly nonlinear systems.

Suppose we can find *constant* X, Y and continuous *functions* F, J, N defined on $\boldsymbol{P}$ such that (14), (15), and (28) hold for all $p \in \boldsymbol{P}$. As described earlier, one can construct a (full order) controller as a function of p and a *constant* $\mathcal{X}$a such that, for the corresponding closed-loop matrices, (8) holds on $\boldsymbol{P}$. As a consequence, if the controller is scheduled along a specific parameter curve (which is measured on-line), it solves the mixed H_2/H_∞ control problem with bounds γ and α. Hence the original dynamic problem is reduced to a static problem of solving linear matrix inequalities over the parameter set $\boldsymbol{P}$.

Due to the (possibly) nonlinear dependence of the data on p and due to the unspecified structure of $\boldsymbol{P}$, solving these inequalities results in a nonlinear problem. If the parameter space has a moderate dimension, simple discretization methods reduce this nonlinear problem to the convex problem of testing the feasibility of a finite number of LMIs.

Let us briefly describe one possible paradigm. Collect the parameters as $a = (A, G, H_1, H_2, F_1)$, $b = (B, E_1, E_2, C, D)$ and the unknowns as $c = (F, J, N)$, $x = (X, Y)$. Define the block diagonal function $g(a, b, c, x)$ by putting the left-hand sides of (14) and (15) (without derivatives) and the negative of the left-hand side of (28) on the diagonal. For *constant* x, the three inequalities (14), (15), (28) can then be written as $g(a(p(.)), b(p(.)), c(p(.)), x) \ll 0$. The generally *quadratic* function g has the following properties: For fixed system data (a, b) it is linear in (c, x), and for fixed (b, x) it is linear in (a, c).

In our case, $a(p), b(p)$ are continuous functions on $\boldsymbol{P}$. The problem is to find a continuous function $c(p)$ and a constant x such that

$$g(a(p), b(p), c(p), x) \ll 0 \quad \text{on} \quad \boldsymbol{P}. \tag{55}$$

Let us now choose a finite number of points $p_j \in \boldsymbol{P}$. If $c(p)$ and x with (55) exist, one can find $\delta_j > 0$ such that, with $c_j := c(p_j)$, the inequalities

$$g(a(p_j), b(p_j), c_j, x) \leq -\delta_j I \tag{56}$$

are satisfied.

For a specific choice of p_j and δ_j, let us now assume that (56) hold for some c_j, x, and δ_j. In addition, suppose that $\boldsymbol{P}$ is open, that the data functions $(a(.), b(.))$ are smooth, and that their derivatives are bounded on $\boldsymbol{P}$. Let

$$g'(p, c, x) := \Big(\|\partial_{p^k} g(a(p), b(p), c, x)\| \Big)_k$$

denote the vector of the norms of the partial derivatives of $g(a(p), b(p), c, x)$ with respect to the components of p. Finally, let $\boldsymbol{P}_j$ denote (large) open convex subsets $\boldsymbol{P}_j$ of $\boldsymbol{P}$ with $p_j \in \boldsymbol{P}_j$ and

$$\sup\{\|g'(p, c_j, x)\| \|p - p_j\| \mid p \in \boldsymbol{P}_j\} < \delta_j. \tag{57}$$

Then the mean value theorem implies $g(a(p), b(p), c_j, x) \ll 0$ on $\boldsymbol{P}_j$. If the sets $\boldsymbol{P}_j$ cover $\boldsymbol{P}$, one can easily find a function $c(p)$ which 'interpolates' the controller parameters c_j such that (55) is verified. Just choose a smooth partition of unity ϕ_j which is subordinated to $\cup_j \boldsymbol{P}_j$, and define $c(p) := \sum_j c_j \phi_j(p)$ [32, Section 6.1]. (Indeed, there exists an $\epsilon > 0$ with $g(a(p), b(p), c_j, x) \leq -\epsilon I$ for all $p \in \boldsymbol{P}_j$ and all j. If p meets $\boldsymbol{P}_{j_k}$ but no other sets, we infer $\sum_k \phi_{j_k}(p) = 1$ and thus $g(a(p), b(p), c(p), x) = \sum_k \phi_{j_k}(p) g(a(p), b(p), c_{j_k}, x) \leq -\epsilon \sum_k \phi_{j_k}(p) I = -\epsilon I$.)

If the sets $\boldsymbol{P}_j$ do not cover $\boldsymbol{P}$, one has to include more points from $\boldsymbol{P}$ in the list p_j. If (55) is solvable, it is not difficult to see that there *always* exists a choice of (sufficiently dense) p_j and (sufficiently small) δ_j such that the corresponding $\boldsymbol{P}_j$ cover $\boldsymbol{P}$: Suppose x and the (bounded) function $c(p)$ satisfy $g(a(p), b(p), c(p), x) \leq -\delta I$ for $p \in \boldsymbol{P}$. Let us introduce the bounds $\hat{x}$ and $\hat{c}$ by $\|x\| < \hat{x}$ and $\sup\{\|c(p)\| \mid p \in \boldsymbol{P}\} < \hat{c}$, and define $s := \sup\{\|g'(p, c, x)\| \mid p \in \boldsymbol{P}, \|c\| < \hat{c}, \|x\| < \hat{x}\}$. For an *arbitrary* choice of $p_j \in \boldsymbol{P}$ and with $\delta_j := \delta$, the LMIs (56) with the constraints $\|c\| < \hat{c}$, $\|x\| < \hat{x}$ have solutions c_j and x. Then $\|p - p_j\| < \delta/(s+1)$ implies $\|g'(p, c_j, x)\| \|p - p_j\| < \delta s/(s+1) < \delta$ and thus $\boldsymbol{P}_j$ satisfying (57) can be chosen to contain at least the ball $\{p \in \boldsymbol{P} : \ \|p - p_j\| < \delta/(s+1)\}$. Since the radius of this ball does not depend on the choice of p_j, it is clear that a sufficiently dense set of points p_j indeed leads to $\boldsymbol{P}_j$ covering $\boldsymbol{P}$. ■

We have proved that we can indeed test whether (55) is solvable by some bounded continuous function $c(p)$ and by some x:

- For $p_j \in \boldsymbol{P}$, $\delta_j > 0$, and some bounds $\hat{c}$, $\hat{x}$, test the feasibility of (56) with the constraints $\|c\| < \hat{c}$, $\|x\| < \hat{x}$. If not feasible, decrease $\delta_j > 0$, increase the bounds $\hat{c}$, $\hat{x}$, and repeat. If (56) is never feasible then stop.
- Otherwise, construct $\boldsymbol{P}_j$ (as large as possible). If the sets $\boldsymbol{P}_j$ cover $\boldsymbol{P}$ then a $c(p)$ can be computed. Otherwise, refine the choice of p_j and return to the first step.

It is important to note that one does not need to start with a *uniform partition* of the parameter set but one can systematically vary the the density of the points p_j according to the sizes of $\boldsymbol{P}_j$, which takes the rate of variation (derivatives) of the system data with respect to the parameter into account. In practice, this might allow to keep the number of points (and hence of LMIs in (56)) reasonably small.

Under certain additional hypothesis on the system data and on the parameter set, one can exploit convexity. In the spirit of [11], let us just describe how to proceed if the (possibly nonlinear) system can be written as

$$\begin{pmatrix} \dot{x} \\ y \\ z_1 \\ z_2 \end{pmatrix} = \sum_j \lambda_j(t) \left(\begin{array}{c|cc} A_j & B_j & G_j \\ \hline C_j & 0 & D_j \\ (H_1)_j & (E_1)_j & (F_1)_j \\ (H_2)_j & (E_2)_j & (F_2)_j \end{array}\right) \begin{pmatrix} x \\ u \\ w \end{pmatrix}$$

where the finitely many (constant) vertices (a_j, b_j) are a priorily available and the continuous convex combination coefficients $\lambda_j(t)$ with $\lambda_j(t) \geq 0$ and $\sum_j \lambda_j(t) = 1$ can be measured on-line. With this parameter dependence, a necessary condition for the solvability of (55) is the existence of c_j and x satisfying

$$g(a_j, b_j, c_j, x) < 0$$

for all vertices.

To reverse the arguments, we assume that (c_j, x) satisfy these LMIs. Now the reason for the distinction of the system parameters a and b becomes evident. Namely, if b_j does not vary and is identical to b, we infer by linearity that $g(\sum_j \lambda_j(t)a_j, b, \sum_j \lambda_j(t)c_j, x) = \sum_j \lambda_j(t)g(a_j, b, c_j, x) \ll 0$ and hence the parameters

$$c(t) := \sum_j \lambda_j(t)c_j$$

can be used to construct a controller which leads to (8). This is the *full generalization* of the results in [3]-[8] to the mixed H_2/H_∞ problem.

If the b_j also depend on j, our approach allows some way out: Try to solve the LMIs $g(a_j, b_j, c, x) < 0$ with a common (constant) parameter c. Then one can exploit $g(\sum_j \lambda_j(t)a_j, \sum_j \lambda_j(t)b_j, c, x) = \sum_j \lambda_j(t)g(a_j, b_j, c, x) \ll 0$ to infer that a controller leading to (8) can be constructed with c. In this case, N is constant, but the other parameters K, L, M are still scheduled along the parameter curve. If we fix $N = 0$, the formulas (29) reveal that L and M are constant as well. All this also works for mixtures: If only (C, D) is constant, schedule J and keep (F, N) constant. Dually, if (B, E_1, E_2) is constant, schedule F and keep (J, N) constant. Other specific structures might allow refinements to avoid conservatism, just by investigating the function g. These remedies have been made possible by our trick of keeping the (transformed) controller parameters c in our problem solution. Even for the pure H_∞ problem, this would not have been possible solely on the basis of the results in [3]-[8].

Let us remark that the nonlinear (discretization) scheduling technique can be combined with those based on convexity, possibly over different regions of the parameter space. There is no need to discuss all the details.

The implementation of hybrid structures can be pursued further along the following lines. As described in Corollary 16, the estimation differential inequality is related to testing whether an initial value problem has a bounded solution. For a specific LPV system, standard comparison results [19] could allow to guarantee the existence and boundedness of the solution for all possible parameter curves $p(.)$, perhaps taking a bound on the derivative $\dot{p}(.)$ into account. This is generally less conservative than using a *constant* and, along each parameter curve, globally valid solution of the algebraic inequality. Then the estimation DLMI is solved by a time function X whereas the control LMI is still solved with a constant Y. The controller is *nonlinear* since it includes the on-line solution of the corresponding Riccati initial value problem (along the actual parameter curve). The robustness and performance bounds γ and α are still guaranteed by Theorem 5, and our general controller construction scheme is valid without change.

Let us include one specific example. Suppose $R(p, X)$ is a Riccati map and we try to solve the differential Riccati inequality $\frac{d}{dt}X(p(t)) + R(p(t), X(p(t)) \ll 0$ with $X(p(t)) \gg 0$. Let us assume that the frozen parameter ARIs have solutions: There exist strictly positive functions $X(p)$ and $\delta(p)$ on $\boldsymbol{P}$ satisfying $R(p, X(p)) + \delta(p)I \ll 0$. Note that, in the case of interest (35), Theorem 10 allows to test the existences of a positive definite solution of the frozen parameter ARI. The chain rule implies that $X(p(t))$ satisfies the desired differential Riccati inequality for all parameter curves $p(.)$ with

$$\|\dot{p}(t)\| \leq \delta(p(t))/\|X'(p(t))\|,$$

where $X'(p) := \left(\|\partial_{p^k} X(p)\|\right)_k$.

Summarizing, we have proposed some initial interpolation and approximation techniques in order to solve, on the basis of Theorem 5, the mixed H_2/H_∞ problem for LPV systems without particular structural dependence of the data on the scheduling parameter. Some observations indicated how to overcome the conservatism introduced by considering only constant solutions of the DLMIs in Theorem 5, as usually done in the literature [3]-[8].

7 Mixed H_2/H_∞ Control and Robust Regulation

In this section we would like to show how to incorporate asymptotic tracking or disturbance rejection objectives in the mixed H_2/H_∞ problem if the system is LTI. Consider

$$\begin{pmatrix} y \\ z_1 \\ z_2 \end{pmatrix} = \left[\begin{array}{c|ccc} A & B & G & G_2 \\ \hline C & 0 & D & D_2 \\ H_1 & E_1 & F_1 & F_{12} \\ H_2 & E_2 & F_2 & F_{22} \end{array}\right] \begin{pmatrix} u \\ w \\ w_2 \end{pmatrix} \tag{58}$$

and suppose that the measured output is partitioned as

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} C & D & D_2 \end{pmatrix} = \begin{pmatrix} C_1 & D_1 & D_{12} \\ C_2 & D_2 & D_{22} \end{pmatrix}.$$

We say that the controller (7) achieves *regulation* if it is stabilizing and satisfies $\lim_{t\to\infty} y_2(t)$ for $w = 0$ and all signals w_2 in the class

$$\mathcal{W} := \{w \mid \dot{w} = Sw\}.$$

Here, S is the so-called signal generator which satisfies

$$\sigma(S) \subset \mathcal{C}^0 \cup \mathcal{C}^+. \tag{59}$$

The controller achieves *robust regulation* if it achieves regulation for all systems which result from (58) by small perturbations in the describing parameters.

This is the classical problem of asymptotically rejecting the disturbance $w_2 \in \mathcal{W}$ from y_2 or letting $C_2 x$ asymptotically track the reference input $-D_{22}w_2$ for all signals w_2 in $\mathcal{W}$. Since we can neglect decaying signals, the hypothesis (59) causes no loss of generality.

Any robust regulator is known to contain a reduplicate model of the signal generator. To be precise, let us construct $\tilde{S}$ as follows: Let

$$q \text{ be the number of components of } y_2.$$

Transform S to Jordan canonical form, choose for each eigenvalue of S the largest Jordan block, and collect q of these blocks on the diagonal of $\tilde{S}$.

If we assume (27) to assure that stabilizing controllers exist, the following result is well-known [12].

Theorem 18 *There exists a controller (7) which achieves robust regulation for (58) iff*

$$\begin{pmatrix} A-\lambda I & B \\ C_2 & 0 \end{pmatrix} \text{ has full row rank for all } \lambda \in \sigma(S).$$

A controller is a robust regulator iff it has a realization

$$\left[\begin{array}{cc|cc} K & L_1 & L & 0 \\ 0 & \tilde{S} & 0 & R \\ \hline M & N_1 & N & 0 \end{array}\right] = \left[\begin{array}{c|cc} K & L_1 & L \\ \hline M & N_1 & N \end{array}\right] \left[\begin{array}{c|cc} \tilde{S} & 0 & R \\ \hline I & 0 & 0 \\ 0 & I & 0 \end{array}\right] \tag{60}$$

for some R (with q columns) such that

$$(\tilde{S}, R) \text{ is controllable} \tag{61}$$

and some $u = \left[\begin{array}{c|cc} K & L_1 & L \\ \hline M & N_1 & N \end{array}\right] \begin{pmatrix} y \\ v \end{pmatrix}$ *stabilizing*

$$\begin{pmatrix} y \\ v \\ z_1 \\ z_2 \end{pmatrix} = \left[\begin{array}{cc|cc} A & 0 & B & G \\ RC_2 & \tilde{S} & 0 & RD_2 \\ \hline C & 0 & 0 & D \\ 0 & I & 0 & 0 \\ H_1 & 0 & E_1 & F_1 \\ H_2 & 0 & E_2 & F_2 \end{array}\right] \begin{pmatrix} u \\ w \end{pmatrix}, \tag{62}$$

the system (58) postcompensated with $v = \left[\begin{array}{c|c} \tilde{S} & R \\ \hline I & 0 \end{array}\right] y_2$.

As a consequence, there exists a controller which achieves robust regulation *and* solves the mixed H_2/H_∞ problem for (58) iff there exists a controller solving the mixed H_2/H_∞ problem for the extended system (62).

Remark. One can in fact prove that the controller (60) achieves regulation for *any* LTI system which is stabilized by (60). Due to the H_∞ constraint implying robust stability, any controller (60) which achieves robust regulation and solves the mixed H_2/H_∞ problem for (58), indeed *achieves robust regulation in the large*: it is a regulator for all systems that result from (58) by perturbations $w = \Delta z_1$ with a (stable) LTI Δ satisfying $\|\Delta\|_\infty \leq \gamma$.

In order to design a robustly regulating mixed H_2/H_∞ controller, one has to solve the LMIs in Theorem 5 for the system (62) *and for some R satisfying (61).* We get an additional variable R which does *not enter linearly.* However, there is a practically relevant special case in which we can always choose a *fixed* $\tilde{R}$ to circumvent this difficulty: S is diagonizable. In this case we can assume w.l.o.g.

$$S = \operatorname{diag}(\lambda_1 I, \ldots, \lambda_s I) \ \text{ such that } \ \tilde{S} = \operatorname{diag}(\lambda_1 I_q, \ldots, \lambda_s I_q)$$

where $\lambda_1, \ldots, \lambda_s$ are the pairwise different eigenvalues of S and I_q is the unit matrix of size q. If R satisfies (61) and is partitioned into $\begin{pmatrix} R_1^T & \cdots & R_s^T \end{pmatrix}^T$, the blocks R_j of size $q \times q$ are nonsingular. With $T = \operatorname{diag}(R_1, \ldots, R_s)$, we then infer $T^{-1}\tilde{S}T = \tilde{S}$ and $T^{-1}R = \begin{pmatrix} I_q & \cdots & I_q \end{pmatrix}^T =: \tilde{R}$. Hence *any* R satisfying (61) can be transformed by a coordinate change in the controller state to $\tilde{R}$, and we can indeed *fix* R to $\tilde{R}$ in Theorem 18.

Hence it remains to apply Theorem 5 for the fixed internal model $(\tilde{S}, \tilde{R})$. The inequalities (14) and (15), (28) are defined with

$$\left(\begin{array}{cc|c} A - sI & 0 & G \\ \tilde{R}C_2 & \tilde{S} - sI & \tilde{R}D_2 \\ \hline C & 0 & D \\ 0 & I & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|c} A - sI & 0 & B \\ \tilde{R}C_2 & \tilde{S} - sI & 0 \\ \hline H_j & 0 & E_j \end{array}\right), \quad j = 1, 2$$

respectively. Hence the extension from (58) to (62) adds the eigenvalues of S as zeros in $\mathcal{C}^0 \cup \mathcal{C}^+$ to the latter pencils. For (14), however, we conclude that, due to the identity block in the last row of the first pencil, the estimation inequalities (38) for the original data $\begin{pmatrix} A - sI & G \\ C & D \end{pmatrix}$ and for the extended data are *identical*; the extension does not influence the solvability of (14).

Summarizing, we have generalized the results in [1, 2] to the mixed H_2/H_∞ problem. Even the specialization to the pure problems are more general since we do not restrict the underlying plant by technical hypotheses and, still, get *explicit* suboptimality tests in the next two sections.

8 The Pure H_∞ Problem

For time-varying parameters we assume again (30) and specialize Corollary 16 by setting $H_2 = 0$, $E_2 = 0$, $F_2 = 0$. We can just dualize the remark after Theorem 6 and conclude that the set of all Y satisfying (15) (for some (F, N)) is identical to the solution set of the differential Riccati inequality

$$\begin{aligned} &-\dot{Y} + (A - BH_{21})Y + Y(A - BH_{21})^T - \gamma BB^T - \\ &\quad - \begin{pmatrix} G - BF_{2\bullet} & YH_{11}^T \end{pmatrix} \begin{pmatrix} -\gamma I & F_{1\bullet}^T \\ F_{1\bullet} & -\gamma I \end{pmatrix}^{-1} \begin{pmatrix} G^T - F_{2\bullet}^T B^T \\ H_{11} Y \end{pmatrix} \ll 0. \end{aligned} \quad (63)$$

For an arbitrary N with $\|F_1+E_1ND\| \ll \gamma$ (whose existence is characterized by (31) and which can be chosen as (33)) and for any solution Y of (63), (Y,F,N) satisfies (15) if

$$F = -(H_{21}Y + \gamma B^T) + \begin{pmatrix} F_{2\bullet} + ND & 0 \end{pmatrix} \begin{pmatrix} -\gamma I & F_{1\bullet}^T \\ F_{1\bullet} & -\gamma I \end{pmatrix}^{-1} \begin{pmatrix} G^T - F_{2\bullet}^T B^T \\ H_{11}Y \end{pmatrix}.$$

Let us now consider the solution Q_T of the perturbed final value problem

$$\begin{aligned} Q_T(T) = 0,\ \dot{Q} + (A - BH_{21})^T Q + Q(A - BH_{21}) - \gamma QBB^TQ + \epsilon I - \\ - \begin{pmatrix} Q(G - BF_{2\bullet}) & H_{11}^T \end{pmatrix} \begin{pmatrix} -\gamma I & F_{1\bullet}^T \\ F_{1\bullet} & -\gamma I \end{pmatrix}^{-1} \begin{pmatrix} (G - BF_{2\bullet})^T Q \\ H_{11} \end{pmatrix} = 0. \end{aligned}$$

Comparison arguments [30, Chapter 10] lead to $0 \leq Q_{T_1}(t) \leq Q_{T_2}(t)$ for $T_1 \leq T_2$ and for t in the existence intervals of both solutions. We say that $Q_\infty(t)$ *exists* if t is in the existence interval of Q_T for all $T \geq t$ and if $Q_T(t)$ remains bounded for $T \to \infty$, and we define $Q_\infty(t) := \lim_{T\to\infty} Q_T(t)$. Note that the limit exists and is positive semidefinite by monotonicity; moreover, $Q_\infty(t)$ is *unique*.

If Y satisfies (63), comparison results [30, Chapter 10] allow to conclude the existence of $\epsilon > 0$ such that Q_∞ exists on $[0,\infty)$ and satisfies $Q_\infty \ll Y^{-1}$; in particular, Q_∞ is bounded. Conversely, if Q_∞ exists on $[0,\infty)$ and is bounded, one can prove (by perturbing the final value and using the techniques in [39]) that (63) has a solution $Y \gg 0$ for which $\sup_{t\geq 0} \|Q_\infty(t) - Y^{-1}(t)\|$ is arbitrarily small. Since the coupling condition in Corollary 16 reduces to $Z^{-1} \gg Y^{-1}$ or $\rho(ZY^{-1}) \ll 1$, we arrive at the following slightly generalized and more explicit version of a basically well-known result [55, 39].

Theorem 19 *There exists a stabilizing LTV controller R with $\|T_1(R)\|_\infty < \gamma$ iff there exists an $\epsilon > 0$ such that the solution Z of (32) and Q_∞ exist and are bounded on $[0,\infty)$, and such that they satisfy $\rho(ZQ_\infty) \ll 1$.*

Under additional hypotheses, a technical result about differential Riccati equations allows to get rid of the perturbation parameter ϵ [39].

If considering LTI systems, we can dualize Theorem 8 for (15) with a basis K_u of the kernel of $\begin{pmatrix} B^T & 0 & E_1^T \end{pmatrix}$. We arrive at the results of [22, 26].

Theorem 20 *There exists a stabilizing LTI controller R with $\|T_1(R)\|_\infty < \gamma$ iff there exists X and Y with*

$$\begin{pmatrix} X & I \\ I & Y \end{pmatrix} > 0 \tag{64}$$

and

$$K_y^T \begin{pmatrix} A^TX + XA & XG & H_1^T \\ G^TX & -\gamma I & F_1^T \\ H_1 & F_1 & -\gamma I \end{pmatrix} K_y < 0,\ K_u^T \begin{pmatrix} AY + YA^T & G & YH_1^T \\ G^T & -\gamma I & F_1^T \\ H_1Y & F_1 & -\gamma I \end{pmatrix} K_u < 0. \tag{65}$$

The only point which needs clarification: (65) imply the existence of N with (31). This follows from Lemma 7. Note that, as a consequence, one can construct a suboptimal controller on the basis of an *arbitrary* N with (31).

Let us now exploit our results in Section 4.2 to derive algebraically verifiable criteria. From now on, assume (27) and recall the relevance of X_γ, U_γ, $P_\gamma = S_X X_\gamma^{-1} S_X^T$ defined on (γ_e, ∞) for the first inequality in (65). Dually, one can construct functions Y_γ, V_γ defined on (δ_e, ∞) with a computable δ_e (Theorem 11) and sharing the properties of X_γ, U_γ in Lemma 13 such that the second inequality in (65) has a solution $Y > 0$ iff $\gamma > \delta_e$, $Y_\gamma > 0$, $V_\gamma > 0$. The set of all possible Y^{-1} has the lower limit point $Q_\gamma = S_Y Y_\gamma^{-1} S_Y^T$ with a (computable) S_Y (Theorem 15).

Finally, define the optimal value of the H_∞ control problem as

$$\gamma_{\text{opt}} := \inf\{\gamma > 0 \mid \text{Exists a stabilizing LTI controller } R \text{ with } \|T_1(R)\|_\infty < \gamma\}.$$

Then we arrive at the following central verifiable characterization of whether a parameter γ in the most general LTI H_∞ control problem is suboptimal.

Corollary 21 *The parameter γ satisfies $\gamma > \gamma_{\text{opt}}$ iff $\gamma > \gamma_e$, $\gamma > \delta_e$, and*

$$f_\gamma := \begin{pmatrix} Y_\gamma & S_Y^T S_X & 0 & 0 \\ S_X^T S_Y & X_\gamma & 0 & 0 \\ 0 & 0 & U_\gamma & 0 \\ 0 & 0 & 0 & V_\gamma \end{pmatrix} > 0. \tag{66}$$

Remark. The proof reveals that

$$\begin{pmatrix} Y_\gamma & S_Y^T S_X \\ S_X^T S_Y & X_\gamma \end{pmatrix} > 0 \Longleftrightarrow \rho(P_\gamma Q_\gamma) < 1. \tag{67}$$

This gives a *coordinate independent* formulation of the coupling between (14), (15) or the two inequalities (65). With the critical parameters γ_c and δ_c for the LMIs (65) (defined by (51) and its dual), we can infer that $\gamma > \gamma_{\text{opt}}$ iff $\gamma > \gamma_c$, $\gamma > \delta_c$, and $\rho(P_\gamma Q_\gamma) < 1$. This characterization is closer to the standard solvability conditions in terms of LMIs (or ARIs for regular problems or AREs for regular problems without imaginary axis zeros) and a spectral radius coupling condition. We also note that U_γ and V_γ in (66) nicely display the influence of the imaginary axis zeros of $\begin{pmatrix} A - sI & G \\ C & D \end{pmatrix}$ or, dually, of $\begin{pmatrix} A - sI & B \\ H & E \end{pmatrix}$ on the optimal value. The functions X_γ, Y_γ or P_γ, Q_γ are not affected by these zeros. We finally stress that the five remarks after Theorem 10 have obvious dual versions; moreover, they highlight cases in which one can choose $S_X = I$ and/or $S_Y = I$.

Corollary 21 implies $\gamma_{\text{opt}} = \inf\{\gamma > \max\{\gamma_e, \delta_e\} \mid f_\gamma > 0\}$. Due to (27), a stabilizing controller exists and, hence, $f_\gamma > 0$ for some large γ. Therefore, f_γ shares the properties of X_γ in Lemma 13. If $f_\gamma > 0$ for all $\gamma > \max\{\gamma_e, \delta_e\}$, we infer $\gamma_{\text{opt}} = \max\{\gamma_e, \delta_e\}$. Otherwise, there exists a γ_1 with $f_{\gamma_1} \not> 0$ which necessarily satisfies $\max\{\gamma_e, \delta_e\} < \gamma_1 \leq \gamma_{\text{opt}}$. The parameter γ_1 is the starting point for the following iteration: Let $\max\{\gamma_e, \delta_e\} < \gamma_j \leq \gamma_{\text{opt}}$ and determine the unique γ_{j+1} such that

$$f_{\gamma_j} + f'_{\gamma_j}(\gamma_{j+1} - \gamma_j) \text{ is positive semidefinite and singular}$$

by solving a symmetric eigenvalue problem. Then $\gamma_j \leq \gamma_{j+1}$, and (γ_j) converges quadratically to γ_{opt} (see Theorem 14).

We have generalized [20] to singular systems with imaginary axis zeros and our previous results in [49] to systems with $F_1 \neq 0$. Hence, for the most general LTI system, we have not only derived a verifiable suboptimality test, but we also obtained a quadratically convergent algorithm to compute γ_{opt}.

9 The Pure H_2 Problem

As mentioned earlier, the pure H_2 problem is covered by Theorem 5 with $H_1 = 0$, $E_1 = 0$, $F_1 = 0$, and $\gamma = 1$. As in Section 8, one could reduce the DLMIs to perturbed Riccati differential equations if the problem is regular ($DD^T \gg 0$, $E_2^T E_2 \gg 0$) which is not pursued here. More importantly, this specialization of our main result leads to an LMI solution of the (generalized) output feedback H_2 control problem. Hence all the techniques in Section 6 discussed for LPV systems can be applied to robust H_2 performance problems, which seems impossible for alternative approaches.

Let us now address, in more detail, the generalized and standard H_2 problem for LTI systems. We concentrate on strictly proper controllers and assume $F_2 = 0$. Let us define the optimal value as

$$\alpha_{\text{opt}} := \inf\{\alpha \mid \text{Exists strictly proper stabilizing LTI } R \text{ with } \|T_2(R)\| < \alpha\}.$$

In fact, the techniques developed so far allow to determine α_{opt} for the generalized H_2 problem by a one parameter search, and they lead to an explicit formula for the genuine H_2 optimal value. We assume again w.l.o.g. (27) such that $\alpha_{\text{opt}} < \infty$.

Since (A, C) is detectable, the solution set

$$\{X^{-1} \mid X > 0 \text{ and } \exists J : \begin{pmatrix} A^T X + XA + JC + (JC)^T & XG + JD \\ (XG + JD)^T & -I \end{pmatrix} < 0\}$$

of the H_2 version of (14) is nonempty. Hence it has a lower limit point which we denote as P (Theorem 15). Exploiting the explicit formula for P, a direct computation shows $P \geq 0$ and $AP + PA^T + GG^T \geq 0$ such that we can write

$$P = UU^T \text{ and } AP + PA^T + GG^T = VV^T.$$

Let us now reformulate the H_2 version of Corollary 5.

Corollary 22 *The parameter α satisfies $\alpha > \alpha_{\text{opt}}$ iff there exist $Z > 0$ and F with*

$$\begin{pmatrix} AZ + ZA^T + BF + (BF)^T & 0 & (H_2Z + E_2F)^T \\ 0 & -\alpha I & (H_2U)^T \\ H_2Z + E_2F & H_2U & -\alpha I \end{pmatrix} < 0 \tag{68}$$

and

$$\alpha I - V^T Z^{-1} V > 0. \tag{69}$$

If $\alpha^2 I > H_2PH_2^T$, the dual version of Theorem 15 implies that there exists a lower limit point Q_α of Z^{-1} where Z varies in the set of all positive definite solutions of (68) (for some F). Again, we can replace Z^{-1} by Q_α.

Corollary 23 *The parameter α satisfies $\alpha > \alpha_{\text{opt}}$ iff $\alpha^2 > \|H_2PH_2^T\|$ and $\alpha I - V^T Q_\alpha V >$* 0.

In contrast to [40], we have reduced the computation of α_{opt} to a *one parameter search* over α. Let us define $\alpha_e := \sqrt{\|H_2PH_2^T\|}$. We extract from Theorem 15 the structure $Q_\alpha = S_Z Z_\alpha^{-1} S_Z^T$ and observe that $\alpha I - V^T Q_\alpha V > 0$ is equivalent to

$$f_\alpha := \begin{pmatrix} \alpha I & V^T S_Z \\ S_Z^T V & Z_\alpha \end{pmatrix} > 0.$$

By Corollary 23, $\alpha_{\text{opt}} = \inf\{\alpha \in (\alpha_e, \infty) \mid f_\alpha > 0\}$. Since f_α shares all the properties in Lemma 13 with Z_α on the interval (α_e, ∞), we can hence compute the optimal value by a quadratically convergent algorithm (Theorem 14). Apart from $F_2 = 0$ and $N = 0$, we stress again that we do not need to restrict the system by technicalities.

The genuine H_2 problem involves the trace operator. Since this functional is linear, we can even go further and provide an explicit formula for the optimal value. Indeed, a slight adaption of the proof of Corollary 22 leads to the following result.

Corollary 24 *There exists a strictly proper stabilizing LTI controller R with $\|T_2(R)\|_2^2 < \alpha$ iff there exist $Z > 0$ and F with*

$$\begin{pmatrix} AZ + ZA^T + BF + (BF)^T & (H_2Z + E_2F)^T \\ H_2Z + E_2F & -I \end{pmatrix} < 0 \tag{70}$$

and $\alpha >$ trace$(H_2PH_2^T)$ + trace$(V^T Z^{-1} V)$.

Let Q denote the lower limit point of the set of all Z^{-1} where $Z > 0$ solves (70) (for some F). Then we conclude that the optimal value of the H_2 control problem equals

$$\sqrt{\text{trace}[H_2PH_2^T + (AP + PA^T + GG^T)Q]}.$$

This generalizes the (dual) results of [53] to system which are not restricted with respect to imaginary axis zeros. It is interesting to observe that, similarly as for the coupling condition in the H_∞ problem, these zeros do not influence the lower limit points P and Q and hence they don't have an effect on the optimal value.

10 Conclusion

We have provided a solution of the mixed H_2/H_∞ problem for time-varying systems and discussed various specializations to linear parametrically-varying and time-invariant systems, as well as to the pure (generalized) H_2 and H_∞ problems. For completely general linear time-invariant systems, we showed how to include a robust regulation objective in the mixed H_2/H_∞ problem. Moreover, in the pure problems, we have obtained algebraic solvability tests for the corresponding linear matrix inequalities, and arrived at pretty deep insights into the structure of their solutions. This enabled us to devise quadratically convergent algorithms for computing the optimal values, or to provide an explicit formula in the H_2 case.

Apart from our main results, Theorems 4 and 5, we believe that the technique of keeping transformed versions of the controller parameters in output feedback problems is the most relevant contribution of this article with further potentials.

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Appendix: Proofs

Proof of Theorem 6. Due to $DD^T = I$, we can eliminate the matrix $JC + (JC)^T$ with the transformation

$$(14) \to \begin{pmatrix} I & -C^T D & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} (14) \begin{pmatrix} I & 0 & 0 \\ -D^T C & I & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Since $(H_1 + E_1 NC) - (F_1 + E_1 ND)D^T C = H_1 - F_1 D^T C$, we get

$$\begin{pmatrix} \dot{X} + A^T X + XA - XGD^T C - C^T DG^T X - \gamma C^T C & * & * \\ G^T X + D^T J^T + \gamma D^T C & -\gamma I & * \\ H_1 - F_1 D^T C & F_1 + E_1 ND & -\gamma I \end{pmatrix} \ll 0.$$

Using the fine partitions (30) leads to

$$\begin{pmatrix} \dot{X} + (A - G_2 C)^T X + X(A - G_2 C) - \gamma C^T C & * & * & * \\ G_1^T X & -\gamma I & 0 & * \\ G_2^T X + \gamma C + J^T & 0 & -\gamma I & * \\ H_1 - F_{\bullet 2} C & F_{\bullet 1} & F_{\bullet 2} + E_1 N & -\gamma I \end{pmatrix} \ll 0.$$

For fixed (X, N), we view this as an inequality in J and apply Lemma 3. We infer that a solution J exists iff

$$\begin{pmatrix} \dot{X} + (A - G_2 C)^T X + X(A - G_2 C) - \gamma C^T C & XG_1 & H_1^T - C^T F_{\bullet 2}^T \\ G_1^T X & -\gamma I & F_{\bullet 1}^T \\ H_1 - F_{\bullet 2} C & F_{\bullet 1} & -\gamma I \end{pmatrix} \ll 0 \qquad (71)$$

and (after further refining the partition as in (30))

$$\begin{pmatrix} -\gamma I & 0 & F_{11}^T & F_{21}^T \\ 0 & -\gamma I & F_{12}^T & F_{22}^T + N^T \\ F_{11} & F_{12} & -\gamma I & 0 \\ F_{21} & F_{22} + N & 0 & -\gamma I \end{pmatrix} \ll 0. \tag{72}$$

A specific solution is given by J defined in (34). If viewing (72) as an inequality in N, we apply Lemma 3 to conclude that a solution exists iff

$$\begin{pmatrix} -\gamma I & F_{11}^T & F_{21}^T \\ F_{11} & -\gamma I & 0 \\ F_{21} & 0 & -\gamma I \end{pmatrix} \ll 0, \quad \begin{pmatrix} -\gamma I & 0 & F_{11}^T \\ 0 & -\gamma I & F_{12}^T \\ F_{11} & F_{12} & -\gamma I \end{pmatrix} \ll 0,$$

and a special solution is given by (33). These two inequalities are clearly equivalent to (31).

Let us assume (31) from now on. By $\begin{pmatrix} -\gamma I & F_{\bullet 1}^T \\ F_{\bullet 1} & -\gamma I \end{pmatrix} \ll 0$, (71) is equivalent to the differential Riccati inequality (35). Standard comparison arguments [30, Chapter 10] reveal that, for all sufficiently small $\epsilon > 0$, the solution of (32) satisfies $0 \ll Z \ll X^{-1}$ on the interval of existence. Consequently, the solution Z exists and the inequalities hold on the whole interval $[0, \infty)$. Conversely, with Z satisfying (32), we infer $Z \gg 0$ and $X = Z^{-1}$ satisfies (35). ■

Proof of Lemma 9. Assume w.l.o.g. $S = I$. Given a basis K of the kernel of C_s, we choose

$$K_y = \left(\begin{array}{ccccc} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K \\ \hline 0 & I & 0 & 0 & 0 \\ -C_r & 0 & 0 & I & 0 \\ -D_r & 0 & 0 & 0 & -D_s K \\ \hline 0 & 0 & I & 0 & 0 \end{array}\right).$$

Partition $X = \begin{pmatrix} X_r & Y^T \\ Y & Z \end{pmatrix}$ to infer by a simple computation that (38) is given as

$$\begin{pmatrix} R_1(X_r) & R_2(Y)^T & R_3(Y)^T K \\ R_2(Y) & -\gamma I & G_s^T Z K \\ K^T R_3(Y) & K^T Z G_s & K^T[A_s^T Z + Z A_s - \gamma D_s^T D_s] K \end{pmatrix} < 0 \tag{73}$$

with

$$\left(\begin{array}{c} R_1(X_r) \\ \hline R_2(Y) \\ \hline R_3(Y) \end{array}\right) = \left(\begin{array}{ccc} A_r^T X_r + X_r A_r - \gamma(C_r^T C_r + D_r^T D_r) & X_r G_r & * \\ G_r^T X_r & -\gamma I & F_r^T \\ H_r - F_{r1} C_r - F_{r2} D_r & F_r & -\gamma I \\ \hline G_s^T Y + \gamma C_r & 0 & F_{r1}^T \\ \hline [A_s^T Y + Y A_r] - \gamma D_s^T D_r & Y G_r & (H_s - F_{r2} D_s)^T \end{array}\right).$$

If (73) has the (positive definite) solution X, we infer by canceling rows/columns that its left-upper block X_r satisfies $R_1(X_r) < 0$ what proves necessity.

Now suppose that there exists a (positive definite) X_r with $R_1(X_r) < 0$. Since G_s^T has full row rank, the first block of $R_2(Y)$ can be arbitrarily assigned by varying Y. Applying Lemma 3, some (explicitly given) Y leads to

$$\begin{pmatrix} R_1(X_r) & R_2(Y)^T \\ R_2(Y) & -\gamma I \end{pmatrix} < 0. \tag{74}$$

Due to (74), a Schur complement argument reveals that (73) is equivalent to

$$K^T[\tilde{A}^T Z + Z\tilde{A} + \frac{1}{\gamma} Z G_s G_s^T Z + \tilde{Q}]K < 0 \tag{75}$$

with $\tilde{A} := A_s - G_s R_2(Y) R_1(X_r)^{-1} R_3(Y)^T/\gamma$, $\tilde{Q} := -R_3(Y)R_1(X_r)^{-1}R_3(Y)^T + R_3(Y)R_1(X_r)^{-1}R_2(Y)^T R_2(Y) R_1(X_r)^{-1} R_3(Y)^T/\gamma - \gamma D_s^T D_s$. By (40), there exist Z and J such that $(\tilde{A}+JC_s)^T Z + Z(\tilde{A}+JC_s) + \frac{1}{\gamma} Z G_s G_s^T Z + \tilde{Q} < 0$. If varying J, the symmetric Z can be made *arbitrarily large*. (This follows from results in almost disturbance decoupling and is the dual version of a slight generalization of [46, Proposition 4.6 (b)]; see also [49].) If we recall that $C_s K = 0$ and that K has full column rank, Z obviously satisfies (75). If $X_r > 0$, Z can be made so large to get $X > 0$. ■

Proof of Theorem 10. We first prove necessity and assume that $X_r > 0$ satisfies (42). Let use rewrite $Q := X_r$ for ease of notation, partition Q as A_r, and define $\tilde{Q} := \begin{pmatrix} Q_1 & Q_{12} \\ Q_{21} & Q_2 \end{pmatrix}$, $\tilde{A} := \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix}$, $\tilde{I} := \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$, $\tilde{G} := \begin{pmatrix} G_1 \\ 0 \end{pmatrix}$, $\tilde{C} := \begin{pmatrix} C_1 & C_2 \end{pmatrix}$, $\tilde{H} := \begin{pmatrix} \tilde{H}_1 & \tilde{H}_2 \end{pmatrix}$, with $\tilde{I}$ partitioned as $\tilde{A}$. Obviously,

$$(-(\tilde{A} + \epsilon\tilde{I}), \tilde{G}) \text{ is stabilizable for all } \epsilon > 0. \tag{76}$$

Just by canceling the third block row/column of (42), we infer

$$\begin{pmatrix} (\tilde{A}+\epsilon\tilde{I})^T\tilde{Q} + \tilde{Q}(\tilde{A}+\epsilon\tilde{I}) + \delta\tilde{I} - \gamma\tilde{C}^T\tilde{C} & \tilde{Q}\tilde{G} & \tilde{H}^T \\ \tilde{G}^T\tilde{Q} & -\gamma I & F_r^T \\ \tilde{H} & F_r & -\gamma I \end{pmatrix} < 0,$$

first for $\epsilon = 0$, $\delta = 0$ and, a posteriori, for some fixed $\delta > 0$ and all small $\epsilon > 0$. Consequently, $F_\gamma > 0$. Moreover, due to (76), there exists [48, Theorem 2] some $Q > \tilde{Q} > 0$ with

$$(\tilde{A}+\epsilon\tilde{I})^T Q + Q(\tilde{A}+\epsilon\tilde{I}) + \delta\tilde{I} - \gamma\tilde{C}^T\tilde{C} + \begin{pmatrix} Q\tilde{G} & \tilde{H}^T \end{pmatrix} F_\gamma^{-1} \begin{pmatrix} \tilde{G}^T Q \\ \tilde{H} \end{pmatrix} = 0 \tag{77}$$

and

$$\sigma\left((\tilde{A}+\epsilon\tilde{I}) + \begin{pmatrix} \tilde{G} & 0 \end{pmatrix} F_\gamma^{-1} \begin{pmatrix} \tilde{G}^T Q \\ \tilde{H} \end{pmatrix}\right) \subset \mathcal{C}^+. \tag{78}$$

Partitioning $Q = \begin{pmatrix} X & Y^T \\ Y & Z \end{pmatrix}$ as $\tilde{A}$, the left-upper blocks of the left-hand sides of (77), (78) and (44), (45) coincide. Since the matrix in (78) is block triangular, we conclude (44)-(45). The left- and right-lower blocks of (77) are given by

$$(A_2 + \epsilon I)^T Y + YA_1 + A_{12}^T X - \gamma C_2^T C_1 + \begin{pmatrix} YG_1 & \tilde{H}_2^T \end{pmatrix} F_\gamma^{-1} \begin{pmatrix} G_1^T X \\ \tilde{H}_1 \end{pmatrix} = 0 \tag{79}$$

and

$$(A_2 + \epsilon I)^T Z + Z(A_2 + \epsilon I) + A_{12}^T Y^T + YA_{12} - \gamma C_2^T C_2 + \begin{pmatrix} G_1^T Y^T \\ \tilde{H}_2 \end{pmatrix}^T F_\gamma^{-1} \begin{pmatrix} G_1^T Y^T \\ \tilde{H}_2 \end{pmatrix} = -\delta I. \tag{80}$$

Due to $E_j^*[(A_2 + \epsilon I)^T Z + Z(A_2 + \epsilon I)]E_j = 2\epsilon E_j^* Z E_j > 0$ (by $A_2 E_j = i\omega_j E_j$ and $Z > 0$), we obtain

$$E_j^* \left[A_{12}^T Y^T + YA_{12} - \gamma C_2^T C_2 + \begin{pmatrix} G_1^T Y^T \\ \tilde{H}_2 \end{pmatrix}^T F_\gamma^{-1} \begin{pmatrix} G_1^T Y^T \\ \tilde{H}_2 \end{pmatrix} \right] E_j = -\delta E_j^* E_j. \tag{81}$$

Now we let ϵ converge to zero. Clearly, for all small $\epsilon \geq 0$, (79) has a unique solution which depends on ϵ and converges for $\epsilon \to 0$ to that of (46). Due to $\delta E_j^* E_j > 0$, (81) leads to (47) in the limit.

For the sufficiency proof, we blockwise construct X_r with $R(X_r) < 0$ using (48) and exploiting the structural dependence of the blocks of $R(X_r)$ on those of X_r as clarified in the remark after Theorem 10. Since $(-A_1, G_1)$ is stabilizable, we can find some X which is arbitrarily close to the stabilizing solution of (44) and such that $R_1(X_r) < 0$ [48, Theorem 2]. If X is sufficiently close to the ARE solution, $X > 0$ and (45) still hold. Hence (46) has a solution Y what amounts to $R_{21}(X_r) = 0$. Moreover, $R_2(X_r)$ equals the left-hand side of (80) for $\epsilon = 0$. Due to (47) and [48, Theorem 4], we can find arbitrarily large Z solving the Lyapunov inequality $R_2(X_r) < 0$. Assign the blocks $*$ in X_r arbitrarily. Since $R_3(X_r)$ has the same structure as $R_2(X_r)$, it equals $A_3^T W + WA_3 + R_3$ where neither R_3 nor any other blocks in $R(X_r)$ depend on W. Since $\sigma(A_3) \subset \mathcal{C}^-$, we can render $R_3(X_r)$ so small that $R(X_r) < 0$. Since both Z and W can be chosen arbitrarily large, we can achieve $X_r > 0$ (by $X > 0$). ■

Proof of Theorem 11. It is well-known that (44) has a solution with (45) iff the strict ARI which corresponds to (44) is solvable [48, Theorem 2]. Together with (43), we hence need to characterize those $\gamma > 0$ for which

$$\begin{pmatrix} A_1^T X + XA_1 - \gamma C_1^T C_1 & XG_1 & \tilde{H}_1^T \\ G_1^T X & -\gamma I & F_r^T \\ \tilde{H}_1 & F_r & -\gamma I \end{pmatrix} < 0 \tag{82}$$

has a symmetric solution. Let us replace $C_1^T C_1$ by $A_1^T X_0 + X_0 A_1 + X_0 G_1 G_1^T X_0$. Then

$$\begin{pmatrix} I & X_0 G_1 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} (82) \begin{pmatrix} I & 0 & 0 \\ G_1^T X_0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

is equivalent to (82), and a short computation results for $\Delta = \gamma X_0 - X$ in

$$\begin{pmatrix} -(A_1 + G_1G_1^TX_0)^T\Delta - \Delta(A_1 + G_1G_1^TX_0) & -\Delta G_1 & (\tilde{H}_1 + F_rG_1^TX_0)^T \\ -G_1^T\Delta & -\gamma I & F_r^T \\ \tilde{H}_1 + F_rG_1^TX_0 & F_r & -\gamma I \end{pmatrix} < 0.$$

Since $-(A_1 + G_1G_1^TX_0)$ is exponentially stable, we can apply the LTI version of Theorem 2 and conclude that this inequality has a solution Δ iff the H_∞ norm of $F_r - (\tilde{H}_1 + F_rG_1^TX_0)(sI + A_1 + G_1G_1^TX_0)^{-1}G_1$ is smaller than γ. ■

Proof of Lemma 13. Since (A, C) is detectable, the LMI (14) has a positive definite solution for some large γ, what implies $X_\gamma > 0$, $U_\gamma > 0$ (Theorem 10).

To show the analytic properties, we use the same notations as in the proof of Theorem 10. We let $X = X_\gamma$ satisfy (44)-(45), $Y = Y_\gamma(\epsilon)$ solve (79), and $Z = Z_\gamma(\epsilon)$ be the unique solution of (80) for $\delta = 0$. Then $Q = Q_\gamma(\epsilon) := \begin{pmatrix} X_\gamma & Y_\gamma(\epsilon)^T \\ Y_\gamma(\epsilon) & Z_\gamma(\epsilon) \end{pmatrix}$ satisfies (77)-(78). Moreover, let us introduce $U_\gamma(\epsilon)$ by (52) with Y_γ replaced by $Y_\gamma(\epsilon)$.

Due to (45), the implicit function theorem implies that X_γ is analytic on (γ_c, ∞). There exists an $\epsilon_0 > 0$ such that $-(A_2 + \epsilon I)$ and the matrix in (45) do not have common eigenvalues for $\epsilon > -\epsilon_0$. Therefore, $Y_\gamma(\epsilon)$ and hence also $U_\gamma(\epsilon)$ are analytic on $(\gamma_c, \infty) \times (-\epsilon_0, \infty)$. However, $Z_\gamma(\epsilon)$ and hence $Q_\gamma(\epsilon)$ are only guaranteed to be analytic on $(\gamma_c, \infty) \times (0, \infty)$.

Let us abbreviate the partial derivatives $Q' = \partial_\gamma Q_\gamma(\epsilon)$, $Q'' = \partial_\gamma^2 Q_\gamma(\epsilon)$, and the matrix in (78) as $\hat{A} = \hat{A}_\gamma(\epsilon)$. Differentiating (44) two times with respect to γ leads to

$$\hat{A}^TQ' + Q'\hat{A} - \tilde{C}^T\tilde{C} - \begin{pmatrix} \tilde{G}^TQ \\ \tilde{H} \end{pmatrix}^T F_\gamma^{-2} \begin{pmatrix} \tilde{G}^TQ \\ \tilde{H} \end{pmatrix} = 0,$$

$$\hat{A}^TQ'' + Q''\hat{A} + 2[F_\gamma^{-1}\begin{pmatrix} \tilde{G}^TQ \\ \tilde{H} \end{pmatrix} + \begin{pmatrix} G^TQ' \\ 0 \end{pmatrix}]^T F_\gamma^{-1} [F_\gamma^{-1}\begin{pmatrix} \tilde{G}^TQ \\ \tilde{H} \end{pmatrix} + \begin{pmatrix} G^TQ' \\ 0 \end{pmatrix}] = 0.$$

By $\sigma(\hat{A}) \subset \mathcal{C}^+$ and $F_\gamma > 0$, we get $Q' \geq 0$ and $Q'' \leq 0$.

This implies $X_\gamma' \geq 0$, $X_\gamma'' \leq 0$ and $\partial_\gamma Z_\gamma(\epsilon) \geq 0$, $\partial_\gamma^2 Z_\gamma(\epsilon) \leq 0$. Now $E_j^*(80)E_j$ leads to $\text{diag}_j(2\epsilon E_j^* Z_\gamma(\epsilon)E_j) = U_\gamma(\epsilon)$ (since $\delta = 0$) and hence $\partial_\gamma U_\gamma(\epsilon) \geq 0$, $\partial_\gamma^2 U_\gamma(\epsilon) \leq 0$. All this holds for $\epsilon > 0$. However, since $U_\gamma(\epsilon)$ is in fact analytic on $(\gamma_c, \infty) \times (-\epsilon_0, \infty)$, we can conclude $\partial_\gamma U_\gamma(0) \geq 0$, $\partial_\gamma^2 U_\gamma(0) \leq 0$ by taking the limit $\epsilon \to 0$. ■

Proof of Theorem 15. Assume w.l.o.g. $S = I$ and recap the proofs of Lemma 9 and Theorem 10. In the necessity parts we showed: If X satisfies (38), then its left-upper block Q satisfies (42). Further, the left-upper block Q_1 of Q is a solution of the strict ARI which corresponds to (44). For X_γ satisfying (44)-(45) we infer $Q_1 < X_\gamma$ [48, Theorem 2] what implies $P_\gamma < X^{-1}$ [48, Lemma 14].

In the sufficiency proof of Theorem 10, the blocks Z and W can be chosen arbitrarily large such that X_r^{-1} is arbitrarily close to $P_r(\gamma)$ [48, Lemma 14]. Now X_r satisfies (42). In the construction of Lemma 9, Z can be taken arbitrarily large such that X^{-1} is arbitrarily close to X_r^{-1} [48, Lemma 14]. Hence X^{-1} can be chosen arbitrarily close to P_γ.

Analyticity of P_γ is a consequence of that of X_γ, and the inequalities for the derivates follow from $(X_\gamma^{-1})' = -X_\gamma^{-1}X_\gamma'X_\gamma^{-1}$, $(X_\gamma^{-1})'' = 2X_\gamma^{-1}X_\gamma'X_\gamma^{-1}X_\gamma'X_\gamma^{-1} - X_\gamma^{-1}X_\gamma''X_\gamma^{-1}$ and $X_\gamma > 0$, $X_\gamma' \geq 0$, $X_\gamma'' \leq 0$. ■

Proof of Corollary 21. Let us first prove the relation (67). From $\rho(P_\gamma Q_\gamma) = \rho(S_X X_\gamma^{-1} S_X^T S_Y Y_\gamma^{-1} S_Y^T) = \rho(Y_\gamma^{-1/2} S_Y^T S_X X_\gamma^{-1} S_X^T S_Y Y_\gamma^{-1/2})$ we infer $\rho(P_\gamma Q_\gamma) < 1$ iff $Y_\gamma > S_Y^T S_X X_\gamma^{-1} S_X^T S_Y$. Taking Schur complement leads to (67).

Necessity is now proved as follows. If γ is suboptimal, there exist constant X, Y satisfying (64) and (65). Hence $\gamma > \gamma_e$, $\gamma > \delta_e$, and $X_\gamma > 0$, $U_\gamma > 0$, $Y_\gamma > 0$, $V_\gamma > 0$, and $\rho(X^{-1}Y^{-1}) < 1$. Since $P_\gamma < X^{-1}$ and $Q_\gamma < Y^{-1}$ (strict lower bound property), we obtain $\rho(P_\gamma Q_\gamma) < 1$ and hence (66) by (67).

For the sufficiency part, $\gamma > \gamma_e$, $\gamma > \delta_e$, and $X_\gamma > 0$, $U_\gamma > 0$, $Y_\gamma > 0$, $V_\gamma > 0$ imply that (65) have solutions $X > 0$, $Y > 0$. By (67) we infer $\rho(P_\gamma Q_\gamma) < 1$. Since X^{-1}, Y^{-1} can be chosen close to P_γ, Q_γ (by the limit point property), we have $\rho(X^{-1}Y^{-1}) < 1$ for suitable X, Y what implies (64). ■

Proof of Corollary 22. If $\alpha > \alpha_{\mathrm{opt}}$, Corollary 17 yields the existence of (Y, F) with $AY + YA^T + BF + (BF)^T + GG^T < 0$ and

$$\begin{pmatrix} \alpha^2 I - H_2 P H_2^T & H_2[Y-P] + E_2 F \\ (H_2[Y-P] + E_2F)^T & Y - P \end{pmatrix} > 0.$$

With the transformation $\tilde{Y} := Y - P$, $\tilde{F} := F\tilde{Y}^{-1}$ and the abbreviation $S_\alpha := \alpha^2 I - H_2 P H_2^T$, these inequalities are rewritten to

$$(A + B\tilde{F})\tilde{Y} + \tilde{Y}(A + B\tilde{F})^T + VV^T < 0, \quad \begin{pmatrix} S_\alpha & (H_2 + E_2\tilde{F})\tilde{Y} \\ \tilde{Y}(H_2 + E_2\tilde{F})^T & \tilde{Y} \end{pmatrix} > 0$$

and the second one is equivalent to

$$S_\alpha > 0, \quad \tilde{Y} > 0, \quad I > S_\alpha^{-1/2}(H_2 + E_2\tilde{F})\tilde{Y}(H_2 + E_2\tilde{F})^T S_\alpha^{-1/2}.$$

We conclude that $A + B\tilde{F}$ is exponentially stable. Moreover, defining $M(t) := S_\alpha^{-1/2}(H_2 + E_2\tilde{F})e^{(A+B\tilde{F})t}V$, we infer $I > \int_0^\infty M(t)M(t)^T dt$. Going back to the integral definiton with Riemann sums, one shows $I > \int_0^\infty M(t)^T M(t) dt$. Hence there exists an $X > 0$ with

$$(A + B\tilde{F})^T X + X(A + B\tilde{F}) + (H_2 + E_2\tilde{F})^T S_\alpha^{-1}(H_2 + E_2\tilde{F}) < 0, \quad V^T X V < I.$$

With $Z := X^{-1}$, $F := \tilde{F}X^{-1}$ and recalling the definition of S_α, we obtain

$$\begin{pmatrix} AZ + ZA^T + BF + (BF)^T & (H_2 Z + E_2 F)^T \\ H_2 Z + E_2 F & H_2 U U^T H_2^T - \alpha^2 I \end{pmatrix} < 0, \quad V^T Z^{-1} V < I.$$

Dividing the first inequality by α leads to (68) and (69) for $(Z/\alpha, F/\alpha)$. The converse is obtained by reversing the arguments. ■