WHEN ARE MULTIPLIER RELAXATIONS EXACT?

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Abstract: We revisit the robust performance analysis problem for structured uncertainties whose frequency responses lie in the rather general class of diagonal matrices with blocks that are full ellipsoidal or repeated and contained in intersections of disks or circles or that are located in finitely generated convex sets. Based on the full block S-procedure we suggested LMI relaxations to verify robust performance. Our main purpose is to prove a general computationally verifiable condition for when these relaxations do not involve any conservatism. This allows to show the general exactness of the suggested relaxations for small block-structures, comprising an elementary proof for the structured singular value being exactly computable by convex optimization for three full complex blocks.

Keywords: Robust performance, structured singular value, relaxation analysis, uncertain linear systems, convex optimization

1. INTRODUCTION

During the last twenty years it has been established that a whole variety of robust stability and robust performance conditions can be translated into robust linear algebra tests in the sense as exposed in (El Ghaoui and Lebret, 1997) for classical least-squares approximation. The concept of multipliers (Packard and Doyle, 1993) plays a crucial role in relaxing these typically hard non-convex problems into finite-dimensional convex optimization problems. The general framework of quadratic separators (Iwasaki and Hara, 1996; Iwasaki and Hara, 1998) or full block multipliers (Scherer, 1996; Scherer, 2001) turns out to be particularly useful for constructing relaxation families with varying degrees of conservatism. The power of this procedure has been demonstrated by devising classes of robust performance analysis algorithms for linear parameter-varying systems and Lyapunov functions which depend (piece-wise) rationally on the parameters (Scherer, 1998; Dettori and Scherer, 1998). Unfortunately all these relaxations typically involve conservatism, and only for simple block structures it is known that they are generally exact (Packard and Doyle, 1993; Meinsma *et al.*, 1997). Moreover a reduction of conservatism is not possible uniformly in all problem instances (Toker and Ozbay, 1995). However, for specific practical problem instances it is our experience that, often, no improvement can be gained by enlarging the multiplier set (e.g. from diagonal to full). This raises the issue of developing computational tools for the verification of exactness in order to be sure that no further improvement is possible by whatever refinement is employed.

The present paper is an initial step in this direction. Motivated by the exact characterization of the structured singular value by duality we suggest a novel exactness principle for general relaxations that are based on the full block S-

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procedure, and we demonstrate the validity of this principle for a concrete but rather general class of uncertainty block structures, while extending and unifying the results and techniques of proof as appearing in (Packard and Doyle, 1993; Rantzer, 1996; Meinsma *et al.*, 1997; Vandenberghe and Balakrishnan, 1999; Henrion and Meinsma, 2001). Practicability is demonstrated by means of a simple example.

2. ROBUST PERFORMANCE ANALYSIS

Suppose that some controlled generalized plant (with uncertainty and performance channels) is described by

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} = \left[\begin{array}{c|cc} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \hline \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{array}\right]$$

with $\mathcal{A}$ being Hurwitz. Let the set of uncertainties consist of all proper and stable Δ whose frequency responses satisfy $\Delta(s) \in \boldsymbol{\Delta_c}$ for all $s \in i\mathbb{R} \cup \{\infty\}$ where $\boldsymbol{\Delta_c}$ is some set of complex matrices imposing structure and size. It is well-known (Packard and Doyle, 1993; Zhou *et al.*, 1996) that various practically important robust stability and performance specifications (such as a H_∞-norm bounds or positive-realness conditions) can be reformulated as the following RP test: Verify whether for all $s \in i\mathbb{R} \cup \{\infty\}$ and all $\Delta \in \boldsymbol{\Delta_c}$

$$\det(I - N_{11}(s)\Delta) \neq 0, \quad (1)$$

$$\begin{pmatrix} I \\ \Delta \star N(s) \end{pmatrix}^* P_p \begin{pmatrix} I \\ \Delta \star N(s) \end{pmatrix} < 0. \quad (2)$$

The performance index $P_p = P_p^*$ is supposed to have a positive semi-definite right-lower block. We fix $s \in i\mathbb{R} \cup \{\infty\}$ and abbreviate $H := N(s)$ partitioned into A, B, C, D. It is then rather simple to verify that (1)-(2) are satisfied if there exists a Hermitian multiplier P with

$$\begin{pmatrix} \Delta \\ I \end{pmatrix}^* P \begin{pmatrix} \Delta \\ I \end{pmatrix} \geq 0 \ \text{ for all } \ \Delta \in \boldsymbol{\Delta_c} \quad (3)$$

$$\begin{pmatrix} * \\ * \end{pmatrix}^* P \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} * \\ * \end{pmatrix}^* P_p \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} < 0. \quad (4)$$

To render this test computational we introduce a class $\boldsymbol{\Delta_c}$ which is sufficiently general to cover the most important uncertainty structures as they appear in practice (Iwasaki and Hara, 1998; Peaucelle, 2000), but which has a sufficiently specific description to allow for proving the main result of this paper. With the column partition $I = (E_1 \ \cdots \ E_m)$ and $I = (F_1 \ \cdots \ F_m)$ of the identity matrix we choose for the block-diagonal basic structure

$$\boldsymbol{\Delta_c} := \left\{ \Delta = \sum_{k=1}^{m} E_k \Delta_k F_k^* : \quad \Delta_k \in \boldsymbol{\Delta_k} \right\}$$

where $\boldsymbol{\Delta_k}$ imposes itself a more specific structure on the sub-block Δ_k (with index sets satisfying $K_f \cap K_r \cap K_k = \emptyset$, $K_f \cup K_r \cup K_k = \{1, \ldots, m\}$) as follows:

Full blocks. For $k \in K_f$, $\boldsymbol{\Delta_k}$ is the set of all full blocks $\Delta_k \in \mathbb{C}^{n_k \times m_k}$ constrained as

$$\begin{pmatrix} \Delta_k \\ I \end{pmatrix}^* P^k \begin{pmatrix} \Delta_k \\ I \end{pmatrix} \geq 0$$

with fixed nonsingular $P^k = (P^k)^*$ having negative definite left-upper block. Choose the multiplier set $\boldsymbol{P_k} := \{\tau^k P^k : \ \tau^k \geq 0\}$.

Repeated scalar blocks. For $k \in K_r$ consider the set $\boldsymbol{\delta_k}$ of $\delta_k \in \mathbb{C}$ with

$$\begin{pmatrix} \delta_k \\ 1 \end{pmatrix}^* P_0^k \begin{pmatrix} \delta_k \\ 1 \end{pmatrix} = 0, \ \begin{pmatrix} \delta_k \\ 1 \end{pmatrix}^* P_j^k \begin{pmatrix} \delta_k \\ 1 \end{pmatrix} \geq 0$$

for $j = 1, \ldots, N_k$ with fixed Hermitian matrices $P_j^k \in \mathbb{C}^{2\times 2}$. By Lemma 3 we exclude trivialities with $\det(P_j^k) < 0$ for all j (including $j = 0$ if $P_0^k \neq 0$). If $P_0^k = 0$, $\boldsymbol{\delta_k}$ is a finite intersection of disks and half-planes in $\mathbb{C}$, and if $P_0^k \neq 0$ then it is additionally constrained to either lines or circles. Note that $\boldsymbol{\delta_k}$ is unbounded iff all $(1,1)$ elements of P_j^k are nonnegative. Define

$$\boldsymbol{\Delta_k} = \{\delta_k I_{n_k} : \ \delta_k \in \boldsymbol{\delta_k}\} \quad \text{for bounded} \quad \boldsymbol{\delta_k}$$

and include ∞ (with a natural interpretation for use in linear fractional representations) if $\boldsymbol{\delta_k}$ is unbounded. Now introduce

$$\boldsymbol{P_k} := \left\{ \sum_{j=0}^{N_k} P_j^k \otimes D_j^k : \ D_0^k = (D_0^k)^*, \ D_j^k \geq 0 \right\}.$$

Convex-hull blocks. For $k \in K_h$ we assume $\boldsymbol{\Delta_k} := \mathrm{con}\{\Delta_1^k, \ldots, \Delta_{N_k}^k\}$ such that Δ_k is confined to polytopes of real or complex full or structured matrices. Now $\boldsymbol{P_k}$ is the set of all $P_k = P_k^*$ such that for all $j = 1, \ldots, N_k$

$$\begin{pmatrix} I \\ 0 \end{pmatrix}^* P_k \begin{pmatrix} I \\ 0 \end{pmatrix} \leq 0, \ \begin{pmatrix} \Delta_j^k \\ I \end{pmatrix}^* P_k \begin{pmatrix} \Delta_j^k \\ I \end{pmatrix} \geq 0.$$

Then it is elementary to verify that

$$\boldsymbol{P} = \left\{ \sum_{k=1}^{m} \begin{pmatrix} E_k & 0 \\ 0 & F_k \end{pmatrix} P_k \begin{pmatrix} E_k^* & 0 \\ 0 & F_k^* \end{pmatrix} : \ P_k \in \boldsymbol{P_k} \right\}$$

does indeed satisfy (3), and hence(1)-(2) are guaranteed if there exists some $P \in \boldsymbol{P}$ with (4). This computational robust performance test is rather well-established in the literature (Packard and Doyle, 1993; Iwasaki and Hara, 1996; Scherer, 1996; Iwasaki and Hara, 1998; Peaucelle, 2000; Scherer, 2001) and known to be only sufficient for general problem instances. In practice, however, it does indeed happen pretty often that this relaxation does not involve any conservatism, which motivates to identify numerically verifiable conditions for its exactness.

3. WHEN ARE RELAXATIONS EXACT?

Suppose there does not exist any $P \in \boldsymbol{P}$ that satisfies (4). By a standard Farkas duality result there exist Lagrange multipliers $M \geq 0$, $m^k \geq 0$, $M_j^k \geq 0$, $\hat{M}_k \geq 0$ which do not all vanish and that satisfy (with $U_k = \begin{pmatrix} E_k^* & 0 \end{pmatrix}$, $V_k = \begin{pmatrix} F_k^* A & F_k^* B \end{pmatrix}$)

$$\langle P_p, \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} M \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^* \rangle \geq 0 \qquad (5)$$

as well as

$$\langle P^k, \begin{pmatrix} U_k \\ V_k \end{pmatrix} M \begin{pmatrix} U_k & V_k \end{pmatrix} \rangle - m^k = 0,$$

$$\begin{pmatrix} U_k & V_k \end{pmatrix} [(P_j^k)^T \otimes M] \begin{pmatrix} U_k^* \\ V_k^* \end{pmatrix} - M_j^k = 0,$$

$$\begin{pmatrix} U_k \\ V_k \end{pmatrix} M \begin{pmatrix} U_k^* & V_k^* \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} \hat{M}_k \begin{pmatrix} I & 0 \end{pmatrix} - \\ - \sum_{j=1}^{N_k} \begin{pmatrix} \Delta_j^k \\ I \end{pmatrix} M_j^k \begin{pmatrix} \Delta_j^k \\ I \end{pmatrix}^* = 0$$

for $k \in K_f$, $k \in K_r$, $k \in K_h$ respectively. We infer that $M \neq 0$ which leads us to our main result.

Theorem 1. Suppose there exists an infeasibility certificate such that M has rank one. Then there exists some $\hat{\Delta} \in \boldsymbol{\Delta_c}$ at which either $\Delta \star H$ is not well-posed or for which

$$\begin{pmatrix} I \\ \hat{\Delta} \star H \end{pmatrix}^* P_p \begin{pmatrix} I \\ \hat{\Delta} \star H \end{pmatrix} \not< 0.$$

Proof. Decompose $M = mm^*$, partition $m = \mathrm{col}(w, \xi)$ according to the columns of $(A\ B)$, and define the column vectors $z = Aw + B\xi$, $\eta = Cw + D\xi$, $w_k := E_k^* w = U_k m$, $z_k := F_k^* z = V_k m$. For $k \in K_f$ we have

$$0 \leq \langle P^k, \begin{pmatrix} U_k \\ V_k \end{pmatrix} mm^* \begin{pmatrix} U_k \\ V_k \end{pmatrix}^* \rangle = \begin{pmatrix} w_k \\ z_k \end{pmatrix}^* P^k \begin{pmatrix} w_k \\ z_k \end{pmatrix}.$$

One can prove $w_k = \Delta_k z_k$ for some $\Delta_k \in \boldsymbol{\Delta_k}$. For $k \in K_r$ we infer for $Q_j^k := (P_j^k)^T$ that

$$0 \leq \begin{pmatrix} U_k^* \\ V_k^* \end{pmatrix}^* Q_j^k \otimes mm^* \begin{pmatrix} U_k^* \\ V_k^* \end{pmatrix} = \begin{pmatrix} w_k^* \\ z_k^* \end{pmatrix}^* Q_j^k \begin{pmatrix} w_k^* \\ z_k^* \end{pmatrix}.$$

If $z_k \neq 0$ it is not difficult to show $w_k = z_k \delta_k = [\delta_k I] z_k = \Delta_k z_k$ with $\Delta_k \in \boldsymbol{\Delta_k}$. The same holds if $z_k = 0$, $w_k = 0$ with any $\delta_k \in \boldsymbol{\delta_k}$, and if $z_k = 0$, $w_k \neq 0$ for $\delta_k = \infty$, since then all $(1,1)$ elements of Q_j^k are nonnegative which implies $\infty \in \boldsymbol{\delta_k}$.
For $k \in K_h$ we conclude from $z_k z_k^* = \sum_{j=1}^{N_k} M_j^k$ that $\ker(z_k^*) \subset \ker(M_j^k)$ and hence there exist α_j^k with $M_j^k = z_k \alpha_j^k z_k^*$. If $z_k \neq 0$ we have $\alpha_j^k \geq 0$ and $\sum_{j=1}^{N_k} \alpha_j^k = 1$, and we infer from $w_k z_k^* = \sum_{j=1}^{N_k} \Delta_j^k M_j^k$ by right-multiplication with z_k^* and division by $z_k^* z_k$ that $w_k = \Delta_k z_k$ for $\Delta_k = \sum_{j=1}^{N_k} \alpha_j^k \Delta_j^k$. If $z_k = 0$ then $M_j^k = 0$ and hence $w_k w_k^* + \hat{M}_k = 0$ and hence $w_k = 0$ and hence $w_k = \Delta_k z_k$ for any $\Delta_k \in \boldsymbol{\Delta_k}$.
Let us assume that the constructed sub-blocks Δ_k define $\hat{\Delta} \in \boldsymbol{\Delta_c}$. If $\Delta \star H$ is not well-posed at $\hat{\Delta}$ the proof is finished. If $\Delta \star H$ is well-posed at $\hat{\Delta}$ we infer $\eta = [\hat{\Delta} \star H]\xi$. With (5) we get

$$0 \leq \begin{pmatrix} \xi \\ \eta \end{pmatrix}^* P_p \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \xi^* \begin{pmatrix} I \\ \hat{\Delta} \star H \end{pmatrix}^* P_p \begin{pmatrix} I \\ \hat{\Delta} \star H \end{pmatrix} \xi.$$

It remains to show $\xi \neq 0$; if $\xi = 0$ then $z = Aw$, $w = \hat{\Delta} z$ and thus $w = 0$, $z = 0$ (by well-posedness) and thus $m = 0$, a contradiction. ∎

Theorem 2. The suggested relaxation is always exact in case that either one of the following conditions on the number of blocks hold:

- $|K_f| = 2$, $|K_r| = 0$ and $|K_h| = 0$.
- $|K_f| = 0$, $|K_r| = 1$, $|K_h| = 0$ & $N_k = 1$, $k \in K_r$.
- $|K_f| = 0$, $|K_r| = 0$, $|K_h| = 1$ & $N_k = 2$, $k \in K_h$.

Proof.
• The infeasibility test reduces to the existence of some nonzero $M \geq 0$ with (5) and

$$\langle \begin{pmatrix} A_k & B_k \end{pmatrix} P^k \begin{pmatrix} U_k \\ V_k \end{pmatrix}, M \rangle \geq 0 \quad \text{for} \quad k \in K_f.$$

We can apply Lemma 4 to infer that there exists a vector $m \neq 0$ such that the same relations are true if we replace M with mm^*. Hence there exists a rank one infeasibility certificate.
• Choose $k \in K_r$ and assume $P_0^k \neq 0$. Then infeasibility is equivalent to the existence of a nonzero $M \geq 0$ with (5) and

$$\begin{pmatrix} U_k & V_k \end{pmatrix} [(P_j^k)^T \otimes M] \begin{pmatrix} U_k^* \\ V_k^* \end{pmatrix} = 0 \quad \text{and} \quad \geq 0$$

for $j = 0, 1$ respectively. By Lemma 5 and due to (5) we find some nonzero m such that the same relations hold for $M = mm^*$ and such that, in addition

$$\langle \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^* P_p \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}, mm^* \rangle \geq 0.$$

Hence there is a rank one infeasibility certificate. The proof is somewhat more tricky in case that P_0^k vanishes since then we cannot directly apply Lemma 5. Since $E_k = I$, $F_k = I$ and if we recall the definition of U_k, V_k, infeasibility is equivalent to the existence of a nonzero $M \geq 0$ with (5) and $L = K[(P_1^k)^T \otimes M]K^* \geq 0$ with the additional abbreviation $K = (I\ 0\ A\ B)$. If there exists some $\delta \in \mathbb{C}$ with

$$\begin{pmatrix} \delta^* \\ 1 \end{pmatrix}^* (P_1^k)^T \begin{pmatrix} \delta^* \\ 1 \end{pmatrix} \geq 0, \ \det(I - \delta^* A^*) = 0$$

or if A^* is singular (in case that $\boldsymbol{\delta_k}$ is unbounded), we infer that $\Delta \star H$ is not well-posed at δI. Otherwise we apply Lemma 3 to conclude the existence of a Hermitian $X \leq 0$ with

$$\begin{pmatrix} I \\ A^* \end{pmatrix}^* (P_1^k)^T \otimes X \begin{pmatrix} I \\ A^* \end{pmatrix} = L.$$

We can extend X by zero blocks to obtain some M_0 with $K[(P_1^k)^T \otimes M_0]K^* = L$ and with

$$\langle \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^* P_p \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}, M_0 \rangle = \langle R_p, CXC^* \rangle \leq 0$$

where $R_p \geq 0$ is the right-lower block of P_p. Hence $\hat{M} := M - M_0 \geq M \geq 0$ is nonzero and satisfies (5) as well as $(U_k\ V_k)[(P_1^k)^T \otimes \hat{M}](U_k\ V_k)^* = 0$. The proof can now be finished as for $P_0^k \neq 0$.
• This corresponds to an uncertainty structure with one real repeated block and is hence a consequence of the second item. ■

For verifying robust performance this test has to be performed frequency-wise, with the risk of missing relevant frequencies to arrive at misleading conclusions. Just due to

$$\Delta \star N(s) = \begin{pmatrix} \frac{1}{s}I & 0 \\ 0 & \Delta \end{pmatrix} \star \begin{pmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{pmatrix}$$

it is straightforward to guarantee (1)-(2) for all s in some intersections of disks or circles (possibly containing ∞ and then degenerating to half-planes and lines) by a suitable multiplier relaxation. We can directly apply Theorems 1 and 2 to obtain conditions for when these relaxations are non-conservative. A specialization to the case when the original uncertainties are absent leads to a whole class of novel variations of the KYP Lemma (for polynomials) on intervals as recently discussed in (Meinsma *et al.*, 1997; Nesterov, 2000; Genin *et al.*, 2000; Alkire and Vandenberghe, 2001; Sturm and Zhang, 2001).

For full and repeated blocks we have generalized in this section the exactness characterization of the structured singular value in (Packard and Doyle, 1993; Meinsma *et al.*, 1997) to more general sets, with unifying proofs that allow for unbounded repeated uncertainties. We would like to particularly stress the simplicity of the proof of the first item in Theorem 2 which corresponds to the considerably more involved exposition in (Packard and Doyle, 1993) for the case of three full complex blocks. In contrast to (Meinsma *et al.*, 1997; Henrion and Meinsma, 2001) we managed to handle *intersections* of disks and circles to describe repeated blocks, which leads to a common treatment of the KYP Lemma for continuous- and discrete-time (Rantzer, 1996; Vandenberghe and Balakrishnan, 1999). After the submission of this paper we became aware of an independent proof of the second item in Theorem 2 in (Iwasaki and Hara, 2003). For convex hull blocks Theorem 1 seems to be new.

Despite the just described improvements, the main purpose of this paper is to rather suggest the following general principle for the absence of a relaxation gap.

Principle. If there exists an infeasibility certificate (Lagrange multiplier) corresponding to the full S-procedure multiplier inequality (4) which has rank one then the relaxation is non-conservative.

In addition to the cases discussed above, this principle has been successfully confirmed for problems in which one minimizes a linear functional of some parameters that enter the performance index affinely (such as computing optimal robust H_∞-norm bounds). We strongly believe in the existence of many more interesting problem instances for which it can be turned into a concrete result. Let us finally demonstrate that it can be often beneficially applied in practice.

4. AN EXAMPLE

For some tracking interconnection with three one-dimensional uncertainty blocks (one dynamic and two real parametric) and a one-dimensional performance channel we have computed the following description of the closed-loop interconnection

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ e \end{pmatrix} = \left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 2.5 & 1 & 0 & 0 & -1 \\ 0 & 4 & -0.1 & -10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -2 & 0 & 0 & 2 & 0 \\ \hline 0.5 & 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & -1.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.5 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ d \end{pmatrix}$$

where $w = \Delta z$ and $\Delta = diag(\delta_1, \delta_2, \delta_3) \in \mathrm{diag}(RH_\infty, \mathbb{R}, \mathbb{R})$ with $\|\delta_1\|_\infty \leq 0.9$, $|\delta_2| \leq 1.2$ $|\delta_3| \leq 1.2$. We employ the relaxation procedure as sketched at the end of Section 3 to compute bounds on the worst-case H_∞-norm of $d \to e$ on a partition of the frequency interval $[1, 10]$ into forty subintervals and for two classes of multipliers, one as they appear in standard structured singular value theory and one in which we combine the two real uncertainties into a hull block with full block multipliers. Figure 1 depicts the guaranteed bounds and reveals a discrepancy of the bounds in $[1, 2]$ due to the different multiplier classes. More importantly, on the basis of Theorem 1 we indicate in which frequency regions the least conservative relaxation is actually exact.

5. CONCLUSIONS

We provided a test for non-conservatism of LMI robust performance algorithms that is based on a general rank one exactness principle for relaxation based on the full block S-procedure. This allowed to prove in an independent elementary fashion the relaxation's exactness for two full complex blocks and for one repeated complex block in arbitrary disks or half-planes, possibly intersected with lines or circles. A nonstrict version of the second item in Theorem 2 has been independently obtained in (Iwasaki and Hara, 2003).

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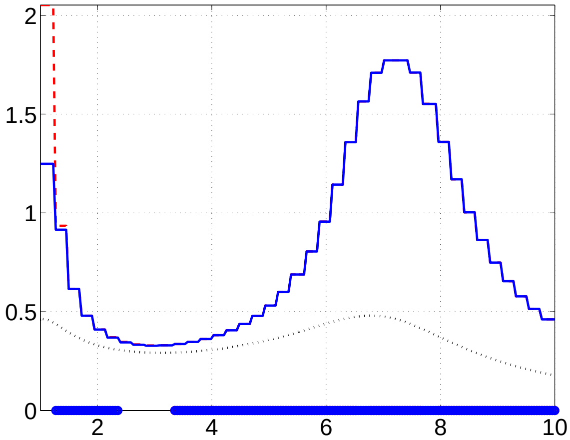

Fig. 1. Performance levels plotted versus frequency: nominal performance (dotted), robust performance with standard scalings (dash-dotted), full scalings (full). Thick parts of axis: Least conservative relaxation is exact.

REFERENCES

Alkire, B. and L. Vandenberghe (2001). Convex optimization problems involving finite autocorrelation sequences. *Mathematical Programming, Series B*.

Dettori, M. and C.W. Scherer (1998). Robust stability analysis for parameter dependent systems using full block S-procedure. In: *Proc. 37th IEEE Conf. Decision and Control*. Tampa, FL. pp. 2798–2799.

El Ghaoui, L. and H. Lebret (1997). Robust solutions to least-squares problems with uncertain data. *SIAM Journal on Matrix Analysis and Applications* **18**(4), 1035–1064.

Genin, Y., Y. Hachez, Y. E. Nesterov and P. Van Dooren (2000). Convex optimization over positive polynomials and filter design. In: *Proceedings of the 2000 UKACC International Conference on Control*. Cambridge University.

Henrion, D. and G. Meinsma (2001). Rank-one LMIs and Lyapunov's inequality. *IEEE Trans. Aut. Contr.* **46**(8), 1285 –1288.

Horn, R.A. and C.R. Johnson (1985). *Matrix Analysis*. Cambrigde University Press. New York.

Iwasaki, T. and S. Hara (1996). Well-posedness of feedback systems: insights into exact robustness analysis. In: *Proc. 35th IEEE Conf. Decision and Control*. Kobe, Japan. pp. 1863–1868.

Iwasaki, T. and S. Hara (1998). Well-posedness of feedback systems: Insights into exact robustness analysis and approximate computations. *IEEE Trans. Autom. Contr.* **43**, 619–630.

Iwasaki, T. and S. Hara (2003). Generalization of Kalman-Yakubovič-Popov lemma for restricted frequency inequalities. In: *Proc. American Control Conf.*. Denver, Colorado USA.

Meinsma, G., G. Shrivastava and M. Fu (1997). A dual formulation of mixed μ and on the losslessness of (D, G) scaling. *IEEE Trans. Aut. Contr.* **42**(7), 1032 –1036.

Nesterov, Y. (2000). Squared functional systems and optimization problems. In: *High Performance Optimization*. pp. 405–440. Kluwer Academic Press. Dordrecht.

Packard, A. and J. Doyle (1993). The complex structured singular value. *Automatica* **29**, 71–109.

Peaucelle, D. (2000). Unified Formulation for Robust Analysis and Synthesis with Parameter Dependent Lyapunov Functions. Ph.d.. Universit Paul Sabatier.

Rantzer, A. (1996). On the Kalman-Yakubovich-Popov lemma. *Syst. Contr. Letters* **28**, 7–10.

Scherer, C.W. (1996). Mixed H_2/H_∞ control for time-varying and linear parametrically-varying systems. In: *Int. J. of Robust and Nonlinear Control*. pp. 929–952.

Scherer, C.W. (1998). Robust performance analysis for parameter dependent systems using tensor product splines. In: *Proc. 37th IEEE Conf. Decision and Control*. pp. 2216–2221. Tampa, Fl.

Scherer, C.W. (2001). LPV control and full block multipliers. *Automatica* **37**, 361–375.

Sturm, J.F. and S. Zhang (2001). On cones of nonnegative quadratic functions.

Toker, O. and H. Ozbay (1995). On the NP-hardness of the purely complex μ computation, analysis/synthesis, and some related problems in multidimensional systems. In: *Proc. American Control Conf.*. pp. 447–451.

Vandenberghe, L. and V. Balakrishnan (1999). Semidefinite programming duality and linear system theory: connections and implications for computation. In: *Proc. 38th IEEE Conf. Decision and Control*. Phoenix, Arizona. pp. 989 –994.

Zhou, K., J. Doyle and K. Glover (1996). *Robust and Optimal Control*. Prentice Hall. Upper Saddle River, New Jersey.

6. AUXILIARY RESULTS

Lemma 3. For $P = P^* \in \mathbb{C}^{2\times 2}$ define

$$\boldsymbol{\delta} = \left\{ \delta \in \mathbb{C} : \begin{pmatrix} \delta \\ 1 \end{pmatrix}^* P \begin{pmatrix} \delta \\ 1 \end{pmatrix} \geq 0 \right\}.$$

If $\boldsymbol{\delta}$ contains at least two points and is not $\mathbb{C}$ then $\det(P) < 0$. If q denotes the $(1,1)$ element of P, $\boldsymbol{\delta}$ is the closed exterior of a circle for $q > 0$, a half-plane for $q = 0$, and a closed disk for $q < 0$. Suppose $I - \delta A$ is non-singular for all $\delta \in \boldsymbol{\delta}$, and A is non-singular if $\boldsymbol{\delta}$ is unbounded. Then for every $Q \geq 0$ there exists a unique $X \leq 0$ with

$$\begin{pmatrix} I \\ A \end{pmatrix}^* [P \otimes X] \begin{pmatrix} I \\ A \end{pmatrix} = Q.$$

Lemma 4. Let $A_j \in \mathbb{C}^{n\times n}$ be Hermitian and suppose the nonzero $X \geq 0$ satisfies $\mathrm{Tr}(A_j X) \geq 0$ for $j = 1, 2, 3$. Then there exists a vector $x \neq 0$ with $x^* A_j x \geq 0$ for $j = 1, 2, 3$.

Proof. Decompose $X = U^*U$ with U of full row rank n. Note that $0 \leq \mathrm{Tr}(A_j X) = \mathrm{Tr}(U^* A_j U)$. Define

$$M := [U^*A_1U - \frac{1}{n}\mathrm{Tr}(U^*A_1U)I] + \\ + i[U^*A_2U - \frac{1}{n}\mathrm{Tr}(U^*A_2U)I]$$

with trace zero. There exists a unitary matrix V with $\mathrm{diag}(V^*MV) = 0$ (Horn and Johnson, 1985). Since $0 \leq \mathrm{Tr}(U^*A_3U) = \mathrm{Tr}(V^*[U^*A_3U]V)$, there must exist at least one column v of V with $v^*[U^*A_3U]v \geq 0$. With $x = Uv$ the proof is concluded by observing

$$0 = \begin{pmatrix} \mathrm{Re}(v^*Mv) \\ \mathrm{Im}(v^*Mv) \end{pmatrix} = \begin{pmatrix} x^*A_1x - \mathrm{Tr}(A_1X)/n \\ x^*A_2x - \mathrm{Tr}(A_2X)/n \end{pmatrix}$$

and hence $x^*A_1x \geq 0$ as well as $x^*A_2x \geq 0$. ■

Lemma 5. For complex A, B and Hermitian C, $P, \hat{P} \in \mathbb{C}^{2\times 2}$ with $\det(P) < 0$, $\det(\hat{P}) < 0$, suppose the nonzero $X \geq 0$ satisfies

$$\begin{pmatrix} A \\ B \end{pmatrix}^* P \otimes X \begin{pmatrix} A \\ B \end{pmatrix} = 0, \begin{pmatrix} A \\ B \end{pmatrix}^* \hat{P} \otimes X \begin{pmatrix} A \\ B \end{pmatrix} \geq 0. \quad (6)$$

Then the same holds for X replaced by xx^* with some vector $x \neq 0$ satisfying $x^*Cx \geq \langle C, X\rangle$.

Proof. We exploit (Horn and Johnson, 1985, Theorem 4.5.19) to infer that there exists a nonsingular $S \in \mathbb{C}^{2\times 2}$ such that either (Case 1)

$$S^*PS = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; S^*\hat{P}S = \pm \begin{pmatrix} 0 & \alpha \\ \alpha^* & \beta \end{pmatrix}, \; \alpha \neq 0$$

or (Case 2)

$$S^*PS = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \; S^*\hat{P}S = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \; \alpha, \beta \neq 0.$$

Decompose $X = U^*U$ with U of full row rank $n > 0$ and abbreviate $L = (A^*U^* \; B^*V^*)^*$.

Case 1. If denoting the elements of S^{-1} by t_{11}, t_{12}, t_{21}, t_{22} we infer with $F := t_{11}UA + t_{12}UB$, $G = t_{21}UA + t_{21}UB$ for all $M = M^*$ that

$$L^*(P \otimes M)L = \pm(F^*MG + G^*MF), \quad (7)$$

$$L^*(\hat{P} \otimes M)L = \pm(\alpha F^*MG + \alpha^*G^*MF + \beta G^*MG). \quad (8)$$

Let us first concentrate on (7). For $M = I$ we infer from (6) that $F^*G + G^*F = 0$. Choose a unitary V of dimension n such that

$$V^*G = \begin{pmatrix} G_1 \\ 0 \end{pmatrix} \quad \text{with } G_1 \text{ of full row rank } m$$

and partition the rows of V^*F accordingly into the blocks F_1, F_2. We observe $0 = F^*G + G^*F = F_1^*G_1 + G_1^*F_1$. Since $\ker(G_1) \subset \ker(F_1)$ there exists a square T of size m with $F_1 = TG_1$. From $0 = F_1^*G_1 + G_1^*F_1 = G_1^*T^*G_1 + G_1^*TG_1$ we conclude $T + T^* = 0$ since G_1 has full row rank. Choose a set of orthonormal eigenvectors w_j with eigenvalue λ_j of T where $\lambda_j^* + \lambda_j = 0$. With the standard unit vectors e_j (of suitable length) define the orthonormal set of vectors

$$x_j = V\begin{pmatrix} w_j \\ 0 \end{pmatrix}, \quad x_j = V\begin{pmatrix} 0 \\ e_{j-m+1} \end{pmatrix}$$

for $j = 1, \ldots, m$ and $j = m+1, \ldots, n$ respectively. If $j \leq m$ then $G^*x_j = G_1^*w_j$ and $F^*x_j = F_1^*w_j = G_1^*T^*w_j = \lambda_j G_1^*w_j$ and hence $F^*x_jx_j^*G + G^*x_jx_j^*F = (\lambda_j + \lambda_j^*)(G_1^*w_jw_j^*G_1) = 0$. For $j > m$ we have $x_j^*G = 0$ and thus the same is true. With (7) for $M = x_jx_j^*$ we conclude

$$(x_j^*L)^*P(x_j^*L) = 0, \quad j = 1, \ldots, n. \quad (9)$$

Let us now turn to (8). For $M = I$ we infer from (6) that $\alpha F^*G + \alpha^*G^*F + \beta G^*G \geq 0$ hence $\alpha F_1^*G_1 + \alpha^*G_1^*F_1 + \beta G_1^*G_1 \geq 0$ hence $G_1^*(\alpha T^* + \alpha^*T + \beta I)G_1 \geq 0$ hence $\alpha T^* + \alpha^*T + \beta I \geq 0$ hence $\alpha\lambda_j^* + \alpha^*\lambda_j + \beta \geq 0$ hence $\alpha F^*x_jx_j^*G + \alpha^*G^*x_jx_j^*F + \beta G^*x_jx_j^*G = (\alpha\lambda_j + \alpha^*\lambda_j^* + \beta)(G_1^*w_jw_j^*G_1) \geq 0$ for all $j = 1, \ldots, n$. Again with (8) for $M = x_jx_j^*$ we conclude

$$(x_j^*L)^*\hat{P}(x_j^*L) \geq 0, \quad j = 1, \ldots, n. \quad (10)$$

Finally, if $\max_{j=1,\ldots,n} x_j^*UCU^*x_j = x_{j_0}^*UCU^*x_{j_0}$, then $x = \sqrt{n}U^*x_{j_0}$ does the desired job since

$$\langle C, X\rangle = \mathrm{Tr}(U^*CU) = \sum_{j=1}^{n} x_j^*UCU^*x_j \leq x^*Cx.$$

Case 2. As in Case 1 we get for all $M = M^*$ that

$$L^*(P \otimes M)L = G^*MG - F^*MF, \quad (11)$$

$$L^*(\hat{P} \otimes M)L = \alpha F^*MF + \beta G^*MG. \quad (12)$$

With (11) for $M = I$ we infer $G^*G - F^*F = 0$ due to (6). Choose a unitary V as in Case 1 and observe $0 = G^*G - F^*F = G_1^*G_1 - F^*F$. Since $\ker(G_1) \subset \ker(F)$ there exists some T_1 with $F = T_1G_1$, and $0 = G_1^*(I - T_1^*T_1)G_1$ implies $T_1^*T_1 = I$. Extend to a unitary $(T_1 \; T_2)$ and define $T = (T_1 \; T_2)V^*$ to conclude $F = TG$. With a set of orthonormal eigenvectors x_j of T^* with eigenvalues λ_j we infer $F^*x_jx_j^*F = G^*T^*x_jx_j^*TG = G^*x_jx_j^*G$ (since $|\lambda_j| = 1$) and thus, with (11) for $M = x_jx_j^*$, we arrive at (9). Let us finally show (10). For $M = I$ in (12) we infer $\alpha F^*F + \beta G^*G \geq 0$ or $(\beta - \alpha)G^*G \geq 0$. If $G^*G = 0$ then $F^*F = 0$ and hence $G = 0$, $F = 0$. If $G^*G \neq 0$ then $\beta \geq \alpha$ and thus $\alpha F^*x_jx_j^*F + \beta G^*x_jx_j^*G = (\beta - \alpha)G^*x_jx_j^*G \geq 0$. In both cases (10) follows from (12) for $M = x_jx_j^*$, and the proof is finished as in Case 1. ■