Introduction to Basic Concepts

- Test properness and stability of inverse
- Fractional transformations

Test Properness and Stability of Inverse

Suppose $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is stabilizable/detectable realization.

G has a proper inverse iff

$$G(\infty) = D \text{ is non-singular.}$$

If D is invertible, G^{-1} has a realization

$$G^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right].$$

Let G be **stable**. Then it has a proper and stable inverse if and only if

$$\det(G(\lambda)) \neq 0 \text{ for all } \lambda \in \mathbb{C}^0 \cup \mathbb{C}^+ \cup \{\infty\}$$

if and only if

$$D \text{ is invertible and } A - BD^{-1}C \text{ is stable.}$$

Fractional Transformations

Lower linear fractional trafo $S(P,K)$ of P and K:

Partition

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \text{ such that } P_{22}K \text{ is square,}$$

check whether $I - P_{22}K$ has an inverse, and set

$$S(P,K) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}.$$

Upper linear fractional trafo $S(\Delta,P)$ of Δ and P:

Partition

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \text{ such that } P_{11}\Delta \text{ is square,}$$

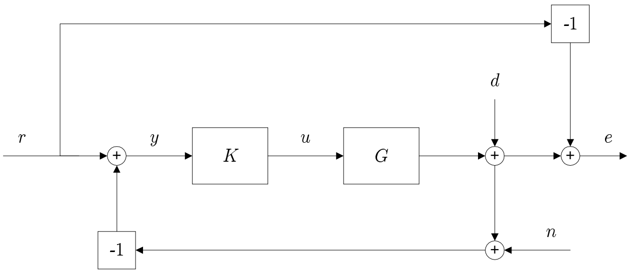
check whether $I - P_{11}\Delta$ has an inverse, and set

$$S(\Delta,P) := P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12}.$$

Stabilizing Controllers

- General framework to handle interconnections
- Definition of stabilizing controllers
- Tests of whether a controller is stabilizing
- Definition of generalized plants
- Tests of whether an interconnection is generalized plant

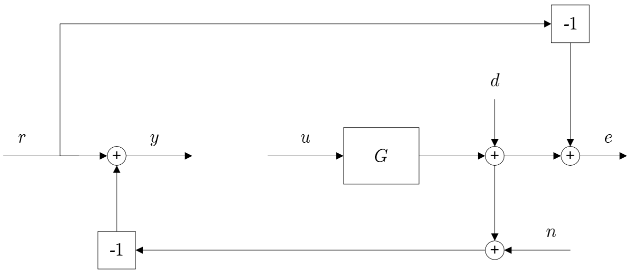
A Tracking Configuration



Have explicitly specified signals of interest:

- Signals that affect the interconnection and cannot be influenced by the controller: r, d, n
- Signals that characterize whether the controller achieves the desired goal: e should be kept as small as possible for all inputs r, d, n in a certain class
- Signals via which the plant can be controlled: u
- Signals that are available for control: y

Corresponding Open-Loop Interconnection



Straightforward derivation of algebraic relations:

$$\begin{pmatrix} e \\ \hline y \end{pmatrix} = \left(\begin{array}{ccc|c} I & 0 & -I & G \\ \hline -I & -I & I & -G \end{array}\right) \begin{pmatrix} d \\ n \\ r \\ \hline u \end{pmatrix}.$$

Obtain input-output description of closed-loop interconnection with $u = Ky$.

General Framework

In an arbitrary interconnection, let

- w denote the signal that affects the system and cannot be influence by the controller.
  w is called **generalized disturbance**.
- z denote the signal that allows to characterize whether a controller has certain desired properties.
  z is called **controlled variable**.
- u denote the output signal of the controller.
  u is called **control input**.
- y denote the signal that enters the controller.
  y is called **measurement output**.

In example:

$$w = \begin{pmatrix} d \\ n \\ r \end{pmatrix}, \quad z = e.$$

General Framework

After disconnecting the controller, any open-loop interconnection is described as

$$\begin{pmatrix} z \\ y \end{pmatrix} = P \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix}.$$

The closed-loop interconnection is obtained by reconnecting the controller as

$$u = Ky.$$

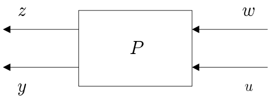
General Formula for closed-loop interconnection:

$$z = [P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}]w = S(P, K)w.$$

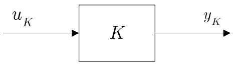
Assumption

Throughout the course, both P and K are LTI systems.

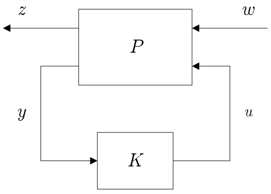
General Framework in Pictures



General open-loop interconnection



General controller



General closed-loop interconnection

State-Space Descriptions

Open-loop interconnection P:

$$\begin{aligned}\dot x &= Ax + B_1 w + B_2 u\\ z &= C_1 x + D_{11} w + D_{12} u\\ y &= C_2 x + D_{21} w + D_{22} u\end{aligned}$$

Controller K:

$$\begin{aligned}\dot x_K &= A_K x_K + B_K u_K\\ y_K &= C_K x_K + D_K u_K\end{aligned}$$

Assumption: Both stabilizable and detectable.

Closed-loop interconnection with $u = y_K$ and $u_K = y$:

$$\begin{aligned}\dot\xi &= \mathcal{A}\xi + \mathcal{B}w\\ z &= \mathcal{C}\xi + \mathcal{D}w\end{aligned}$$

Realization of Closed-Loop Interconnection?

Merge descriptions of P and K as

$$\left(\begin{array}{c}\dot x\\ \dot x_K\\ \hline z\\ \hline y_K\\ y\end{array}\right) = \underbrace{\left(\begin{array}{cc|c|cc} A & 0 & B_1 & B_2 & 0\\ 0 & A_K & 0 & 0 & B_K\\ \hline C_1 & 0 & D_{11} & D_{12} & 0\\ \hline 0 & C_K & 0 & 0 & D_K\\ C_2 & 0 & D_{21} & D_{22} & 0\end{array}\right)}_{\begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}_1 & \boldsymbol{B}_2\\ \boldsymbol{C}_1 & \boldsymbol{D}_{11} & \boldsymbol{D}_{12}\\ \boldsymbol{C}_2 & \boldsymbol{D}_{21} & \boldsymbol{D}_{22}\end{pmatrix}} \left(\begin{array}{c}x\\ x_K\\ \hline w\\ \hline u\\ u_K\end{array}\right)$$

Formula for realization of closed-loop system:

$$\begin{pmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{pmatrix} = \begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}_1\\ \boldsymbol{C}_1 & \boldsymbol{D}_{11}\end{pmatrix} + \begin{pmatrix}\boldsymbol{B}_2\\ \boldsymbol{D}_{12}\end{pmatrix}[I - \boldsymbol{D}_{22}]^{-1}\begin{pmatrix}\boldsymbol{C}_2 & \boldsymbol{D}_{21}\end{pmatrix}.$$

We have

$$\begin{pmatrix}y_K\\ y\end{pmatrix} = \left(\begin{array}{c|c}\boldsymbol{C}_2 & \boldsymbol{D}_{21}\end{array}\right)\left(\begin{array}{c}x\\ x_K\\ \hline w\end{array}\right) + \boldsymbol{D}_{22}\begin{pmatrix}u\\ u_K\end{pmatrix}.$$

With $\begin{pmatrix}u\\ u_K\end{pmatrix} = \begin{pmatrix}y_K\\ y\end{pmatrix}$ obtain

$$[I - \boldsymbol{D}_{22}]\begin{pmatrix}y_K\\ y\end{pmatrix} = \left(\begin{array}{c|c}\boldsymbol{C}_2 & \boldsymbol{D}_{21}\end{array}\right)\left(\begin{array}{c}x\\ x_K\\ \hline w\end{array}\right).$$

If $I - \boldsymbol{D}_{22}$ is non-singular, obtain

$$\begin{pmatrix}y_K\\ y\end{pmatrix} = [I - \boldsymbol{D}_{22}]^{-1}\left(\begin{array}{c|c}\boldsymbol{C}_2 & \boldsymbol{D}_{21}\end{array}\right)\left(\begin{array}{c}x\\ x_K\\ \hline w\end{array}\right).$$

End up with

$$\left(\begin{array}{c}\dot x\\ \dot x_K\\ \hline z\end{array}\right) = \left[\left(\begin{array}{c|c}\boldsymbol{A} & \boldsymbol{B}_1\\ \hline \boldsymbol{C}_1 & \boldsymbol{D}_{11}\end{array}\right) + \begin{pmatrix}\boldsymbol{B}_2\\ \boldsymbol{D}_{12}\end{pmatrix}[I - \boldsymbol{D}_{22}]^{-1}\left(\begin{array}{c|c}\boldsymbol{C}_2 & \boldsymbol{D}_{21}\end{array}\right)\right]\left(\begin{array}{c}x\\ x_K\\ \hline w\end{array}\right).$$

Fundamental Definition

$K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ **stabilizes** $P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$ if the matrix

$$\begin{pmatrix} I & -D_K \\ -D_{22} & I \end{pmatrix}$$

is **non-singular** and

$$\begin{pmatrix} A & 0 \\ 0 & A_K \end{pmatrix} + \begin{pmatrix} B_2 & 0 \\ 0 & B_K \end{pmatrix} \begin{pmatrix} I & -D_K \\ -D_{22} & I \end{pmatrix}^{-1} \begin{pmatrix} 0 & C_K \\ C_2 & 0 \end{pmatrix}$$

is **stable**.

Remarks.

- Recall: Realizations must be stabilizable/detectable.
- Same definition if w and z are absent.
- Much simpler formulas if $D_K = 0$ or $D_{22} = 0$.

Example

In tracking configuration, take the plant model

$$G(s) = \frac{200}{10s+1}\,\frac{1}{(0.05s+1)^2}$$

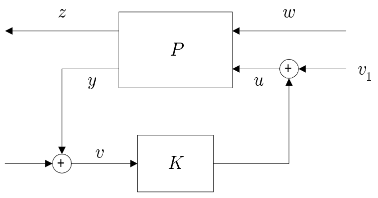
and the controller

$$K(s) = \frac{0.1s+1}{(0.65s+1)(0.03s+1)}.$$

Test whether K stabilizes P in μ-tools:

```
G=nd2sys( [200],conv([10 1],conv([0.05 1],[0.05 1])) );
K=nd2sys( [0.1 1], conv([0.65 1],[0.03 1]) );
systemnames='G';
inputvar='[d;n;r;u]';
outputvar='[G+d-r;r-n-d-G]';
input_to_G='[u]';
sysoutname='P';
cleanupsysic='yes';
sysic
S=starp(P,K);
[A,B,C,D]=unpck(S);
eig(A)
```

Input-Output Description



Theorem
K stabilizes P if and only if the interconnection defined through the relations

$$\begin{pmatrix} z \\ y \end{pmatrix} = P \begin{pmatrix} w \\ u \end{pmatrix},\ u = Kv + v_1,\ v = y + v_2$$

defines a **proper and stable transfer matrix**

$$\begin{pmatrix} w \\ v_1 \\ v_2 \end{pmatrix} \to \begin{pmatrix} z \\ u \\ v \end{pmatrix}.$$

More Explicitly

Merge description to

$$\begin{pmatrix} z \\ v_1 \\ v_2 \end{pmatrix} = \left(\begin{array}{c|cc} P_{11} & P_{12} & 0 \\ \hline 0 & I & -K \\ -P_{21} & -P_{22} & I \end{array}\right) \begin{pmatrix} w \\ u \\ v \end{pmatrix}.$$

System $\begin{pmatrix} w \\ v_1 \\ v_2 \end{pmatrix} \to \begin{pmatrix} z \\ u \\ v \end{pmatrix}$ proper if and only if

$\begin{pmatrix} I & -K \\ -P_{22} & I \end{pmatrix}$ or $I - P_{22}K$ have proper inverse.

Then $\begin{pmatrix} w \\ v_1 \\ v_2 \end{pmatrix} \to \begin{pmatrix} z \\ u \\ v \end{pmatrix}$ is described by transfer matrix

$$\left(\begin{array}{c|cc} P_{11} + P_{12}K(I-P_{22}K)^{-1}P_{21} & P_{12}(I-KP_{22})^{-1} & P_{12}K(I-P_{22}K)^{-1} \\ \hline K(I-P_{22}K)^{-1}P_{21} & (I-KP_{22})^{-1} & K(I-P_{22}K)^{-1} \\ (I-P_{22}K)^{-1}P_{21} & (I-P_{22}K)^{-1}P_{22} & (I-P_{22}K)^{-1} \end{array}\right).$$

Have to check stability of all nine blocks.

Input-Output Description

Explicit Input-Output Stability Test
K stabilizes P if and only if

$$I - P_{22}K \text{ has a proper inverse}$$

and the following transfer matrix is stable:

$$\left(\begin{array}{c|cc} P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} & P_{12}(I - KP_{22})^{-1} & P_{12}K(I - P_{22}K)^{-1} \\ \hline K(I - P_{22}K)^{-1}P_{21} & (I - KP_{22})^{-1} & K(I - P_{22}K)^{-1} \\ (I - P_{22}K)^{-1}P_{21} & (I - P_{22}K)^{-1}P_{22} & (I - P_{22}K)^{-1} \end{array}\right).$$

Remarks.

- In general, one really has to check all nine blocks.
- If some of the blocks of P or K are stable, might suffice to check only a subset. **Useful**: Formulas for rearranging block matrices.
- w, z absent: K stabilizes P_{22} if and only if

$$\begin{pmatrix} I & -K \\ -P_{22} & I \end{pmatrix} \text{ has proper and stable inverse.}$$

Why is Input-Output Characterization Valid?

$$\begin{pmatrix} z \\ v_1 \\ v_2 \end{pmatrix} = \left(\begin{array}{c|cc} P_{11} & P_{12} & 0 \\ \hline 0 & I & -K \\ -P_{21} & -P_{22} & I \end{array}\right) \begin{pmatrix} w \\ u \\ v \end{pmatrix}$$

admits **stabilizable** and **detectable** realization:

$$\begin{pmatrix} \dot{x} \\ \dot{x}_K \\ \hline z \\ \hline v_1 \\ v_2 \end{pmatrix} = \left(\begin{array}{c|c|c} \boldsymbol{A} & \boldsymbol{B}_1 & \boldsymbol{B}_2 \\ \hline \boldsymbol{C}_1 & \boldsymbol{D}_{11} & \boldsymbol{D}_{12} \\ \hline -\boldsymbol{C}_2 & -\boldsymbol{D}_{21} & I - \boldsymbol{D}_{22} \end{array}\right) \begin{pmatrix} x \\ x_K \\ \hline w \\ \hline u \\ v \end{pmatrix}.$$

$$\begin{pmatrix} w \\ v_1 \\ v_2 \end{pmatrix} \rightarrow \begin{pmatrix} z \\ u \\ v \end{pmatrix}$$ is LTI if and only if $I - \boldsymbol{D}_{22}$ non-singular.

Has **stabilizable** and **detectable** realization defined with

$$\left(\begin{array}{c|c|c} \boldsymbol{A} + \boldsymbol{B}_2(I - \boldsymbol{D}_{22})^{-1}\boldsymbol{C}_2 & \boldsymbol{B}_1 + \boldsymbol{B}_2(I - \boldsymbol{D}_{22})^{-1}\boldsymbol{D}_{21} & \boldsymbol{B}_2(I - \boldsymbol{D}_{22})^{-1} \\ \hline \boldsymbol{C}_1 + \boldsymbol{D}_{12}(I - \boldsymbol{D}_{22})^{-1}\boldsymbol{C}_2 & \boldsymbol{D}_1 + \boldsymbol{D}_{12}(I - \boldsymbol{D}_{22})^{-1}\boldsymbol{D}_{21} & \boldsymbol{D}_{12}(I - \boldsymbol{D}_{22})^{-1} \\ \hline (I - \boldsymbol{D}_{22})^{-1}\boldsymbol{C}_2 & (I - \boldsymbol{D}_{22})^{-1}\boldsymbol{D}_{21} & (I - \boldsymbol{D}_{22})^{-1} \end{array}\right).$$

Transfer matrix stable iff $\boldsymbol{A} + \boldsymbol{B}_2(I - \boldsymbol{D}_{22})^{-1}\boldsymbol{C}_2$ stable.

Example

The system

$$P(s) = \begin{pmatrix} 1 & 1/s \\ 1 & 1/(s+1) \end{pmatrix}$$

does not admit a stabilizing controller.

Why? Either one of the functions

$$P_{12}(s)(I - K(s)P_{22}(s))^{-1} = \frac{1}{s}\frac{1}{1 - \frac{K(s)}{s+1}}$$

or

$$P_{12}(s)(I - K(s)P_{22}(s))^{-1}K(s) = \frac{1}{s}\frac{1}{\frac{1}{K(s)} - \frac{1}{s+1}}$$

is unstable.

Generalized Plant

We distinguish those P that admit a stabilizing controller.

Definition
If there exists at least one controller $u = Ky$ that stabilizes $\begin{pmatrix} z \\ y \end{pmatrix} = P\begin{pmatrix} w \\ u \end{pmatrix}$, we call P a **generalized plant**.

Violation indicates that interconnection contains unstable components that cannot be stabilized.

Requires to change components, interconnection, actuator or sensor location, ...

Must always be verified. Is there a simple test?

Generalized Plant: State-Space Test

Theorem
The system with stabilizable/detectable realization

$$\begin{pmatrix} \dot{x} \\ \hline z \\ y \end{pmatrix} = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right) \begin{pmatrix} x \\ \hline w \\ u \end{pmatrix}$$

is a generalized plant if and only if

$$(A, B_2) \text{ is stabilizable}, \quad (A, C_2) \text{ is detectable}.$$

Nontrivial condition due to presence of channel $w \to z$.

If valid, design controller on the basis of F, L that render $A + B_2F$ and $A + LC_2$ stable.

Always true if w and z are absent. System then looks as

$$\begin{pmatrix} \dot{x} \\ \hline y \end{pmatrix} = \left(\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right) \begin{pmatrix} x \\ \hline u \end{pmatrix}.$$

Construct Stabilizing Controller

Take F, L such that $A + B_2F$, $A + LC_2$ stable. Define

$$\begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix} = \begin{pmatrix} A + B_2F + LC_2 + LD_{22}F & -L \\ F & 0 \end{pmatrix}.$$

Why does this controller stabilize?

Since D_K vanishes, $\begin{pmatrix} I & -D_K \\ -D_{22} & I \end{pmatrix}$ is non-singular.

The matrix

$$\begin{pmatrix} A & 0 \\ 0 & A_K \end{pmatrix} + \begin{pmatrix} B_2 & 0 \\ 0 & B_K \end{pmatrix} \begin{pmatrix} I & -D_K \\ -D_{22} & I \end{pmatrix}^{-1} \begin{pmatrix} 0 & C_K \\ C_2 & 0 \end{pmatrix}$$

equals

$$\begin{pmatrix} A & B_2F \\ -LC_2 & A + B_2F + LC_2 \end{pmatrix}.$$

This is stable! (Look at error dynamics.)

Generalized Plant: Input-Output Test

As a consequence of our discussion, there always exists a controller $u = Ky$ that stabilizes $y = P_{22}u$.

One can find a minimal realization of P_{22} and construct a state-space controller.

Theorem
Let $u = Ky$ be any controller that stabilizes $y = P_{22}u$. Then P is a generalized plant if and only if this controller K stabilizes the open-loop interconnection P.

Easy to check.

Suppose P is a generalized plant. Then

- Any controller that stabilizes P_{22} also stabilizes P.
- To check whether K is stabilizing, it suffices to verify only **four** instead of **nine** transfer matrices.

Summary

Extract open-loop interconnection described with P.

Test whether P is generalized plant:

- Find a **stabilizable/detectable** state-space realization of P and check whether (A, B_2) is stabilizable and (A, C_2) is detectable.
- Find any K that stabilizes P_{22}, and verify whether this K stabilizes P.

Only proceed if P is a generalized plant.

Closed-loop system given by

$$z = S(P, K) = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] w$$

with formulas for $S(P, K)$ and for $\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$.

Back to Tracking Example

Open-loop interconnection

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \left(\begin{array}{ccc|c} I & 0 & -I & G \\ \hline -I & -I & I & -G \end{array}\right)$$

is generalized plant.

State-space test:

Let $G(s) = C_G(sI - A_G)^{-1}B_G + D_G$ be minimal realization.

Then $P(s)$ equals

$$\left(\begin{array}{c} C_G \\ \hline -C_G \end{array}\right)(sI - A_G)^{-1}\left(\begin{array}{ccc|c} 0 & 0 & 0 & B_G \end{array}\right) + \left(\begin{array}{ccc|c} I & 0 & -I & D_G \\ \hline -I & -I & I & -D_G \end{array}\right).$$

Hence P admits minimal realization

$$\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{c|cccc} A_G & 0 & 0 & 0 & B_G \\ \hline C_G & I & 0 & -I & D_G \\ -C_G & -I & -I & I & -D_G \end{array}\right].$$

(A, B_2) is stabilizable, (A, C_2) is detectable. Finished!

Back to Tracking Example

Input-output test

Let K stabilize $P_{22} = -G$. Hence

$$\begin{pmatrix} (I - KP_{22})^{-1} & K(I - P_{22}K)^{-1} \\ (I - P_{22}K)^{-1}P_{22} & (I - P_{22}K)^{-1} \end{pmatrix} =$$

$$= \begin{pmatrix} (I + KG)^{-1} & K(I + GK)^{-1} \\ -(I + GK)^{-1}G & (I + GK)^{-1} \end{pmatrix}$$

is well-defined and stable.

Look at nine blocks. Note that P_{11}, P_{21} are stable.

'Critical' blocks

$$P_{12}(I - KP_{22})^{-1} = G(I + KG)^{-1} = (I + GK)^{-1}G$$

$$P_{12}K(I - KP_{22})^{-1} = KG(I + KG)^{-1} = I - (I + KG)^{-1}$$

$$P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

are seen to be stable. Finished!

3 Robust Stability

- A SISO example to demonstrate main ideas
- General framework for uncertain systems
- Type/Structure/Size of uncertainties
- Pulling out uncertainties
- Determinant test for robust stability
- Small-gain theorem for full block uncertainties
- Small-SSV theorem for structured uncertainties
- Digression: The Structured Singular Value (SSV)

Robust Stability - A SISO Example

Choose $G(s)$ and $K(s)$ in tracking configuration as earlier.

Have seen that K stabilizes P formed with G.

Suppose the model G deviates from the actual plant H.

Suppose we can **bound the deviation** as

$$|G(i\omega) - H(i\omega)| < 1 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}.$$

Fundamental robust stability question:

Does K stabilize the interconnection if replacing G by H?

Introduce Uncertainty

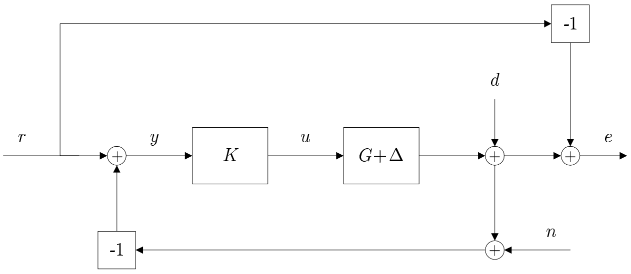
Introduce plant-model mismatch $\Delta(s) := H(s) - G(s)$.

Actual plant given as

$$H(s) = G(s) + \Delta(s)$$

with some proper stable $\Delta(s)$ satisfying

$$|\Delta(i\omega)| < 1 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}.$$



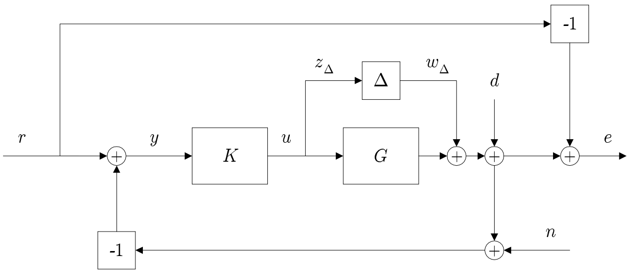
Fundamental robust stability question:

Does K stabilize the interconnection in Figure for all Δ?

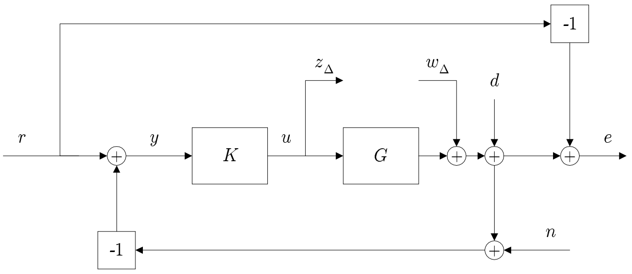
Is there a Δ for which K does not stabilize interconnection?

Transfer Function Seen by Uncertainty

Rewrite interconnection:



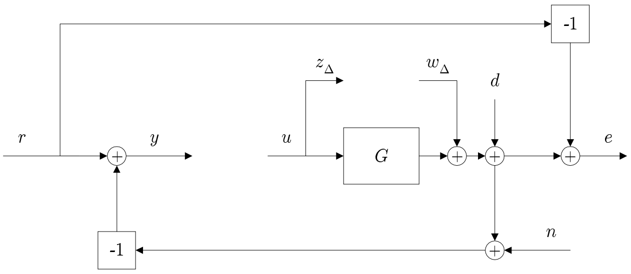
Disconnect uncertainty:



Transfer function seen by Δ: $w_\Delta \to z_\Delta$.

How to Compute?

Disconnect controller and uncertainty:



Collect $w_p = \begin{pmatrix} d \\ n \\ r \end{pmatrix}$, $z_p = e$.

Leads to interconnection description

$$\begin{pmatrix} z_\Delta \\ z_p \\ y \end{pmatrix} = P \begin{pmatrix} w_\Delta \\ w_p \\ u \end{pmatrix}.$$

Performance channel $w_p \to z_p$.

Uncertainty channel $w_\Delta \to z_\Delta$.

General Framework

Open-loop interconnection described as

$$\begin{pmatrix} z_\Delta \\ z_p \\ y \end{pmatrix} = P \begin{pmatrix} w_\Delta \\ w_p \\ u \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \begin{pmatrix} w_\Delta \\ w_p \\ u \end{pmatrix}.$$

Reconnect controller as $u = Ky$ to obtain closed-loop interconnection described as

$$\begin{pmatrix} z_\Delta \\ z_p \end{pmatrix} = S(P, K) \begin{pmatrix} w_\Delta \\ w_p \end{pmatrix}.$$

Reconnect uncertainty as $w_\Delta = \Delta z_\Delta$ to obtain perturbed closed-loop interconnection described as

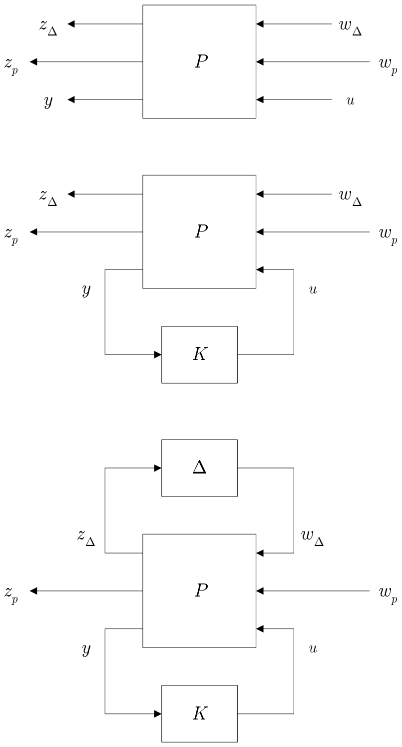
$$z_p = S(\Delta, S(P, K)) w_p = S(S(\Delta, P), K) w_p.$$

For $S(P, K)$, use abbreviation

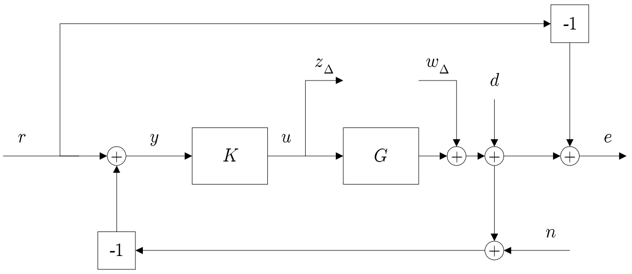
$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} = \begin{pmatrix} M & N_{12} \\ N_{21} & N_{22} \end{pmatrix}.$$

M is transfer matrix seen by Δ.

General Framework in Pictures



Back to Example



Closed-loop interconnection described as

$$\begin{pmatrix} z_\Delta \\ z_p \end{pmatrix} = \begin{pmatrix} M & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} w_\Delta \\ w_p \end{pmatrix}.$$

Reconnecting $w_\Delta = \Delta z_\Delta$ leads to

$$z_p = [N_{22} + N_{21}\Delta(I - M\Delta)^{-1}N_{12}]w_p.$$

Only source for instability: Inverse $(I - M\Delta)^{-1}$!

Transfer Function Seen by Uncertainty

Need to verify whether

$$I - M\Delta \text{ has proper and stable inverse for all } \Delta.$$

Both M and Δ are stable.

Hence $(I - M\Delta)^{-1}$ is stable if Nyquist curve

$$\omega \to M(i\omega)\Delta(i\omega) \text{ does not encircle } 1.$$

True if

$$|M(i\omega)\Delta(i\omega)| < 1 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}.$$

Since $|\Delta(i\omega)| < 1$, this holds if

$$\boxed{|M(i\omega)| \leq 1 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}.}$$

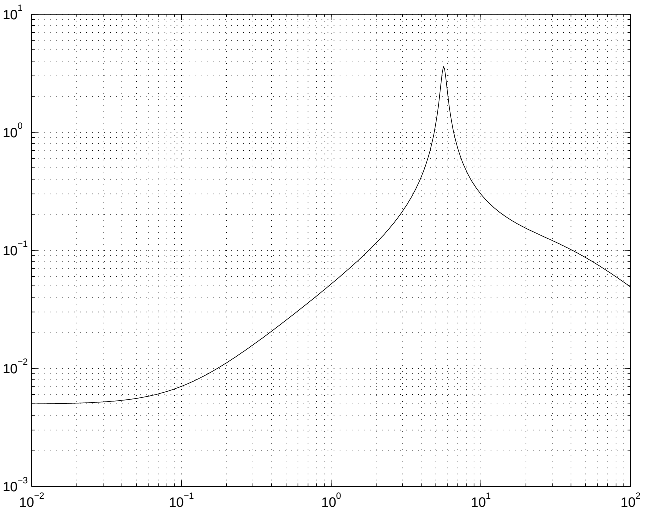
If true, system cannot be destabilized with proper stable uncertainty Δ satisfying $|\Delta(i\omega)| < 1$ for all $\omega \in \mathbb{R} \cup \{\infty\}$.

Back to Specific Example

Compute M and plot $M(i\omega)$ over frequency with code

```
G=nd2sys( [200],conv([10 1],conv([0.05 1],[0.05 1])) );
K=nd2sys( [0.1 1], conv([0.65 1],[0.03 1]) );
systemnames=’G’;
inputvar=’[w;d;n;r;u]’;
outputvar=’[u;w+G+d-r;w+r-n-d-G]’;
input_to_G=’[u]’;
sysoutname=’P’;
cleanupsysic=’yes’;
sysic
N=starp(P,K);
[A,B,C,D]=unpck(N);
eig(A)
M=sel(N,1,1);
om=logspace(-2,4);
Mom=frsp(M,om);
vplot(’liv,lm’,Mom);
grid on
```

Graph of Frequency Response of M



Cannot guarantee robust stability for $|\Delta(i\omega)| < 1$.

Can guarantee robust stability for $|\Delta(i\omega)| < \frac{1}{4}$.

Largest possible bound given by

$$\left[\sup_{\omega\in\mathbb{R}\cup\{\infty\}} |M(i\omega)|\right]^{-1} = \|M\|_\infty^{-1}.$$

Destabilizing Perturbations?

Expected to happen for Δ such that $(I - M\Delta)^{-1}$ unstable.

Find Δ and $i\omega_0$ such that $\det(I - M(i\omega_0)\Delta(i\omega_0)) = 0$.

If M is SISO: Amounts to $1 = M(i\omega_0)\Delta(i\omega_0)$.

First Step

Choose $i\omega_0$ with $|M(i\omega_0)| > 1$.

Complex number $\Delta_0 := \frac{1}{M(i\omega_0)}$ satisfies

$$1 = M(i\omega_0)\Delta_0, \quad |\Delta_0| < 1.$$

Second Step

Determine **real-rational proper and stable** $\Delta(s)$ with

$$\Delta(i\omega_0) = \Delta_0, \quad |\Delta(i\omega)| < 1 \ \text{ for all } \ \omega \in \mathbb{R}\cup\{\infty\}.$$

Conclusion

- Δ is in desired class: $|\Delta(i\omega)| < 1$ for all $\omega \in \mathbb{R}\cup\{\infty\}$.
- $(I - M\Delta)^{-1}$ unstable: $1 = M(i\omega_0)\Delta(i\omega_0)$.

A little Interpolation Lemma

Lemma

Let $\omega_0 \geq 0$ and $\Delta_0 \in \mathbb{C}$. Set

$$\alpha = \pm|\Delta_0|, \quad \beta = i\omega_0\,\frac{\alpha - \Delta_0}{\alpha + \Delta_0}.$$

Then

$$\Delta(s) = \alpha\frac{s-\beta}{s+\beta}$$

is proper, real-rational, and satisfies

$$\Delta(i\omega_0) = \Delta_0, \quad |\Delta(i\omega)| = |\Delta_0| \ \text{ for all } \ \omega \in \mathbb{R}\cup\{\infty\}.$$

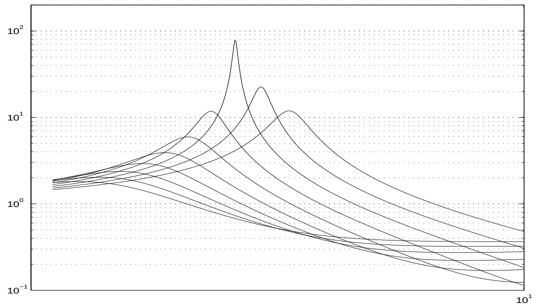
Either for $\alpha = |\Delta_0|$ or for $\alpha = -|\Delta_0|$, Δ is stable.

Example: Matlab Code

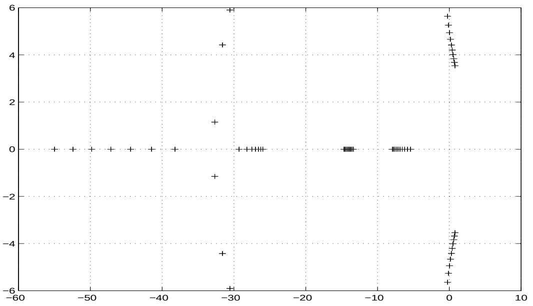
```
Delta0=vunpck(minv(frsp(M,5)));
alpha=abs(Delta0);
beta=real(5*i*(alpha-Delta0)/(alpha+Delta0));
Delta=nd2sys([alpha -alpha*beta],[1 beta]);

spoles(Delta)

m=logspace(0.5,1,500);clf;

for tau=linspace(0,5,10);

D=mmult(tau,Delta);
PDelta=starp(D,P);
cll=starp(PDelta,K);

figure(1);plot(spoles(cll),'+');hold on;
figure(2);vplot('liv,lm',frsp(sel(cll,1,2),om));hold on;

end;
```

Example: Results

Frequency response of $r \rightarrow e$:



Poles of controlled interconnection:



Summary of SISO Example

The SISO example demonstrated ...

- ... how to get to general framework
- ... that the transfer matrix seen by uncertainty determines robust stability
- ... that robust stability can be guaranteed by a simple version of small-gain theorem
- ... how to construct destabilizing perturbations to verify that robust stability test is exact.

Uncertainties?

Typical **types** of uncertainties in system components.

How does **structure** appear?

How to pull uncertainties out of system components?

How to pull uncertainties out of interconnections?

Will arrive at general framework.

Size and structure of uncertainties captured by specifying possible values of frequency response.

Values = Block-diagonal complex matrices bounded by 1.

Parametric Uncertainty

Second order system with parametric uncertainty:

$$\hat{z}(s) = \frac{1}{s^2 + cs + 1}\hat{w}(s), \quad c_1 < c < c_2.$$

Set of all parameters described as

$$c = c_0 + W\delta \ \text{ with } \ -1 < \delta < 1$$

with nominal value $c_0 := \frac{c_1+c_2}{2}$ and weight $W = \frac{c_2-c_1}{2}$.

Uncertain system described as

$$G_\Delta(s) = \frac{1}{s^2 + (c_0 + W\Delta)s + 1}, \quad \Delta \in \mathbf{\Delta}$$

with uncertainty class

$$\mathbf{\Delta} := \{\delta \in \mathbb{R} \mid -1 < \delta < 1\}.$$

Pull out Uncertainty

Input-output description:

$$G_\Delta(s) = \frac{1}{s^2+(c_0+W\Delta)s+1}$$

rewritten to

$$G_\Delta(s) = S\left(\Delta, \underbrace{\frac{1}{s^2+c_0 s+1}\begin{pmatrix}-Ws & s\\ -W & 1\end{pmatrix}}_{G}\right).$$

State-space description:

$$\dot{x} = \begin{pmatrix}0 & 1\\ -1 & -(c_0+W\Delta)\end{pmatrix}x + \begin{pmatrix}0\\ 1\end{pmatrix}w,\quad z = \begin{pmatrix}1 & 0\end{pmatrix}x$$

rewritten to

$$\begin{pmatrix}\dot{x}\\ \hline z_\Delta\\ z\end{pmatrix} = \left(\begin{array}{cc|cc}0 & 1 & 0 & 0\\ -1 & -c_0 & -W & 1\\ \hline 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{array}\right)\begin{pmatrix}x\\ \hline w_\Delta\\ w\end{pmatrix},\quad w_\Delta = \Delta z_\Delta.$$

Can rewrite uncertainty system as

$$G_\Delta = S(\Delta, G).$$

General Affine Parametric Uncertainty

System matrices depend affinely on parameters:

$$\begin{pmatrix}A(\delta) & B(\delta)\\ C(\delta) & D(\delta)\end{pmatrix} = \begin{pmatrix}A_0 & B_0\\ C_0 & D_0\end{pmatrix} + \sum_{j=1}^{k}\delta_j\begin{pmatrix}A_j & B_j\\ C_j & D_j\end{pmatrix} =$$

$$= \begin{pmatrix}A_0 & B_0\\ C_0 & D_0\end{pmatrix} + \sum_{j=1}^{k}\delta_j\begin{pmatrix}L_j^1\\ L_j^2\end{pmatrix}[\delta_j I]\begin{pmatrix}R_j^1 & R_j^2\end{pmatrix} =$$

$$= S\left(\begin{pmatrix}\delta_1 I & & 0\\ & \ddots & \\ 0 & & \delta_k I\end{pmatrix}, \left(\begin{array}{ccc|cc}0 & & 0 & R_1^1 & R_2^2\\ & \ddots & & \vdots & \vdots\\ 0 & & 0 & R_k^1 & R_k^2\\ \hline L_1^1 & \cdots & L_k^1 & A_0 & B_0\\ L_1^2 & \cdots & L_k^2 & C_0 & D_0\end{array}\right)\right)$$

with full rank factorization

$$\begin{pmatrix}A_j & B_j\\ C_j & D_j\end{pmatrix} = \begin{pmatrix}L_j^1\\ L_j^2\end{pmatrix}\begin{pmatrix}R_j^1 & R_j^2\end{pmatrix}.$$

Easily generalized to **rational dependence**.

Dynamic SISO Uncertainty

Frequency domain experiments: Set of responses $\mathcal{H}(\omega)$.

Cover $\mathcal{H}(\omega)$ by set with more appropriate description:

$$\mathcal{H}(\omega) \subset G(i\omega) + W(i\omega)\boldsymbol{\Delta}_c \text{ for all } \omega \in \mathbb{R}\cup\{\infty\}$$

where

- $G(s)$ is real-rational proper
- $\boldsymbol{\Delta}_c$ is open unit disk around 0:
  $$\boldsymbol{\Delta}_c := \{\Delta_c \in \mathbb{C} \mid |\Delta_c| < 1\}$$
- $W(s)$ is real-rational weighting.

Uncertain system described as

$$G_\Delta := S\left(\Delta, \begin{pmatrix}0 & W\\ I & G\end{pmatrix}\right),\quad \Delta \in \boldsymbol{\Delta}$$

with uncertainty class

$$\boldsymbol{\Delta} := \{\Delta \in RH_\infty \mid \Delta(i\omega) \in \boldsymbol{\Delta}_c \text{ for all } \omega \in \mathbb{R}\cup\{\infty\}\}$$

Observe: $\boldsymbol{\Delta}$ is just set of $\Delta \in RH_\infty$ with $\|\Delta\|_\infty < 1$!

Mixture

Can have both parametric and dynamic uncertainty together:

$$G_\Delta(s) = \begin{pmatrix}\frac{1}{s+1} & \frac{1+W_2(s)\Delta_2(s)}{s+2}\\ \frac{1+W_1\Delta_1}{s+3} & \frac{1}{2s+1}\end{pmatrix}$$

with real $|\Delta_1| < 1$, dynamic $\|\Delta_2\|_\infty < 1$.

With nominal model and weighting

$$G(s) = \begin{pmatrix}\frac{1}{s+1} & \frac{1}{s+2}\\ \frac{1}{s+3} & \frac{1}{2s+1}\end{pmatrix},\quad W(s) = \begin{pmatrix}0 & \frac{1}{s+2}W_2(s)\\ \frac{1}{s+3}W_1 & 0\end{pmatrix}$$

rewritten to

$$G_\Delta(s) = G(s) + W(s)\begin{pmatrix}\Delta_1 & 0\\ 0 & \Delta_2(s)\end{pmatrix}.$$

Have to live with block-diagonal structure of uncertainty!

Structure in Uncertainty

Set of uncertainties consists of all

$$\Delta = \begin{pmatrix}\Delta_1 & 0\\ 0 & \Delta_2\end{pmatrix} \in RH_\infty^{2\times 2}$$

with real $|\Delta_1| < 1$ and dynamic $\|\Delta_2\|_\infty < 1$.

Capture **structure and size** of uncertainty as follows:
Define **value set**

$$\boldsymbol{\Delta}_c := \left\{ \begin{pmatrix}\Delta_1 & 0\\ 0 & \Delta_2\end{pmatrix} \mid \Delta_1 \in \mathbb{R},\ \Delta_2 \in \mathbb{C},\ |\Delta_j| < 1 \right\}.$$

Actual class of uncertainties then given by

$$\boldsymbol{\Delta} := \{\Delta \in RH_\infty \mid \Delta(i\omega) \in \boldsymbol{\Delta}_c \text{ for all } \omega \in \mathbb{R}\cup\{\infty\}\}.$$

Uncertain system again described as

$$G_\Delta := S\left(\Delta, \begin{pmatrix}0 & I\\ W & G\end{pmatrix}\right),\quad \Delta \in \boldsymbol{\Delta}.$$

Unstructured Uncertainty

MIMO model accurate at low frequencies.

Accuracy of **all entries** decreases at high frequencies.

With SISO high-pass filter W, actual frequency response is

$$G(i\omega) + [W(i\omega)I]\Delta_c$$

with any **unstructured** complex matrix

$$\Delta_c = \begin{pmatrix}\Delta_{11} & \Delta_{12}\\ \Delta_{21} & \Delta_{22}\end{pmatrix}$$

that is bounded as

$$\|\Delta_c\| = \sigma_{\max}(\Delta_c) < 1.$$

Full block uncertainty or **unstructured uncertainty**.

Fits in general scenario with

$$\boldsymbol{\Delta}_c := \{\Delta_c \in \mathbb{C}^{2\times 2} \mid \|\Delta_c\| < 1\}.$$

Unstructured versus Structured

Individual weightings and bounds on elements:

$$G(i\omega) + \begin{pmatrix}W_{11}(i\omega)\Delta_{11} & W_{12}(i\omega)\Delta_{12}\\ W_{21}(i\omega)\Delta_{21} & W_{22}(i\omega)\Delta_{22}\end{pmatrix}$$

with $|\Delta_{11}| < 1$, $|\Delta_{12}| < 1$, $|\Delta_{21}| < 1$, $|\Delta_{22}| < 1$.

Deviation can be rewritten to

$$\begin{pmatrix}W_{11}(i\omega) & 0 & W_{12}(i\omega) & 0\\ 0 & W_{21}(i\omega) & 0 & W_{22}(i\omega)\end{pmatrix}\begin{pmatrix}\Delta_{11} & 0\\ \Delta_{21} & 0\\ 0 & \Delta_{12}\\ 0 & \Delta_{22}\end{pmatrix}$$

or to

$$\begin{pmatrix}1\ 0\ 1\ 0\\ 0\ 1\ 0\ 1\end{pmatrix}\begin{pmatrix}\Delta_{11} & 0 & 0 & 0\\ 0 & \Delta_{21} & 0 & 0\\ 0 & 0 & \Delta_{12} & 0\\ 0 & 0 & 0 & \Delta_{22}\end{pmatrix}\begin{pmatrix}W_{11}(i\omega) & 0\\ W_{21}(i\omega) & 0\\ 0 & W_{12}(i\omega)\\ 0 & W_{22}(i\omega)\end{pmatrix}.$$

Which should we take?

If we insist on $|\Delta_{jk}| < 1$, have to take second one!

Other Typical Structures how Uncertainties could Enter

Additive uncertainty:

$$G + W_1\Delta W_2 = S\left(\Delta, \begin{pmatrix}0 & W_2\\ W_1 & G\end{pmatrix}\right).$$

Input-output multiplicative uncertainty:

$$(I+\Delta_1)G(I+\Delta_2) = S\left(\begin{pmatrix}\Delta_1 & 0\\ 0 & \Delta_2\end{pmatrix}, \left(\begin{array}{cc|c}0 & G & G\\ 0 & 0 & I\\ \hline I & G & G\end{array}\right)\right)$$

Factor uncertainty: Let G_2 have proper inverse. Then

$$(G_1+\Delta_1)(G_2+\Delta_2)^{-1} =$$

$$= S\left(\begin{pmatrix}\Delta_1\\ \Delta_2\end{pmatrix}, \left(\begin{array}{cc|c}0 & -G_2^{-1} & G_2^{-1}\\ \hline I & -G_1G_2^{-1} & G_1G_2^{-1}\end{array}\right)\right).$$

Can iterate procedure.

Can introduce weightings a posteriori.

Summary: Component Uncertainties

Uncertain components could be described as

$$G_\Delta = S(\Delta, G), \quad \Delta \in \mathbf{\Delta}$$

where G involves nominal system and weightings.

Class of uncertainties is

$$\mathbf{\Delta} := \{\Delta \in RH_\infty \mid \Delta(i\omega) \in \mathbf{\Delta}_c \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}\}$$

with set $\mathbf{\Delta}_c$ of complex matrices defining **structure, size**.

Could work with set $\mathbf{\Delta}_c$ of block-diagonal matrices

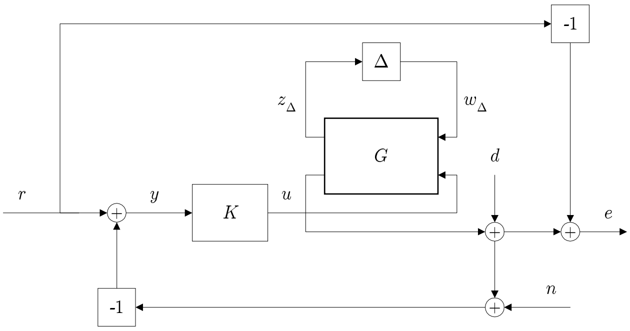
$$\Delta_c = \begin{pmatrix} \delta_1 I & & & & & 0 \\ & \ddots & & & & \\ & & \delta_r I & & & \\ & & & \Delta_1 & & \\ & & & & \ddots & \\ 0 & & & & & \Delta_f \end{pmatrix} \quad \text{with} \quad \delta_j \in \mathbb{R},\ \Delta_j \in \mathbb{C}^{p_j \times q_j}$$

that are bounded as

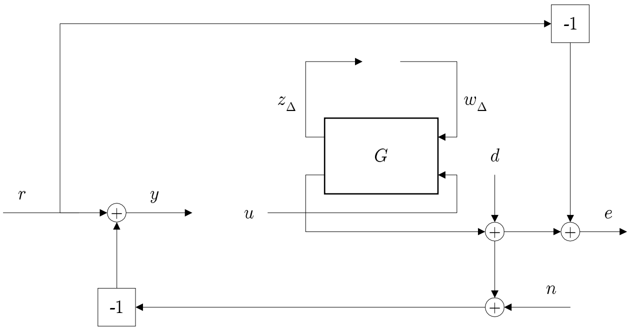
$$\|\Delta_c\| < 1.$$

Pulling Uncertainties out of Interconnections

Pull uncertainties out of component:



Disconnect K and Δ to get P:

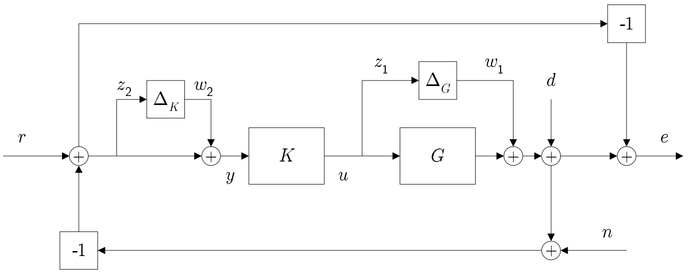


Structure Cannot be Avoided

Multiple unstructured uncertainties in several components

$$\Downarrow$$

Structured uncertainty for interconnection.



Will lead to interconnection uncertainty

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Delta_G & 0 \\ 0 & \Delta_K \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

General Framework

Open-loop interconnection described as

$$\begin{pmatrix} z_\Delta \\ z_p \\ y \end{pmatrix} = P \begin{pmatrix} w_\Delta \\ w_p \\ u \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \begin{pmatrix} w_\Delta \\ w_p \\ u \end{pmatrix}.$$

Reconnect controller as $u = Ky$ to obtain closed-loop interconnection described as

$$\begin{pmatrix} z_\Delta \\ z_p \end{pmatrix} = S(P, K) \begin{pmatrix} w_\Delta \\ w_p \end{pmatrix}.$$

Reconnect uncertainty as $w_\Delta = \Delta z_\Delta$ to obtain perturbed closed-loop interconnection described as

$$z_p = S(\Delta, S(P, K)) w_p = S(S(\Delta, P), K) w_p.$$

For $S(P, K)$, use abbreviation

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} = \begin{pmatrix} M & N_{12} \\ N_{21} & N_{22} \end{pmatrix}.$$

M is transfer matrix seen by Δ.

Fundamental Hypotheses

Controlled uncertain system described as

$$\begin{pmatrix} z_\Delta \\ z_p \\ y \end{pmatrix} = P \begin{pmatrix} w_\Delta \\ w_p \\ u \end{pmatrix}, \; u = Ky, \; w_\Delta = \Delta z_\Delta, \; \Delta \in \boldsymbol{\Delta}.$$

Hypotheses

- P is a generalized plant.
- With value set of complex matrices $\boldsymbol{\Delta}_c$ (structure, size):

  $$\boldsymbol{\Delta} = \{\Delta \in RH_\infty \mid \Delta(i\omega) \in \boldsymbol{\Delta}_c \text{ for } \omega \in \mathbb{R} \cup \{\infty\}\}.$$

  Value set $\boldsymbol{\Delta}_c$ is **star-shaped with center $\mathbf{0}$:**

  $$\Delta_c \in \boldsymbol{\Delta}_c \;\Rightarrow\; \tau\Delta_c \in \boldsymbol{\Delta}_c \text{ for all } \tau \in [0,1].$$

- For all $\Delta_c \in \boldsymbol{\Delta}_c$,

  $$I - P_{11}(\infty)\Delta_c \text{ is non-singular.}$$

Robust Stability

Robust Stabilization

Say that

$$K \text{ robustly stabilizes } S(\Delta, P) \text{ against } \boldsymbol{\Delta}$$

if K stabilizes $S(\Delta, P)$ for any uncertainty Δ contained in class $\boldsymbol{\Delta}$.

Robust Stability Analysis Problem

For given and fixed controller K, test whether it robustly stabilizes $S(\Delta, P)$ against $\boldsymbol{\Delta}$.

Robust Stability Synthesis Problem

Design a controller K that robustly stabilizes $S(\Delta, P)$ against $\boldsymbol{\Delta}$.

Let's turn to analysis.

Reduce to Transfer Matrix Seen by Uncertainty

Recall abbreviation

$$S(P, K) = N = \begin{pmatrix} M & N_{12} \\ N_{21} & N_{22} \end{pmatrix}.$$

Theorem

If K stabilizes P, and if

$I - M\Delta$ has a proper and stable inverse for all $\Delta \in \boldsymbol{\Delta}$,

then K robustly stabilizes $S(\Delta, P)$ against $\boldsymbol{\Delta}$.

Recall that $I - M\Delta$ has proper and stable inverse iff

$$\det(I - M(s)\Delta(s)) \neq 0 \text{ for all } s \in \mathbb{C}^0 \cup \mathbb{C}^+ \cup \{\infty\}.$$

Still hard to test!

Proof

With any $\Delta \in \boldsymbol{\Delta}$, have to check that

$$\begin{pmatrix} z_p \\ y \end{pmatrix} = S(\Delta, P) \begin{pmatrix} w_p \\ y \end{pmatrix}, \; u = Kv + v_1, \; v = y + v_2$$

defines proper and stable system $\begin{pmatrix} w_p \\ v_1 \\ v_2 \end{pmatrix} \to \begin{pmatrix} z_p \\ u \\ v \end{pmatrix}$.

System can be written as

$$\begin{pmatrix} z_\Delta \\ z_p \\ y \end{pmatrix} = P \begin{pmatrix} w_\Delta \\ w_p \\ u \end{pmatrix}, \; u = Kv + v_1, \; v = y + v_2, \; w_\Delta = \Delta z_\Delta.$$

Absorbing controller leads to

$$\left(\begin{array}{c} z_\Delta \\ z_p \\ \hline u \\ v \end{array}\right) = \left(\begin{array}{cc|cc} M & N_{12} & H_{13} & H_{14} \\ N_{21} & N_{22} & H_{23} & H_{24} \\ \hline H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{array}\right) \left(\begin{array}{c} w_\Delta \\ w_p \\ \hline v_1 \\ v_2 \end{array}\right), \quad w_\Delta = \Delta z_\Delta.$$

Since K stabilizes P, all transfer matrices are stable.

Since $(I - M\Delta)^{-1}$ is stable, upper LFT is stable.

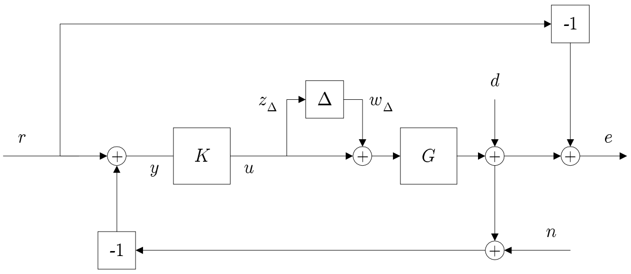
Example: Spinning Satellite

Parametric input multiplicative uncertainty

$$\frac{1}{s^2+a^2}\begin{pmatrix} s-a^2 & a(s+1) \\ -a(s+1) & s-a^2 \end{pmatrix}\left(I+\begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}\right)$$

with

$$\Delta(s)=\begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}\in\mathbb{R}^{2\times 2}.$$



If controlled with $K=I$, transfer matrix seen by Δ is

$$M(s)=\frac{1}{s+1}\begin{pmatrix} -1 & -a \\ a & -1 \end{pmatrix}.$$

Stability Test

Determinant of

$$I-M(s)\Delta=\begin{pmatrix} 1+\frac{\delta_1}{s+1} & \frac{a\delta_2}{s+1} \\ \frac{-a\delta_1}{s+1} & 1+\frac{\delta_2}{s+1} \end{pmatrix}$$

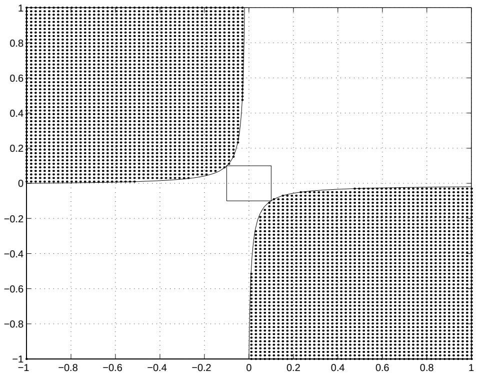
equals

$$\frac{1}{(s+1)^2}\left(s^2+(2+\delta_1+\delta_2)s+(1+\delta_1+\delta_2)+(a^2+1)\delta_1\delta_2\right).$$

Only stable zeros if

$$2+\delta_1+\delta_2>0,\quad (1+\delta_1+\delta_2)+(a^2+1)\delta_1\delta_2>0.$$

For $a=10$, plot region of (δ_1,δ_2) where **not** true:



Reduction to Test on Imaginary Axis

Theorem

Suppose M is proper and stable. If

$$\det(I-M(i\omega)\Delta_c)\neq 0 \text{ for all } \Delta_c\in\mathbf{\Delta}_c,\ \omega\in\mathbb{R}\cup\{\infty\},$$

then

$$I-M\Delta \text{ has a proper and stable inverse for all } \Delta\in\mathbf{\Delta}.$$

Straightforward consequence: Fundamental Result.

Corollary

If K stabilizes P, and if

$$\det(I-M(i\omega)\Delta_c)\neq 0 \text{ for all } \Delta_c\in\mathbf{\Delta}_c,\ \omega\in\mathbb{R}\cup\{\infty\},$$

then K robustly stabilizes $S(\Delta,P)$ against $\mathbf{\Delta}$.

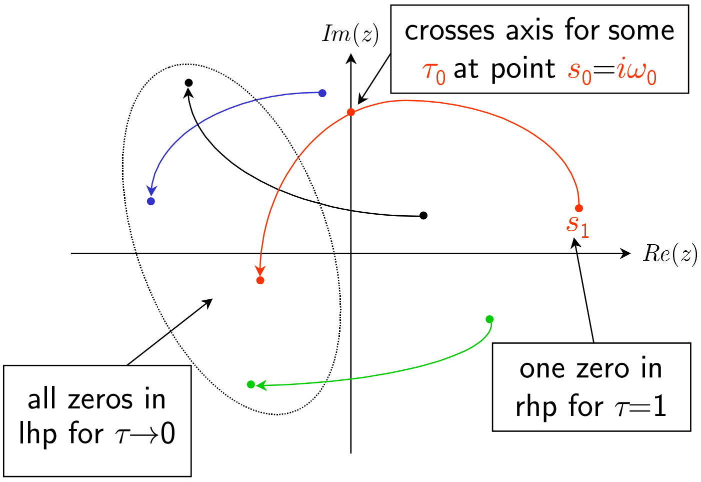
Have to check non-singularity of $I-M(i\omega)\Delta_c$ for all frequencies (including infinity) and for values that can be taken by frequency response of uncertainty.

Idea of Proof

$(I-M(s)\Delta(s))^{-1}$ unstable.

Hence $\det(I-M(s)\Delta(s))$ has zero s_1 in $\mathbb{C}^+$.

Follow zeros of $\det(I-M(s)[\tau\Delta(s)])$ for $\tau=1\ldots0$:



crosses axis for some τ_0 at point $s_0{=}i\omega_0$

one zero in rhp for $\tau{=}1$

all zeros in lhp for $\tau{\to}0$

$\det(I-M(s)[\tau_0\Delta(s)])$ has zero on axis.

Contradiction.

Construction of Destabilizing Perturbations

Suppose test fails. Then there exist frequency $\omega_0 \in \mathbb{R} \cup \{\infty\}$ and complex matrix $\Delta_0 \in \mathbf{\Delta}_c$ such that

$$I - M(i\omega_0)\Delta_0 \ \text{ is singular.}$$

Find proper and stable $\Delta(s)$ with

$\Delta(i\omega_0) = \Delta_0$ and $\Delta(i\omega) \in \mathbf{\Delta}_c$ for all $\omega \in \mathbb{R} \cup \{\infty\}$.

Trivial if Δ_0 is **real matrix**.

Clear that Δ renders $(I - M\Delta)^{-1}$ unstable.

Candidate for destabilizing perturbation.

Test whether K stabilizes $S(\Delta, P)$.

Answer no: Have found destabilizing perturbation.

Answer yes: ??? (This can happen!!!)

Conditions to Guarantee Success

Constructed Δ is **certainly** destabilizing if

- $\omega_0 = \infty$
- or if ω_0 is finite, if

$$\left(\begin{array}{c|cc|c} A - i\omega_0 I & 0 & B_1 & B_2 \\ \hline 0 & \Delta_0 & -I & 0 \\ C_1 & -I & D_{11} & D_{12} \end{array}\right) \ \text{ has full row rank,}$$

and if

$$\left(\begin{array}{c|cc} A - i\omega_0 I & 0 & B_1 \\ \hline 0 & \Delta_0 & -I \\ C_1 & -I & D_{11} \\ \hline C_2 & 0 & D_{21} \end{array}\right) \ \text{ has full column rank.}$$

- or if ω_0 is finite, if $i\omega_0$ is no pole of P, if

$$\begin{pmatrix} I - P_{11}(i\omega_0)\Delta_0 & P_{12}(i\omega) \end{pmatrix} \ \text{ has full row rank,}$$

and if

$$\begin{pmatrix} I - \Delta_0 P_{11}(i\omega_0) \\ P_{21}(i\omega_0) \end{pmatrix} \ \text{ has full column rank.}$$

Example by Hand

Consider

$$y(s) = \frac{1 + \Delta_1(s)}{s + \alpha} u(s).$$

Pull out Δ_1 and α:

$$\left(\begin{array}{c} z_1(s) \\ z_2(s) \\ \hline y(s) \end{array}\right) = \underbrace{\left(\begin{array}{cc|c} 0 & 0 & 1 \\ 1/s & -1/s & 1/s \\ \hline 1/s & -1/s & 1/s \end{array}\right)}_{P(s)} \left(\begin{array}{c} w_1(s) \\ w_2(s) \\ \hline u(s) \end{array}\right)$$

$$\begin{pmatrix} w_1(s) \\ w_2(s) \end{pmatrix} = \underbrace{\begin{pmatrix} \Delta_1(s) & 0 \\ 0 & \alpha \end{pmatrix}}_{\Delta(s)} \begin{pmatrix} z_1(s) \\ z_2(s) \end{pmatrix}.$$

Control P with $K(s) = -1$ to get

$$S(P, K)(s) = M(s) = \begin{pmatrix} -1/(s+1) & 1/(s+1) \\ 1/(s+1) & -1/(s+1) \end{pmatrix}$$
$$= \frac{1}{s+1} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix}.$$

Example By Hand

Consider

$$\det(I - M(i\omega)\Delta(i\omega)) =$$
$$= \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{i\omega + 1} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} \Delta_1(i\omega) & -\alpha \end{pmatrix} \right) =$$
$$= \det\left(1 + \frac{1}{i\omega + 1}(\Delta_1(i\omega) + \alpha) \right).$$

Nonzero if and only if

$$\Delta_1(i\omega) + \alpha \neq -(i\omega + 1).$$

Holds for example for

$$\alpha_1 < \alpha, \quad -1 - \alpha_1 < \mathrm{Re}(\Delta_1(i\omega)).$$

Have RS for stable Δ with frequency response in

$$\mathbf{\Delta}_c = \left\{ \begin{pmatrix} \Delta_1 & 0 \\ 0 & \alpha \end{pmatrix} \mid \alpha_1 < \alpha, \ \mathrm{Re}(\Delta_1) > -1 - \alpha_1 \right\}$$

Definitions for Complex Matrices

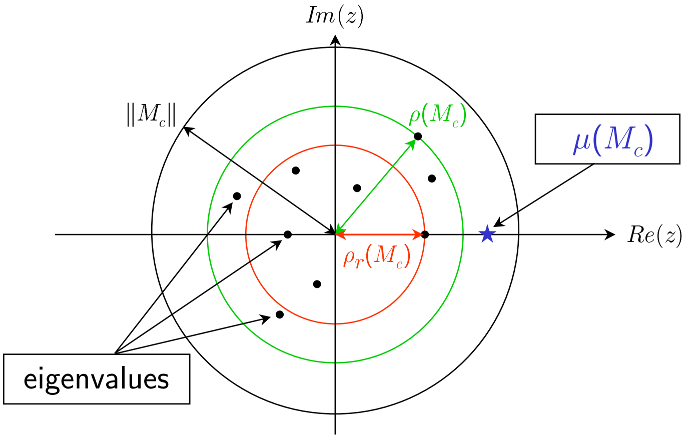
Eigenvalues: $\lambda(M_c) = \{\lambda : \det(\lambda I - M_c) = 0\}$

Spectral Norm: $\|M_c\| = \overline{\sigma}(M_c)$

Spectral Radius: $\rho(M_c) = \max\{|\lambda| : \lambda \in \lambda(M_c)\}$

Real SR: $\rho_r(M_c) = \max\{|\lambda| : \lambda \in \mathbb{R} \cap \lambda(M_c)\}$

Relations: $\rho_r(M_c) \leq \rho(M) \leq \|M_c\|$



Various Sufficient Conditions

$$\det(I - M(i\omega)\Delta_c) \neq 0$$
$$\Updownarrow$$
$$1 \notin \lambda(M(i\omega)\Delta_c)$$
$$\Uparrow$$
$$\rho_r(M(i\omega)\Delta_c) < 1$$
$$\Uparrow$$
$$\rho(M(i\omega)\Delta_c) < 1$$
$$\Uparrow$$
$$\|M(i\omega)\Delta_c\| < 1$$
$$\Uparrow$$
$$\|M(i\omega)\|\|\Delta_c\| < 1.$$

Can continue with

$$\Uparrow$$
$$\|M(i\omega)\| \leq \frac{1}{r}$$

if matrices in $\boldsymbol{\Delta}_c$ are bounded as

$$\|\Delta_c\| < r \quad \text{for all} \quad \Delta_c \in \boldsymbol{\Delta}_c.$$

Small-Gain is Sufficient

Suppose we know that

$$\|\Delta_c\| < r \quad \text{for all} \quad \Delta_c \in \boldsymbol{\Delta}_c$$

or, equivalently,

$$\|\Delta\|_\infty < r \quad \text{for all} \quad \Delta \in \boldsymbol{\Delta}.$$

Corollary
If M is proper and stable, and if

$$\|M\|_\infty \leq \frac{1}{r}$$

then $I - M\Delta$ has proper and stable inverse for all $\Delta \in \boldsymbol{\Delta}$.

Norm condition is **sufficient**! Ignores structure.

Corollary
If K stabilizes P, and if

$$\|M\|_\infty \leq \frac{1}{r}$$

then K robustly stabilizes $S(\Delta, P)$ against $\boldsymbol{\Delta}$.

Full Block Uncertainty: Small-Gain is Exact

Suppose we know that

$\boldsymbol{\Delta}$ **equals** set of all $\Delta \in RH_\infty$ with $\|\Delta\|_\infty < r$.

Theorem
Let M be proper and stable. Then

$$\|M\|_\infty \leq \frac{1}{r}$$

if and only if

$I - M\Delta$ has a proper and stable inverse for **all** $\Delta \in \boldsymbol{\Delta}$.

For full block uncertainty, norm condition is also **necessary**!

This means:

$$\|M\|_\infty > \frac{1}{r}$$

implies that

$I - M\Delta$ has **no** proper, stable inverse for **some** $\Delta \in \boldsymbol{\Delta}$.

Construction of Destabilizing Perturbation

Let $\|M(i\omega_0)\| > \frac{1}{r}$.

Exists $u \neq 0$ with $[M(i\omega_0)M(i\omega_0)^*]u = \|M(i\omega_0)\|^2 u$.

Define $v := \frac{M(i\omega_0)^* u}{\|M(i\omega_0)\|^2}$ and $\Delta_0 := v\frac{u^*}{\|u\|^2}$.

Implies

$$\|\Delta_0\| \leq \frac{\|v\|}{\|u\|} \leq \frac{1}{\|u\|}\frac{\|M(i\omega_0)^* u\|}{\|M(i\omega_0)\|^2} \leq \frac{1}{\|M(i\omega_0)\|} < r$$

and

$$[I - M(i\omega_0)\Delta_0]u = u - M(i\omega_0)v =$$

$$= u - \frac{M(i\omega_0)M(i\omega_0)^* u}{\|M(i\omega_0)\|^2} = u - \frac{\|M(i\omega_0)\|^2}{\|M(i\omega_0)\|^2}u = 0.$$

Have constructed complex matrix Δ_0 with

$$\|\Delta_0\| < r \;\text{ and }\; \det(I - M(i\omega_0)\Delta_0) = 0.$$

Construction of Destabilizing Perturbation

Have to find dynamic Δ with $\Delta(i\omega_0) = \Delta_0$ and $\|\Delta\|_\infty < r$.

If u and v are **real**, Δ_0 is **real**. Choose $\Delta(s) := \Delta_0$.

Otherwise apply interpolation Lemma to components of

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_p \end{pmatrix}, \quad \frac{u^*}{\|u\|^2} = \begin{pmatrix} u_1 & \cdots & u_q \end{pmatrix}.$$

Exist $\alpha_j \geq 0$ and $\beta_j \geq 0$ such that

$$u_j(s) = \pm|u_j|\frac{s-\alpha_j}{s+\alpha_j} \;\text{ and }\; v_j(s) = \pm|v_j|\frac{s-\beta_j}{s+\beta_j}$$

satisfy

$$u_j(i\omega_0) = u_j, \quad |u_j(i\omega)| = |u_j|$$

$$v_j(i\omega_0) = v_j, \quad |v_j(i\omega)| = |v_j|.$$

Then the dynamic uncertainty

$$\Delta(s) = \begin{pmatrix} v_1(s) \\ \vdots \\ v_p(s) \end{pmatrix} \begin{pmatrix} u_1(s) & \cdots & u_q(s) \end{pmatrix}$$

does the job.

Summary of Robust Stability Tests

For general uncertain interconnections, have seen

- ... how to pull out uncertainties to compute the transfer matrix M seen by the uncertainties in controlled interconnection.
- ... that robust stability is guaranteed if controller is nominally stabilizing and if $\det(I - M\Delta)$ has no zeros on extended imaginary axis (determinant test).
- ... how to check whether test is tight by constructing destabilizing perturbations.
- ... that small-gain theorem is exact for full block uncertainties.

Continue with **block-diagonal uncertainty structure**.

Will arrive at **structured singular value**.

Standard Block-Diagonal Structure

Assume from now on: $\mathbf{\Delta}_c$ set of all

$$\Delta_c = \begin{pmatrix} p_1 I & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & p_{n_r} I & & & & & & \\ & & & \delta_1 I & & & & & \\ & & & & \ddots & & & & \\ & & & & & \delta_{n_c} I & & & \\ & & & & & & \Delta_1 & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & \Delta_{n_f} \end{pmatrix} \in \mathbb{C}^{p \times q}$$

with

- $p_j \in \mathbb{R}$ with $|p_j| < 1$
- $\delta_j \in \mathbb{C}$ with $|\delta_j| < 1$
- $\Delta_j \in \mathbb{C}^{p_j \times q_j}$ with $\|\Delta_j\| < 1$.

Hence we have

- n_r real repeated blocks
- n_c complex repeated blocks
- n_f full (complex) blocks.

Non-Singularity Margin

For robust stability, have to check

$$\det(I - M(i\omega)\Delta_c) \neq 0 \ \text{ for all } \ \Delta_c \in \boldsymbol{\Delta}_c, \ \omega \in \mathbb{R} \cup \{\infty\}$$

Let $M_c \in \mathbb{C}^{q \times p}$ be a complex matrix.

Test whether

$$\det(I - M_c\Delta_c) \neq 0 \ \text{ for all } \ \Delta_c \in \boldsymbol{\Delta}_c.$$

Answer: Yes or No.

Modification:

Determine largest r such that

$$\det(I - M_c\Delta_c) \neq 0 \ \text{ for all } \ \Delta_c \in r\boldsymbol{\Delta}_c.$$

Answer: Number r_*. (Non-singularity margin.)

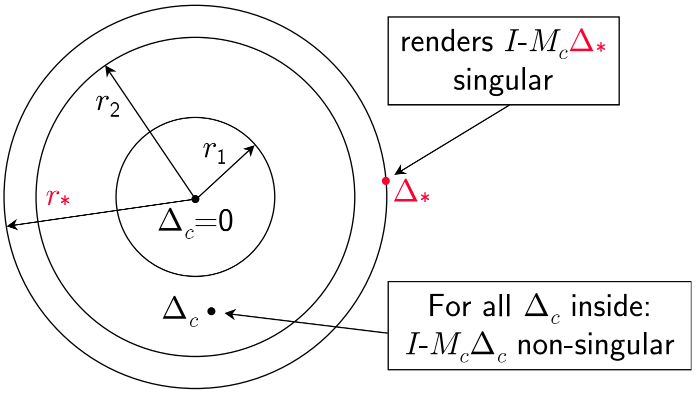
If $r_* < 1$ then answer is no.

If $r_* \geq 1$ then answer is yes.

Non-Singularity Margin

Largest r such that

$$\det(I - M_c\Delta_c) \neq 0 \ \text{ for all } \ \Delta_c \in r\boldsymbol{\Delta}_c.$$



renders I-$M_c\Delta_*$ singular

For all Δ_c inside: I-$M_c\Delta_c$ non-singular

The Structured Singular Value

Definition
The structured singular value (SSV) of the matrix M_c with respect to the set $\boldsymbol{\Delta_c}$ is the number

$$\frac{1}{\sup\{r \mid \det(I - M_c\Delta_c) \neq 0 \ \text{ for all } \ \Delta_c \in r\boldsymbol{\Delta}_c\}}.$$

and denoted as

$$\mu_{\boldsymbol{\Delta_c}}(M_c).$$

Upper bound on SSV:

Suppose that $\mu_{\boldsymbol{\Delta_c}}(M_c) \leq \gamma_1$.

Then $I - M_c\Delta_c$ is non-singular for all Δ_c in the class $\frac{1}{\gamma_1}\boldsymbol{\Delta_c}$.

Lower bound on SSV:

Suppose that $\mu_{\boldsymbol{\Delta_c}}(M_c) > \gamma_2$.

Then there exists $\Delta_c \in \frac{1}{\gamma_2}\boldsymbol{\Delta_c}$ such that $I - M_c\Delta_c$ singular.

SSV Test of Robust Stability

Theorem
Suppose M is proper and stable. Then

$$\mu_{\boldsymbol{\Delta_c}}(M(i\omega)) \leq 1 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}$$

if and only if

$I - M\Delta$ has a proper and stable inverse for all $\Delta \in \boldsymbol{\Delta}$.

Destabilizing perturbations: Construct blockwise.

Consequence: Fundamental result of course.

Corollary
If K stabilizes P, and if

$$\mu_{\boldsymbol{\Delta_c}}(M(i\omega)) \leq 1 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}$$

then K robustly stabilizes $S(\Delta, P)$ against $\boldsymbol{\Delta}$.

SSV Test of Hypothesis on P_{11}

In general hypothesis, had to test whether

$$I - P_{11}(\infty)\Delta_c \ \text{ is non-singular}$$

for all $\Delta_c \in \mathbf{\Delta}_c$.

True if and only if

$$\mu_{\mathbf{\Delta_c}}(P_{11}(\infty)) \leq 1.$$

Test of whether upper LFT with Δ is well-defined can be performed with SSV.

Remark. Many more applications of SSV!

Other Value Sets: Weightings

With complex U and V want to test

$$\det(I - M_c\hat{\Delta}_c) \neq 0 \ \text{ for all } \ \hat{\Delta}_c \in U\mathbf{\Delta}_c V.$$

True if and only if

$$\mu_{\mathbf{\Delta_c}}(VM_cU) \leq 1.$$

Why?

Just follows from

$$\det(I - M_c[U\Delta_c V]) \neq 0$$

if and only if

$$\det(I - [VM_cU]\Delta_c) \neq 0.$$

Special Value Sets: Explicit Computation

$\mathbf{\Delta_c} = \{\Delta_c \in \mathbb{C}^{p\times q} \mid \|\Delta_c\| < 1\}$ - one full complex block:

$$\mu_{\mathbf{\Delta_c}}(M_c) = \|M_c\|.$$

Note the consequence:

$$\det(I - M_c\Delta_c) \neq 0 \ \text{ for all } \ \Delta_c \in \mathbb{C}^{p\times q}, \ \|\Delta_c\| < 1$$

if and only if

$$\|M_c\| \leq 1.$$

$\mathbf{\Delta_c} = \{\delta I \mid \delta \in \mathbb{C}, |\delta| < 1\}$ - one repeated complex block:

$$\mu_{\mathbf{\Delta_c}}(M_c) = \rho(M_c)$$

$\mathbf{\Delta_c} = \{\delta I \mid \delta \in \mathbb{R}, |\delta| < 1\}$ - one repeated real block:

$$\mu_{\mathbf{\Delta_c}}(M_c) = \rho_r(M)$$

Why?

One full complex block:

$$\begin{aligned}\|M_c\| \leq 1 &\Rightarrow \det(I - M_c\Delta_c) \neq 0 \ \text{ for all } \ \|\Delta_c\| < 1\\ &\Rightarrow \mu_{\mathbf{\Delta_c}}(M_c) \leq 1.\end{aligned}$$

Suppose $\lambda := \|M_c\| > 1$.

Exists u with $u^*u = 1$ and $M_cM_c^*u = \lambda^2 u$.

$\Delta_c = \dfrac{1}{\lambda}(M_c^*u)u^*$ satisfies $\|\Delta_c\| < 1$ & $(I - M_c\Delta_c)u = 0$.

This implies $\mu_{\mathbf{\Delta_c}}(M_c) > 1$.

One complex repeated block:

$$\begin{aligned}\rho(M_c) \leq 1 &\Leftrightarrow \text{All eigenvalues inside unit disk}\\ &\Leftrightarrow \det(\tfrac{1}{\delta}I - M_c) \neq 0 \ \text{ for } \ |\tfrac{1}{\delta}| > 1\\ &\Leftrightarrow \det(I - M_c\delta) \neq 0 \ \text{ for } \ |\delta| < 1\\ &\Leftrightarrow \mu_{\mathbf{\Delta_c}}(M_c) \leq 1.\end{aligned}$$

One real repeated block: Similar.

Dependence on Value Set

In general, μ grows with uncertainty set:

$$\boldsymbol{\Delta}_1 \subset \boldsymbol{\Delta}_2 \ \text{ implies } \ \mu_{\boldsymbol{\Delta}_1}(M_c) \leq \mu_{\boldsymbol{\Delta}_2}(M_c).$$

In general we have

$$\{\delta I \mid \delta \in \mathbb{R},\ |\delta| < 1\} \subset \boldsymbol{\Delta_c} \subset \{\Delta_c \in \mathbb{C}^{p\times q} \mid \|\Delta_c\| < 1\}.$$

Leads to simple **bounds**

$$\rho_r(M_c) \leq \mu_{\boldsymbol{\Delta_c}}(M_c) \leq \|M_c\|.$$

Calculation of Improved Lower Bound

Let $\boldsymbol{U}$ denote unitary elements in $\boldsymbol{\Delta_c}$:

$$\boldsymbol{U} = \{U \in \boldsymbol{\Delta_c} : U^*U = I\}.$$

If $U \in \boldsymbol{U}$ then $\mu_{\boldsymbol{\Delta_c}}(M_c) = \mu_{\boldsymbol{\Delta_c}}(M_cU)$ and hence

$$\rho_r(M_cU) \leq \mu_{\boldsymbol{\Delta_c}}(M_cU) = \mu_{\boldsymbol{\Delta_c}}(M_c).$$

Consequently

$$\sup_{U\in\boldsymbol{U}} \rho_r(M_cU) \leq \mu_{\boldsymbol{\Delta_c}}(M_c)$$

That's how lower bound is calculated in μ-tools.

No real blocks: Equality with complex spectral radius:

$$\max_{U\in\boldsymbol{U}} \rho(M_cU) = \mu_{\boldsymbol{\Delta_c}}(M_c).$$

Continuity

If there are real blocks ($n_r \geq 1$), then

$$\mu_{\boldsymbol{\Delta_c}}(M_c) \ \text{ can be discontinuous in } \ M_c.$$

Hence

$$\omega \to \mu_{\boldsymbol{\Delta_c}}(M(i\omega)) \ \text{ can have jumps!}$$

Example: One real repeated block:

$$\rho_{\mathbb{R}}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} = \begin{cases} 0 & \text{for } \ m \neq 0 \\ 1 & \text{for } \ m = 0. \end{cases}$$

If there are no real blocks ($n_r = 0$),

$$\mu_{\boldsymbol{\Delta_c}}(M_c) \ \text{ is continuous in } \ M_c.$$

Hence

$$\omega \to \mu_{\boldsymbol{\Delta_c}}(M(i\omega)) \ \text{ does **not** have jumps.}$$

Improve Upper Bound - Idea

Suppose $\det(D) \neq 0$, $D\Delta_c = \Delta_c D$ for all $\Delta_c \in \boldsymbol{\Delta_c}$.

Then

$$\begin{aligned}
\det(I - M_c\Delta_c) &= \underbrace{\det(D^{-1})\det(D)}_{1}\det(I - M_c\Delta_c) \\
&= \det(D^{-1}[I - M_c\Delta_c]D) \\
&= \det(I - D^{-1}[M_c\Delta_c]D) \\
&= \det(I - [D^{-1}M_cD]\Delta_c)
\end{aligned}$$

and hence

$$\mu_{\boldsymbol{\Delta_c}}(M_c) = \mu_{\boldsymbol{\Delta_c}}(D^{-1}M_cD).$$

Therefore

$$\mu_{\boldsymbol{\Delta_c}}(M_c) = \mu_{\boldsymbol{\Delta_c}}(D^{-1}M_cD) \leq \|D^{-1}M_cD\|.$$

Observe that $\|D^{-1}M_cD\|$ **varies** with D.

Minimize $\|D^{-1}M_cD\|$ to get best bound.

Notation: D is called **scaling**.

Structure of Scalings

Recall structure of Δ_c in $\boldsymbol{\Delta_c}$:

$$\Delta_c = \begin{pmatrix} p_1 I & & & & & & & & & 0 \\ & \ddots & & & & & & & & \\ & & p_{n_r} I & & & & & & & \\ & & & \delta_1 I & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & \delta_{n_c} I & & & & \\ & & & & & & \Delta_1 & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & \Delta_{n_f-1} & \\ 0 & & & & & & & & & \Delta_{n_f} \end{pmatrix}.$$

Choose $\boldsymbol{D}$ as set of matrices

$$D = \begin{pmatrix} P_1 & & & & & & & & & 0 \\ & \ddots & & & & & & & & \\ & & P_{n_r} & & & & & & & \\ & & & D_1 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & D_{n_c} & & & & \\ & & & & & & d_1 I & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & d_{n_f-1} I & \\ 0 & & & & & & & & & I \end{pmatrix}$$

where

- P_j, D_j Hermitian positive definite
- d_j positive real scalars.

Have $D\Delta = \Delta D$ and $\det(D) \neq 0$.

Calculation of Improved Upper Bound

We infer

$$\mu_{\boldsymbol{\Delta_c}}(M_c) \leq \inf_{D \in \boldsymbol{D}} \|D^{-1} M_c D\|.$$

Optimization problem nicely solvable: LMIs.

Gap might be large.

Gap can be reduced with more scalings.

Latter technique used in μ-tools.

Nice non-trival result:

Suppose that

$$2(n_r + n_c) + n_f \leq 3.$$

Then upper bound computed in μ-tools is **equal** to SSV.

Summary of SSV

Value set $\boldsymbol{\Delta}_c$ and matrix M_c define SSV $\mu_{\boldsymbol{\Delta_c}}(M_c)$.

Can calculate lower bound and upper bound

$$l_{\boldsymbol{\Delta_c}}(M_c) < \mu_{\boldsymbol{\Delta_c}}(M_c) \leq u_{\boldsymbol{\Delta_c}}(M_c).$$

Interpretation of upper bound:

$$\det(I - M_c \Delta_c) \neq 0 \text{ for all } \Delta_c \in \frac{1}{u_{\boldsymbol{\Delta_c}}(M_c)} \boldsymbol{\Delta}_c.$$

Interpretation of lower bound:

$$\det(I - M_c \Delta_c) = 0 \text{ for some } \Delta_c \in \frac{1}{l_{\boldsymbol{\Delta_c}}(M_c)} \boldsymbol{\Delta}_c.$$

Upper bound can be calculated with set of scalings $\boldsymbol{D}$ (with structure opposite to that of $\boldsymbol{\Delta}$) as

$$u_{\boldsymbol{\Delta_c}}(M_c) = \inf_{D \in \boldsymbol{D}} \|D^{-1} M_c D\|.$$

Summary of SSV Robust Stability Test

M real-rational, proper, stable.

Suppose

$$\mu_{\boldsymbol{\Delta_c}}(M(i\omega)) \leq \gamma_1 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}.$$

Then

$$I - M\Delta \text{ has proper, stable inverse for all } \Delta \in \frac{1}{\gamma_1} \boldsymbol{\Delta}.$$

Suppose

$$\mu_{\boldsymbol{\Delta_c}}(M(i\omega_0)) > \gamma_2 \text{ for some } \omega_0 \in \mathbb{R} \cup \{\infty\}.$$

Then

there exists $\Delta \in \frac{1}{\gamma_2} \boldsymbol{\Delta}$ for which $I - M\Delta$ does **not** have proper, stable inverse.

Example

Choose

$$M(s) = \begin{pmatrix} \frac{1}{2s+1} & 1 & \frac{s-2}{2s+4} \\ -1 & \frac{s}{s^2+s+1} & \frac{1}{(s+1)^2} \\ \frac{3s}{s+5} & \frac{-1}{4s+1} & 1 \end{pmatrix}.$$

Three cases: $\mathbf{\Delta}_c$ all matrices with $\|\Delta_c\| < 1$ structured as

two full complex blocks:

$$\Delta_c = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}, \quad \Delta_1 \in \mathbb{C}^{2\times 2}, \quad \Delta_2 \in \mathbb{C}$$

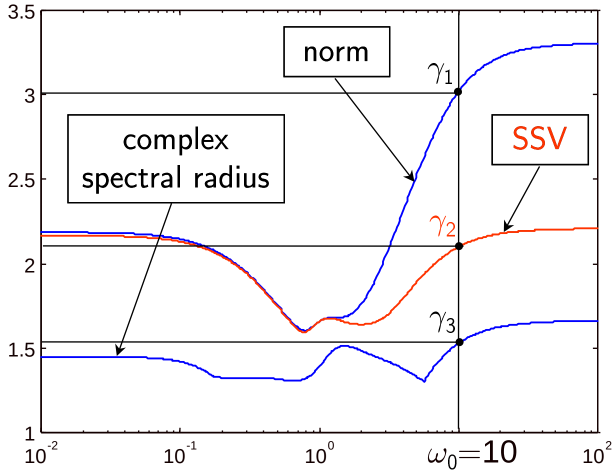
one repeated complex block, one full complex block:

$$\Delta_c = \begin{pmatrix} \delta_1 I_2 & 0 \\ 0 & \Delta_2 \end{pmatrix}, \quad \delta_1 \in \mathbb{C}, \quad \Delta_2 \in \mathbb{C}$$

one repeated real block, one full complex block:

$$\Delta_c = \begin{pmatrix} \delta_1 I_2 & 0 \\ 0 & \Delta_2 \end{pmatrix}, \quad \delta_1 \in \mathbb{R}, \quad \Delta_2 \in \mathbb{C}$$

Two Full Complex



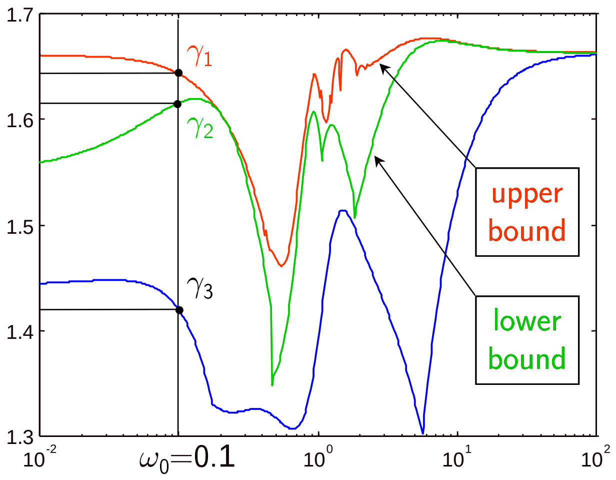
For all $\Delta_c \in \frac{1}{\gamma_2}\mathbf{\Delta}_c$: $\det(I - M(i\omega_0)\Delta_c) \neq 0$

For $\gamma < \gamma_2$ exists $\Delta_c \in \frac{1}{\gamma}\mathbf{\Delta}_c$: $\det(I - M(i\omega_0)\Delta_c) = 0.$

Margin γ_1 for one full complex 3×3-block.

Margin γ_3 for one repeated complex 3×3-block.

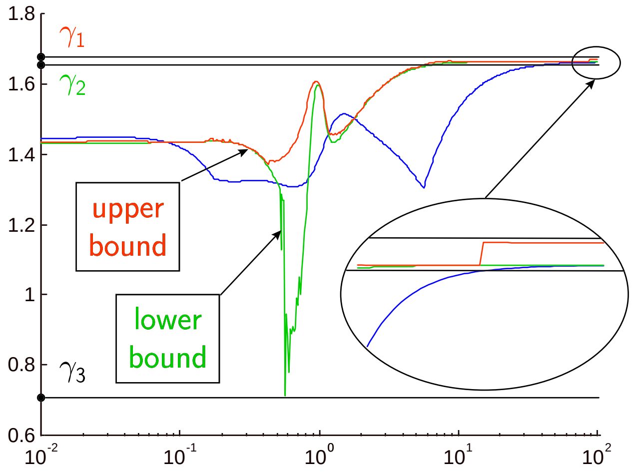
One Complex Repeated, One Full



For all $\Delta_c \in \frac{1}{\gamma_1}\mathbf{\Delta}_c$: $\det(I - M(i\omega_0)\Delta_c) \neq 0$

Exists $\Delta_c \in \frac{1}{\gamma_2}\mathbf{\Delta}_c$: $\det(I - M(i\omega_0)\Delta_c) = 0.$

One Real Repeated, One Full



Upper bound $\leq \gamma_1$ for all frequencies:

$(I - M\Delta)^{-1}$ proper, stable for all uncertainties $\Delta \in \frac{1}{\gamma_1}\mathbf{\Delta}$.

Lower bound $> \gamma_2$ at some frequency:

Can destabilize $(I - M\Delta)^{-1}$ with some $\Delta \in \frac{1}{\gamma_2}\mathbf{\Delta}$.

Nominal and Robust Performance Analysis

Nominal Performance

- Response to persistent signals
- Disturbance suppression and model matching
- The role of weightings

Robust Performance Analysis

- General framework for robust performance analysis
- SSV test for robust performance
- Scaling properties of SSV

Summary: Robust stability and performance analysis

Nominal Performance

Suppose K stabilizes P and leads to closed-loop system

$$z = S(P,K)w.$$

Generalized disturbances:

- Decaying:

$$w \text{ is of finite energy with norm } \|w\|_2.$$

- Persistent: Sinusoid

$$w(t) = w_0 e^{i\omega_0 t} \text{ with norm } \|w\|_{\mathrm{RMS}} := \|w_0\|.$$

- Persistent: Sum of sinusoids

$$w(t) = \sum_{j=1}^{N} w_j e^{i\omega_j t} \text{ with norm } \|w\|_{\mathrm{RMS}} := \sqrt{\sum_{j=1}^{N} \|w_j\|^2}$$

where ω_j are pairwise different.

Concentrate on second two cases.

Response to Sums of Sinusoids

Response of $S(P,K) = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right]$ to $w(t) = \sum_{j=1}^{N} w_j e^{i\omega_j t}$:

$$z(t) = \underbrace{\sum_{j=1}^{N} S(P,K)(i\omega_j) w_j e^{i\omega_j t}}_{\text{steady-state response } z_s(t)} + \sum_{j=1}^{N} \mathcal{C} e^{\mathcal{A}t} (i\omega_j I - \mathcal{A})^{-1} \mathcal{B} w_j.$$

Norm of steady-state response:

$$\|z_s\|_{\mathrm{RMS}} = \sqrt{\sum_{j=1}^{N} \|S(P,K)(i\omega_j) w_j\|^2}$$

$$\leq \|S(P,K)\|_\infty \sqrt{\sum_{j=1}^{N} \|w_j\|^2} = \|S(P,K)\|_\infty \|w\|_{\mathrm{RMS}}.$$

Leads to following alternative interpretation of H_∞-norm:

$$\sup_{z=S(P,K)w,\ \|w\|_{\mathrm{RMS}}>0} \frac{\|z_s\|_{\mathrm{RMS}}}{\|w\|_{\mathrm{RMS}}} = \|S(P,K)\|_\infty.$$

Effects of Weightings

Signal based interpretation of

$$\|ZS(P,K)W\|_\infty \leq 1$$

with real-rational proper stable weightings W and Z.

Suppose that, up to transients, disturbances given as

$$w(t) = \sum_{j=1}^{N} W(i\omega_j) w_j e^{i\omega_j t} \text{ with } \sum_{j=1}^{N} \|w_j\|^2 \leq 1.$$

Then $z = S(P,K)w$ satisfies, up to transients,

$$z(t) = \sum_{j=1}^{N} z_j e^{i\omega_j t} \text{ with } \sum_{j=1}^{N} \|Z(i\omega_j) z_j\|^2 \leq 1.$$

For sinusoids: Amplitudes in frequency dependent ellipsoids.

Loop-Shaping

Let W, Z be real-rational proper stable weightings and W_m be real-rational proper stable **ideal model**.

Loop-shaping interpretation of

$$\|Z\,[S(P,K)-W_m]\,W\|_\infty \leq 1$$

with real-rational proper stable weightings W and Z:

Weighted bound on frequency response:

$$\|Z(i\omega)[S(P,K)(i\omega)-W_m(i\omega)]W(i\omega)\| \leq 1$$

for all $\omega \in \mathbb{R}\cup\{\infty\}$.

- $W_m \neq 0$: Shape $S(P,K)$ by pushing it towards ideal model W_m over frequency band.
- $W_m = 0$: Shape $S(P,K)$ by rendering it small over frequency band.

Put into Generalized Plant Framework

Define

$$\tilde{P} = \begin{pmatrix} Z[P_{11}-W_m]W & ZP_{12} \\ P_{21}W & P_{22} \end{pmatrix}.$$

Since $S(\tilde{P},K) = Z[S(P,K)-W_m]W$, infer that

$$\|Z[S(P,K)-W_m]W\|_\infty \leq 1$$

if and only if

$$\|S(\tilde{P},K)\|_\infty \leq 1.$$

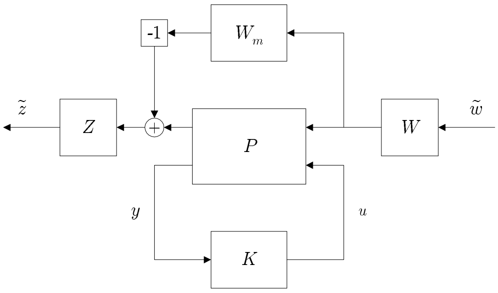
Remarks.

- Performance weightings are auxiliary filters.
- Require K to stabilize $\tilde{P}$.
  Hence $\tilde{P}$ has to be generalized plant.
- If also P is generalized plant:
  Any K that stabilizes $\tilde{P}$ also stabilizes P and vice-versa.

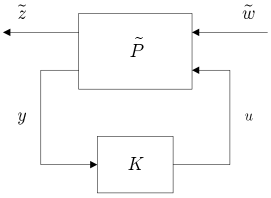
Assume: Performance weightings incorporated in P.

Incorporate Performance Weightings

Interconnection with performance weightings:



Incorporated in generalized plant:



General Paradigm: Robust Performance

Controlled uncertain system described as

$$\begin{pmatrix} z_\Delta \\ z_p \\ y \end{pmatrix} = P \begin{pmatrix} w_\Delta \\ w_p \\ u \end{pmatrix}, \ u = Ky,\ w_\Delta = \Delta z_\Delta,\ \Delta \in \mathbf{\Delta}.$$

Hypotheses

- P is a generalized plant.
- $\mathbf{\Delta_c}$ set of block-diagonal matrices Δ_c with $\|\Delta_c\| < 1$. Class of uncertainties is

  $$\mathbf{\Delta} = \{\Delta \in RH_\infty \mid \Delta(i\omega) \in \mathbf{\Delta_c} \ \text{ for } \ \omega \in \mathbb{R}\cup\{\infty\}\}.$$

- For all $\Delta_c \in \mathbf{\Delta_c}$,

  $$I - P_{11}(\infty)\Delta_c \ \text{ is non-singular.}$$

- Performance specification:
  H_∞-norm of channel $w_p \to z_p$ is smaller than one.

Robust Performance

In case that

$$K \text{ stabilizes } P_\Delta = S(\Delta, P) \text{ and } \|S(P_\Delta, K)\|_\infty \le 1$$

for all uncertainties Δ in the class $\boldsymbol{\Delta}$, we say that

K achieves **robust performance** for $S(\Delta, P)$ against $\boldsymbol{\Delta}$.

Robust Performance Analysis Problem
For given and fixed controller K, test whether it achieves robust performance for $S(\Delta, P)$ against $\boldsymbol{\Delta}$.

Robust Performance Synthesis Problem
Design a controller K that achieves robust performance for $S(\Delta, P)$ against $\boldsymbol{\Delta}$.

Let's turn to analysis.

Frequency by Frequency Test

Recall the notation

$$\begin{pmatrix} z_\Delta \\ z_p \end{pmatrix} = S(P, K) \begin{pmatrix} w_\Delta \\ w_p \end{pmatrix} = \begin{pmatrix} M & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} w_\Delta \\ w_p \end{pmatrix}$$

for the unperturbed controlled system. Recall also

$$S(P_\Delta, K) = S(\Delta, N) = N_{22} + N_{21}\Delta(I - M\Delta)^{-1}N_{12}.$$

Controller achieves **nominal performance** if

$$\|N_{22}(i\omega)\| \le 1 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}.$$

Controller achieves **robust stability** if

$$\mu_{\boldsymbol{\Delta_c}}(M(i\omega)) \le 1 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}.$$

Controller achieves **robust performance** if, in addition,

$$\|N_{22}(i\omega) + N_{21}(i\omega)\Delta_c(I - M(i\omega)\Delta_c)^{-1}N_{12}(i\omega)\| \le 1$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$ and for all $\Delta_c \in \boldsymbol{\Delta_c}$.

Have to check frequency by frequency conditions!

Excursion: Linear Algebra Problem

Given set $\boldsymbol{\Delta_c}$ and **complex** matrix

$$N_c = \begin{pmatrix} M_c & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \text{ with } N_{22} \text{ of size } q_2 \times p_2.$$

Test whether

$$\mu_{\boldsymbol{\Delta_c}}(M_c) \le 1$$

and

$$\|N_{22} + N_{21}\Delta_c(I - M_c\Delta_c)^{-1}N_{12}\| \le 1 \text{ for all } \Delta_c \in \boldsymbol{\Delta_c}.$$

Trick:

$$\|N_{22} + N_{21}\Delta_c(I - M_c\Delta_c)^{-1}N_{12}\| \le 1$$

if and only if

$$\det(I - S(\Delta_c, N_c)\hat{\Delta}_c) \ne 0 \text{ for all } \hat{\Delta}_c \in \mathbb{C}^{p_2 \times q_2},\ \|\hat{\Delta}_c\| < 1.$$

Excursion: Line of Reasoning

Define $\hat{\boldsymbol{\Delta}}_{\boldsymbol{c}} := \{\hat{\Delta}_c \in \mathbb{C}^{p_2 \times q_2} \mid \|\hat{\Delta}_c\| < 1\}$.

For all $\Delta_c \in \boldsymbol{\Delta_c}$

$$\det(I - M_c\Delta_c) \ne 0 \text{ and } \|S(\Delta_c, N_c)\| \le 1$$

if and only if for all $\Delta_c \in \boldsymbol{\Delta_c}$, $\hat{\Delta}_c \in \hat{\boldsymbol{\Delta}}_{\boldsymbol{c}}$

$$\det(I - M_c\Delta_c) \ne 0 \text{ and } \det(I - S(\Delta_c, N_c)\hat{\Delta}_c) \ne 0$$

if and only if for all $\Delta_c \in \boldsymbol{\Delta_c}$, $\hat{\Delta}_c \in \hat{\boldsymbol{\Delta}}_{\boldsymbol{c}}$

$$\det \begin{pmatrix} I - M_c\Delta_c & -N_{12}\hat{\Delta}_c \\ -N_{21}\Delta_c & I - N_{22}\hat{\Delta}_c \end{pmatrix} \ne 0$$

if and only if for all $\Delta_c \in \boldsymbol{\Delta_c}$, $\hat{\Delta}_c \in \hat{\boldsymbol{\Delta}}_{\boldsymbol{c}}$

$$\det \left(I - \begin{pmatrix} M_c & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} \Delta_c & 0 \\ 0 & \hat{\Delta}_c \end{pmatrix} \right) \ne 0.$$

Excursion: Main Loop Theorem

Define set of **extended** matrices

$$\mathbf{\Delta_e} := \left\{ \begin{pmatrix} \Delta_c & 0 \\ 0 & \hat{\Delta}_c \end{pmatrix} :\ \Delta_c \in \mathbf{\Delta_c},\ \hat{\Delta}_c \in \mathbb{C}^{p_2\times q_2},\ \|\hat{\Delta}_c\| < 1 \right\}.$$

Previous structure extended with **one full block**.

Main Loop Theorem
Conditions

$$\mu_{\mathbf{\Delta_c}}(M_c) \leq 1 \text{ and } \|S(\Delta_c, N_c)\| \leq 1 \text{ for all } \Delta_c \in \mathbf{\Delta_c}$$

equivalent to

$$\mu_{\mathbf{\Delta}_e}(N_c) \leq 1.$$

Desired test reduced to just another SSV-test on matrix N_c for extended structure $\mathbf{\Delta_e}$.

Other than Bound 1 - Rescale

SSV-inequality

$$\mu_{\mathbf{\Delta_e}}\left(N_c \begin{pmatrix} \gamma_1 I & 0 \\ 0 & \gamma_2 I \end{pmatrix} \right) \leq \gamma_3$$

equivalent to

$$\mu_{\mathbf{\Delta_c}}(M_c) \leq \frac{\gamma_3}{\gamma_1}$$

and

$$\|S(\Delta_c, N_c)\| \leq \frac{\gamma_3}{\gamma_2} \text{ for all } \Delta_c \in \frac{\gamma_1}{\gamma_3}\mathbf{\Delta_c}.$$

Can investigate trade-off between

uncertainty size and worst norm $\|S(\Delta_c, N_c)\|$

by varying γ_1, γ_2 and computing SSV giving γ_3.

Main Analysis Results in SSV Theory

Theorem

Let $N = \begin{pmatrix} M & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$ be proper, stable.

For all $\Delta \in \mathbf{\Delta}$,

$$(I - M\Delta)^{-1} \in RH_\infty \text{ and } \|S(\Delta, N)\|_\infty \leq 1$$

if and only if

$$\mu_{\mathbf{\Delta}_e}(N(i\omega)) \leq 1 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}.$$

Immediate consequence: Central result.

Corollary
If K stabilizes P, and if

$$\mu_{\mathbf{\Delta}_e}(N(i\omega)) \leq 1 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\},$$

then K achieves robust performance for $S(\Delta, P)$ against $\Delta \in \mathbf{\Delta}$.

Summary

Perturbed controlled system

$$\begin{pmatrix} z_\Delta \\ z_p \end{pmatrix} = \underbrace{\begin{pmatrix} M & N_{12} \\ N_{21} & N_{22} \end{pmatrix}}_{N} \begin{pmatrix} w_\Delta \\ w_p \end{pmatrix}, \quad w_\Delta = \Delta z_\Delta$$

with proper and stable Δ that is **only restricted through** $\Delta(i\omega) \in \mathbf{\Delta_c}$ for all $\omega \in \mathbb{R} \cup \{\infty\}$.

Summary
Controller K stabilizing P achieves

- Robust stability if

$$\mu_{\mathbf{\Delta_c}}(M(i\omega)) \leq 1 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}.$$

- Nominal performance if

$$\|N_{22}(i\omega)\| \leq 1 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}.$$

- Robust performance if

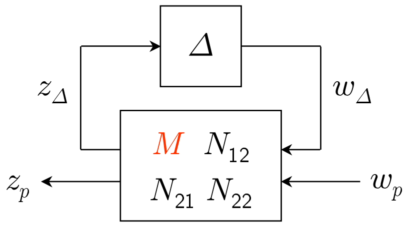
$$\mu_{\mathbf{\Delta_e}}(N(i\omega)) \leq 1 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}.$$

Summary in Pictures

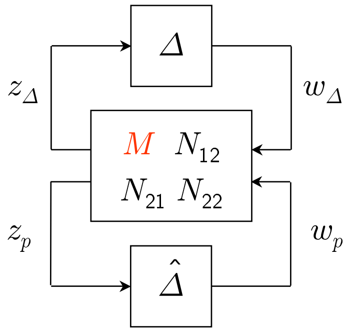
RS: SSV test left-uppoer block M block of N.

NP: SV test on right-lower block N_{22} of N.

RP: SSV test on whole N with extended structure.



RP test is identical to RS test for system



Example

Suppose the controlled system is described with

$$N(s) = \left(\begin{array}{ccc|c} \frac{1}{2s+1} & 1 & \frac{s-2}{2s+4} & \frac{s-0.1}{s+1} \\ -1 & \frac{s}{s^2+s+1} & \frac{1}{(s+1)^2} & 0.1 \\ \frac{3s}{s+5} & \frac{-1}{4s+1} & 1 & \frac{10}{s+4} \\ \hline \frac{1}{s+2} & \frac{0.1}{s^2+s+1} & \frac{s-1}{s+1} & 1 \end{array}\right)$$

and $\boldsymbol{\Delta_c}$ is set of Δ_c with $\|\Delta_c\| < 1$ and

$$\Delta_c = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}, \quad \Delta_1 \in \mathbb{C}^{2\times 2}, \quad \Delta_2 \in \mathbb{C}.$$

Extended set $\boldsymbol{\Delta_e}$ consists of Δ_e with $\|\Delta_e\| < 1$ and

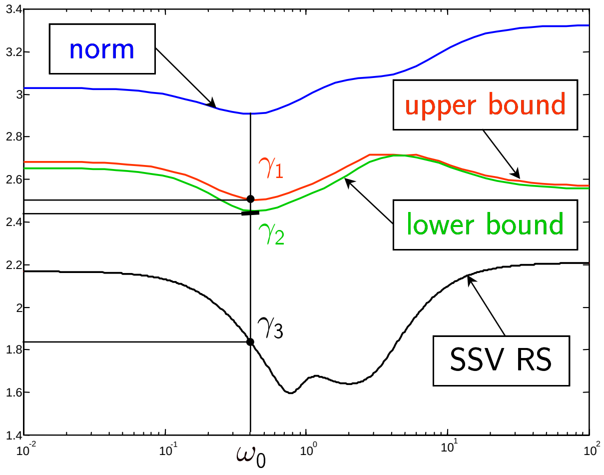
$$\Delta_e = \left(\begin{array}{cc|c} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ \hline 0 & 0 & \hat{\Delta} \end{array}\right), \quad \Delta_1 \in \mathbb{C}^{2\times 2}, \ \Delta_2 \in \mathbb{C}, \ \hat{\Delta} \in \mathbb{C}.$$

RS: Plot $\omega \to \mu_{\boldsymbol{\Delta_c}}(M(i\omega))$.

NP: Plot $\omega \to \|N_{22}(i\omega)\|$.

RP: Plot $\omega \to \mu_{\boldsymbol{\Delta_e}}(N(i\omega))$.

Frequency-Wise Interpretation of SSV Plot of N



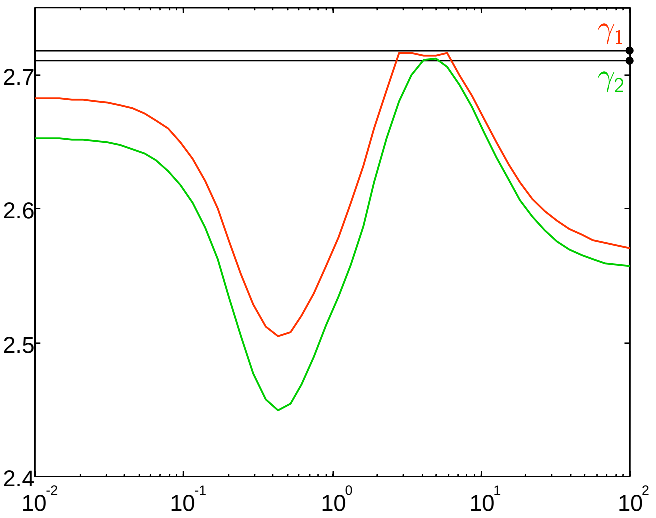
$\mu_{\boldsymbol{\Delta}_e}(N(i\omega_0)) \le \gamma_1$ implies

$$\|S(\Delta_c, N(i\omega_0))\| \le \gamma_1 \quad \text{for all} \quad \Delta_c \in \frac{1}{\gamma_1}\boldsymbol{\Delta_c}.$$

$\mu_{\boldsymbol{\Delta}_e}(N(i\omega_0)) > \gamma_2$ implies

$$\|S(\Delta_c, N(i\omega_0))\| > \gamma_2 \quad \text{for some} \quad \Delta_c \in \frac{1}{\gamma_2}\boldsymbol{\Delta_c}.$$

Interpretation for all Frequencies



Upper bound $\le \gamma_1 = 2.72$ for all frequencies:

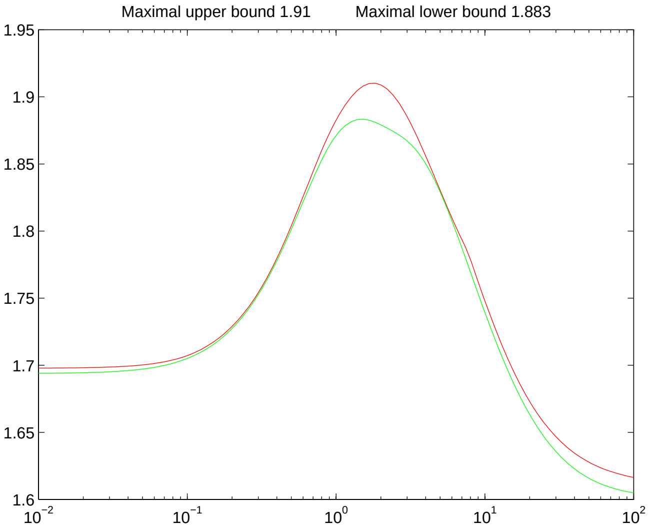
$$\|S(\Delta, N)\|_\infty \le 2.72 \quad \text{for all} \quad \Delta \in 0.367\boldsymbol{\Delta}.$$

Lower bound $> \gamma_2 = 2.71$ for some frequency:

$$\|S(\Delta, N)\|_\infty > 2.71 \quad \text{for some} \quad \Delta \in 0.369\boldsymbol{\Delta}.$$

Bring RP Level below 2

Multiply first block column of N with 0.5 - SSV plot:



Upper bound ≤ 1.92 for all frequencies:

$$\|S(\Delta, N)\|_\infty \leq 1.92 \text{ for all } \Delta \in \underbrace{\frac{0.5}{1.92}}_{\approx\, 0.26} \boldsymbol{\Delta}.$$

Nominal and Robust Controller Design

- Preparation: Riccati equations and inequalities
- State-space computation of H_∞-norm
- Bounded Real Lemma

- State-feedback H_∞ control
- Output-feedback H_∞ control

- Robust controller design is SSV-Minimization
- Scalings/controller iteration as heuristic approach

Summary of course

What can be expected using linear matrix inequalities?

Symmetric Matrices and Properties

Let A be Hermitian: $A = A^*$.

- All eigenvalues are real.
- A positive definite: All eigenvalues positive.
  Write $A > 0$.
- A negative definite: All eigenvalues negative.
  Write $A < 0$.
- A positive semi-definite: All eigenvalues non-negative.
  Write $A \geq 0$.
- A negative semi-definite: All eigenvalues non-positive.
  Write $A \leq 0$.

Semi-Ordering:

$A < B$ means that A, B Hermitian and $A - B < 0$.

Congruence Transformations:

- T full column rank: $A < B$ implies $T^TAT < T^TBT$.
- T non-singular: $A < B$ if and only if $T^TAT < T^TBT$.

Lyapunov Inequalities

Suppose the symmetric X satisfies

$$A^TX + XA < 0.$$

Then A is stable if and only if $X > 0$.

Proof of if: $Ax = \lambda x$, $x \neq 0$, implies

$$\text{Re}(\lambda)(x^*Xx) < 0.$$

Since $X > 0$, infer Re$(\lambda) < 0$.

Proof of only if: Set $P = A^TX + XA$. Then

$$X = -\int_0^\infty e^{A^Tt}Pe^{At}\,dt.$$

Since $P < 0$, formula implies $X > 0$.

Suppose the symmetric X satisfies

$$A^TX + XA \leq 0.$$

If A is stable then $X \geq 0$. **Converse not true.**

ARE and ARI

Data: Matrices A, Q symmetric, $R \geq 0$ of size $n \times n$.

Consider algebraic Riccati inequality (ARI)

$$A^T X + XA + XRX + Q < 0$$

and algebraic Riccati equation (ARE)

$$A^T X + XA + XRX + Q = 0$$

in **symmetric** unknown X.

Main point: Characterization of solvability

Test in terms of A, Q, R whether the ARE or the ARI does have a solution X.

Special solutions of ARE:

X_- stabilizing solution: $A + RX_-$ stable.

X_+ anti-stabilizing solution: $A + RX_-$ anti-stable.

Hamiltonians

Define the Hamiltonian matrix

$$H = \begin{pmatrix} A & R \\ -Q & -A^T \end{pmatrix}.$$

Why? If X solves ARE then

$$H \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \begin{pmatrix} A + RX & R \\ 0 & -(A + RX)^T \end{pmatrix}.$$

Immediate consequence:

If ARE has stabilizing solution X_-, then H does not have eigenvalues on imaginary axis.

If ARE has anti-stabilizing solution X_+, then H does not have eigenvalues on imaginary axis.

(A, R) is controllable: Converse holds as well!

Main Result: (A, R) Controllable

Suppose (A, R) is **controllable**. Equivalent are

- $\begin{pmatrix} A & R \\ -Q & -A^T \end{pmatrix}$ has no eigenvalues on imaginary axis.
- ARE has a stabilizing solution X_-.
- ARE has a antistabilizing solution X_+.
- ARI has a solution.

Any solution X of ARE or ARI satisfies

$$X_- \leq X \leq X_+.$$

Stabilizing and anti-stabilizing solutions are unique.

Among all solutions of ARE (ARI), the

- Stabilizing solution is smallest
- Anti-stabilizing solution is largest.

If (A, R) is only stabilizable, X_+ does generally not exist.

All the other statements remain valid.

Main Result: (A, R) Stabilizable

Suppose (A, R) is **stabilizable**. Equivalent are

- $\begin{pmatrix} A & R \\ -Q & -A^T \end{pmatrix}$ has no eigenvalues on imaginary axis.
- ARE has a stabilizing solution X_-.
- ARI has a solution.

Any solution X of ARE or ARI satisfies

$$X_- \leq X.$$

The stabilizing solution is unique.

Similar result if $(-A, R)$ is stabilizable. Trick:

$$\begin{aligned} & A^T X + XA + XRX + Q = \\ & \quad = (-A^T)(-X) + (-X)(-A) + (-X)R(-X) + Q. \end{aligned}$$

Reduced to result above.

Construction of (Anti) Stabilizing Solution

Schur decomposition: Can compute unitary T with

$$T^*HT = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}$$

where

$$M_{11} \text{ is of size } n \times n \text{ and stable.}$$

Partition T into four $n \times n$ blocks as

$$T = \begin{pmatrix} U & * \\ V & * \end{pmatrix}.$$

Can show that U is non-singular, and that

$$X_- = VU^{-1}$$

is stabilizing solution of ARE.

If M_{11} has all its eigenvalues in $\mathbb{C}^+$, same construction leads to anti-stabilizing solutin X_+.

Monotonicity

Suppose

$$A^TX + XA + XRX + Q = P.$$

Subtract

$$A^TZ + ZA + ZRZ + Q = 0$$

to get

$$(A + RZ)(X - Z) + (X - Z)(A + RZ)^T + \\ + (X - Z)R(X - Z) = P.$$

If $P = 0$ or $P < 0$, obtain with $R \geq 0$

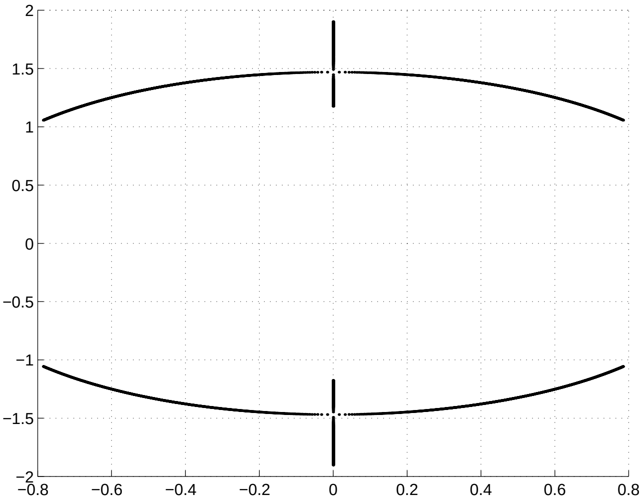
$$(A + RZ)(X - Z) + (X - Z)(A + RZ)^T \leq 0.$$

$A + RZ$ is stable: Then $X - Z \geq 0$.

$A + RZ$ is anti-stable: Then $X - Z \leq 0$.

Example: Eigenvalues of Hamiltonian

```
A=[0 1;1 1];R=[0 0;0 1];Q=[-2 1;1 4];
for fa=linspace(1,2,1000);
   H=[A R;-Q*fa -A'];plot(eig(H),'.');
end;
```



Example: Compute Solution

```
[n,n]=size(A);H=[A R;-Q -A'];[T,D]=eig(H);
Z=[];
for j=1:2*n;
   if real(D(j,j))<0;Z=[Z T(:,j)];end;
end;
U=Z(1:n,:);V=Z(n+1:2*n,:);X=V*U^(-1);

A'*X+X*A+X*R*X+Q =
  1.0e-014 *
   -0.5329    0.1776
    0.3553    0.5329
eig(A+R*X) =
  -0.7849 + 1.0564i
  -0.7849 - 1.0564i
X =
   -2.7189   -2.7321
   -2.7321   -2.5697
eig(X) =
   -5.3774
    0.0888
```

Computation of H_∞-Norm

Suppose $M(s) = C(sI - A)^{-1}B$ stable. Do not compute

$$\|M\|_\infty = \sup_{\omega \in \mathbb{R} \cup \{\infty\}} \|M(i\omega)\|$$

by solving optimization problem.

Alternative: For $\gamma > 0$, test

$$\|C(sI - A)^{-1}B\|_\infty < \gamma.$$

in terms of A, B, C, γ. How?

With substitution

$$B \to \frac{1}{\gamma}B \text{ or } C \to \frac{1}{\gamma}C$$

reduced to bound 1.

Reduce to Test on Hamiltonian

$\|M\|_\infty < 1$ if and only if

$$\lambda_{\max}\left(M(s)^* M(s) - I\right) < 0, \quad \forall s \in \mathbb{C}^0$$

if and only if

$$\lambda_{\max}\left(M(-s)^T M(s) - I\right) < 0, \quad \forall s \in \mathbb{C}^0$$

if and only if

$$\det\left(M(-s)^T M(s) - I\right) \neq 0, \quad \forall s \in \mathbb{C}^0$$

if and only if

$$\det\left(\begin{array}{cc|c} A - sI & 0 & B \\ -C^T C & -A^T - sI & 0 \\ \hline 0 & B^T & -I \end{array}\right) \neq 0, \quad \forall s \in \mathbb{C}^0$$

if and only if (Schur)

$$\det\left(\begin{pmatrix} A & BB^T \\ -C^T C & -A^T \end{pmatrix} - sI\right) \neq 0, \quad \forall s \in \mathbb{C}^0$$

if and only if

$$\underbrace{\begin{pmatrix} A & BB^T \\ -C^T C & -A^T \end{pmatrix}}_{\text{Hamiltonian}} \text{ has no eigenvalues in } \mathbb{C}^0.$$

H_∞-Bound In Terms of Hamiltonian

Theorem

Suppose $M(s) = C(sI - A)^{-1}B$ with A stable. Then

$$\|M\|_\infty < 1$$

if and only if

$$H = \begin{pmatrix} A & BB^T \\ -C^T C & -A^T \end{pmatrix} \text{ has no eigenvalues in } \mathbb{C}^0.$$

- Test bound $\gamma > 0$ with
$$BB^T \to \frac{1}{\gamma}BB^T \text{ or } C^T C \to \frac{1}{\gamma}C^T C.$$
- Not difficult to generalize to $D \neq 0$.
- Have actually obtained:

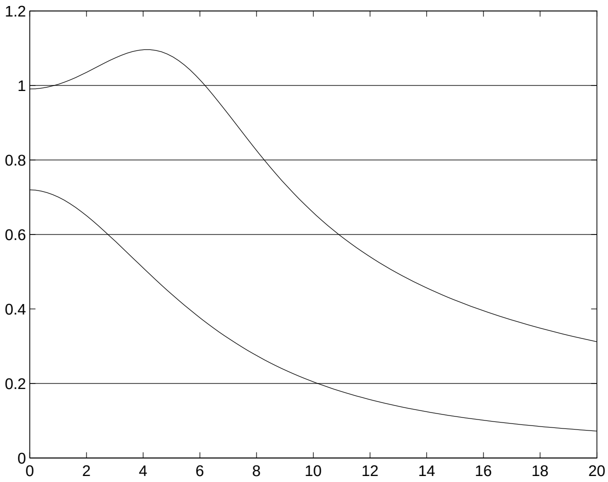
Set of eigenvalues of H on the imaginary axis

equals

Set of all $i\omega$ for which 1 is a singular value of $M(i\omega)$.

SVs of Frequency Response and EVs of Hamiltonian

Plot of singular values of $C(i\omega I - A)^{-1}B$:



Eigenvalues of Hamiltonian:

1.2	3.9	-3.9	$-1.9 - 4.4i$	$-1.9 + 4.4i$	$1.9 - 4.4i$	$1.9 + 4.4i$
1	$-6.2i$	$6.2i$	-3.5	3.5	$-0.9i$	$0.9i$
0.8	$-8.3i$	$8.3i$	-4.1	4.1	-1.8	1.8
0.6	$-10.9i$	$10.9i$	-5.6	5.6	$-2.8i$	$2.8i$
0.2	$-30.7i$	$30.7i$	-10.5	10.5	$-10.2i$	$10.2i$

The Bounded Real Lemma

Let $M(s) = C(sI - A)^{-1}B$ with A stable.

Theorem
$\|M\|_\infty < 1$ if and only if exists solution of ARI

$$A^TX + XA + XBB^TX + C^TC < 0$$

if and only exists stabilizing solution of ARE

$$A^TX + XA + XBB^TX + C^TC = 0.$$

Dualize: $\|C(sI - A)^{-1}B\|_\infty = \|B^T(sI - A^T)^{-1}C^T\|_\infty$.

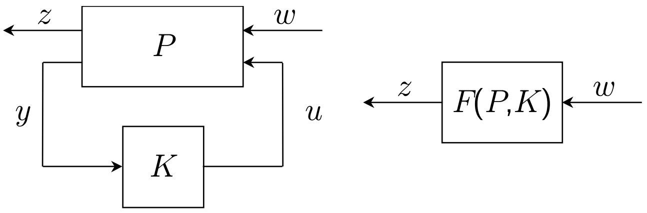
Theorem
$\|M\|_\infty < 1$ if and only if exists solution of ARI

$$AY + YA^T + BB^T + YC^TCY < 0$$

if and only exists stabilizing solution of ARE

$$AY + YA^T + BB^T + YC^TCY = 0.$$

General Setup for Nominal Controller Design



Only Performance Channel - No Uncertainty

Minimize H_∞ norm of transfer matrix $z = S(P,K)w$ by designing stabilizing controller K.

Includes many specific configurations:

- Shaping closed-loop transfer matrices
- Match a given desired model
- Signal-based specifications (disturbance and noise suppression, command tracking)
- One- and two-degree of freedom controllers

Weightings absorbed in P.

System Descriptions

Generalized Plant:

$$P\left\{\begin{aligned} \dot{x} &= Ax + B_1w + B_2u\\ z &= C_1x + D_{11}w + D_{12}u\\ y &= C_2x + D_{21}w \end{aligned}\right.$$

Controller:

$$K\left\{\begin{aligned} \dot{x}_K &= A_Kx_K + B_Ky\\ u &= C_Kx_K + D_Ky \end{aligned}\right.$$

Controlled System:

$$S(P,K)\left\{\begin{aligned} \dot{\xi} &= \mathcal{A}\xi + \mathcal{B}w\\ z &= \mathcal{C}\xi + \mathcal{D}w \end{aligned}\right.$$

with $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ given by

$$\left(\begin{array}{cc|c} A + B_2D_KC_2 & B_2C_K & B_1 + B_2D_KD_{21}\\ B_KC_2 & A_K & B_KD_{21}\\ \hline C_1 + D_{12}D_KC_2 & D_{12}C_K & D_{11} + D_{12}D_KD_{21} \end{array}\right).$$

Formulation of the H_∞ Problem

Minimize

$$\|S(P,K)\|_\infty = \|\mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}\|_\infty$$

over all controllers such that

$$\mathcal{A} \text{ has all its eigenvalues in } \mathbb{C}^-.$$

Solved via **suboptimal** H_∞ Problem

For fixed number $\gamma > 0$, test whether there exists a controller such that

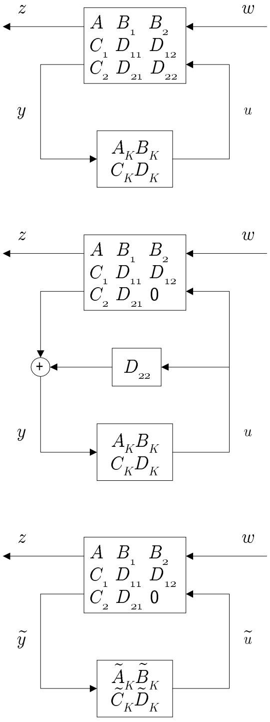
$$\|\mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}\|_\infty < \gamma, \quad \lambda(\mathcal{A}) \subset \mathbb{C}^-.$$

Any such controller is called γ-suboptimal.

Compute minimal possible γ by bisection.

Why can we assume $D_{22} = 0$?

Can push D_{22} to controller:



Our Hypotheses

Our derivation based on

1. (A, B_2) stabilizable, (A, C_2) detectable.
2. $D_{12}^T \begin{pmatrix} C_1 & D_{12} \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix}$, $\begin{pmatrix} B_1 \\ D_{21} \end{pmatrix} D_{21}^T = \begin{pmatrix} 0 \\ I \end{pmatrix}$.
3. (A, C_1) observable, (A, B_1) controllable.
4. $D_{11} = 0$ and $D_K = 0$ (strictly proper controllers). Implies $\mathcal{D} = 0$.

DGKF Assumptions: Hypotheses 3 relaxed to

- (A, C_1) has no unobservable modes in $\mathbb{C}^0$
- (A, B_1) has no uncontrollable modes in $\mathbb{C}^0$

ARI Solution of Static State-Feedback Problem

Control with $u = Fx$ to obtain closed-loop system

$$\dot{x} = (A + B_2 F)x + B_1 w, \quad z = (C_1 + D_{12} F)x.$$

There exists a gain F such that

$$\lambda(A + B_2 F) \subset \mathbb{C}^-$$
$$\|(C_1 + D_{12} F)(sI - A - B_2 F)^{-1} B_1\|_\infty < 1$$

if and only if the ARI

$$AY + YA^T + B_1 B_1^T - B_2 B_2^T + Y C_1^T C_1 Y < 0$$

has a solution $Y > 0$.

With any positive definite solution Y of the ARI, the gain

$$F = -B_2^T Y^{-1}$$

defines a 1-suboptimal static state-feedback controller.

Proof

Exists F with

$$\lambda(A+B_2F) \subset \mathbb{C}^-, \ \|(C_1+D_{12}F)(sI-A-B_2F)^{-1}B_1\|_\infty < 1$$

if and only if exists F and Y with

$$Y > 0, \quad (A + B_2 F)Y + Y(A + B_2 F)^T + B_1 B_1^T + \\ + Y(C_1 + D_{12} F)^T (C_1 + D_{12} F) Y < 0$$

if and only if exists F and Y with

$$Y > 0, \quad AY + YA^T + B_1 B_1^T + Y C_1^T C_1 Y - \\ - B_2 B_2^T + (FY + B_2^T)^T (FY + B_2^T) < 0$$

if and only if exists Y with

$$Y > 0, \quad AY + YA^T + B_1 B_1^T - B_2 B_2^T + Y C_1^T C_1 Y < 0.$$

ARE Solution of Static State-Feedback Problem

Recall that (A, C_1) observable. Hence ARI

$$AY + YA^T + B_1B_1^T - B_2B_2^T + YC_1^TC_1Y < 0$$

has solution $Y > 0$ if and only if the ARE

$$AY + YA^T + B_1B_1^T - B_2B_2^T + YC_1^TC_1Y = 0$$

has an anti-stabilizing solution Y_+ that satisfies $Y_+ > 0$.

There exists a gain F such that

$$\lambda(A + B_2F) \subset \mathbb{C}^-$$

$$\|(C_1 + D_{12}F)(sI - A - B_2F)^{-1}B_1\|_\infty < 1$$

if and only if the anti-stabilizing solution Y_+ of the ARE

$$AY + YA^T + B_1B_1^T - B_2B_2^T + YC_1^TC_1Y = 0$$

exists and is positive definite.

With Y_+, the gain

$$F := -B_2^TY_+^{-1}$$

defines a 1-suboptimal static state-feedback controller.

Output-Feedback Problem: Derivation

Controller A_K (with same size as A), B_K, C_K satisfies

$$\lambda(\mathcal{A}) \subset \mathbb{C}^- \quad \text{and} \quad \|\mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B}\|_\infty < 1$$

if and only if exists $\mathcal{X}$ with

$$\mathcal{X} > 0, \quad \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} + \mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X} + \mathcal{C}^T\mathcal{C} < 0.$$

Partition $\mathcal{X}$ and $\mathcal{X}^{-1}$ according to $\mathcal{A}$:

$$\mathcal{X} = \begin{pmatrix} X & U \\ U^T & * \end{pmatrix}, \quad \mathcal{X}^{-1} = \begin{pmatrix} Y & V \\ V^T & * \end{pmatrix}.$$

Can assume that U, V are non-singular. We infer that

$$\mathcal{R} = \begin{pmatrix} X & U \\ I & 0 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} I & 0 \\ Y & V \end{pmatrix} \quad \text{satisfy} \quad \mathcal{S}\mathcal{X} = \mathcal{R}.$$

Includes relation $XY + UV^T = I$ which is required later.

Left- and right-multiply inequalities with $\mathcal{S}$ and $\mathcal{S}^T$:

$$\mathcal{R}\mathcal{S}^T > 0, \quad \mathcal{S}\mathcal{A}^T\mathcal{R}^T + \mathcal{R}\mathcal{A}\mathcal{S}^T + \mathcal{R}\mathcal{B}\mathcal{B}^T\mathcal{R}^T + \mathcal{S}\mathcal{C}^T\mathcal{C}\mathcal{S}^T < 0.$$

Output-Feedback Problem: Derivation

With the substitution

$$L = X^{-1}UB_K \quad \text{and} \quad F = C_KV^TY^{-1}$$

a straightforward computation leads to

$$\mathcal{S}\mathcal{R}^T = \begin{pmatrix} X & I \\ I & Y \end{pmatrix}$$

$$\mathcal{R}\mathcal{A}\mathcal{S}^T = \left(\begin{array}{c|c} X(A + LC_2) & UA_KV^T + X(A + LC_2 + B_2F)Y \\ \hline A & (A + B_2F)Y \end{array}\right)$$

$$\mathcal{R}\mathcal{B} = \begin{pmatrix} X(B_1 + LD_{21}) \\ B_1 \end{pmatrix}$$

$$\mathcal{C}\mathcal{S}^T = \begin{pmatrix} C_1 & (C_1 + D_{12}F)Y \end{pmatrix}.$$

Output-Feedback Problem: Derivation

Two inequalities

$$\mathcal{R}\mathcal{S}^T > 0, \quad \mathcal{S}\mathcal{A}^T\mathcal{R}^T + \mathcal{R}\mathcal{A}\mathcal{S}^T + \mathcal{R}\mathcal{B}\mathcal{B}^T\mathcal{R}^T + \mathcal{S}\mathcal{C}^T\mathcal{C}\mathcal{S}^T < 0$$

equivalent to

$$\begin{pmatrix} X & I \\ I & Y \end{pmatrix} > 0, \quad \begin{pmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{pmatrix} < 0$$

with

$$R_{11} = (A + LC_2)^TX + X(A + LC_2) + \\ + X(B_1 + LD_{21})(B_1 + LD_{21})^TX + C_1^TC_1$$

$$R_{22} = (A + B_2F)Y + Y(A + B_2F)^T + B_1B_1^T + \\ + Y(C_1 + D_{12}F)^T(C_1 + D_{12}F)Y$$

$$R_{12} = UA_KV^T + XAY + \\ + UB_KC_2Y + XB_2C_KV^T + XB_1B_1^T + C_1^TC_1Y + A^T.$$

Implies: $R_{11} < 0$, $R_{22} < 0$.

Is implied by: $R_{11} < 0$, $R_{22} < 0$, $R_{12} = 0$.

ARI Solution of Output-Feedback Problem

Exists controller A_K, B_K, C_K that satisfies

$$\lambda(\mathcal{A}) \subset \mathbb{C}^- \quad \text{and} \quad \|\mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{B}\|_\infty < 1$$

if and only if exist X and Y that satisfy

$$\begin{pmatrix} X & I \\ I & Y \end{pmatrix} > 0$$

$$A^TX + XA + C_1^TC_1 - C_2^TC_2 + XB_1B_1^TX < 0$$

$$AY + YA^T + B_1B_1^T - B_2B_2^T + YC_1^TC_1Y < 0.$$

With X and Y, construct 1-suboptimal controller as follows:

Choose non-singular U, V with $XY + UV^T = I$ and take

$$B_K = -U^{-1}C_2^T, \quad C_K = -B_2^TV^{-T}, \quad A_K \text{ with } R_{12} = 0.$$

Output-Feedback Control: Riccati Equation Solution

Theorem

Exists 1-suboptimal H_∞ controller if and only if ARE's

$$A^TX + XA + XB_1B_1^TX + C_1^TC_1 - C_2^TC_2 = 0$$

$$AY + YA^T + YC_1^TC_1Y + B_1B_1^T - B_2B_2^T = 0$$

have anti-stabilizing solutions X_+ and Y_+ that satisfy the coupling condition

$$\begin{pmatrix} X_+ & I \\ I & Y_+ \end{pmatrix} > 0.$$

Construct controller as follows:

Choose non-singular U, V with $X_+Y_+ + UV^T = I$ and take

$$A_K = -U^{-1}[A^T + X_+AY_+ + X_+(B_1B_1^T - B_2B_2^T) + \\ + (C_1^TC_1 - C_2^TC_2)Y_+]V^{-T}$$

$$B_K = -U^{-1}C_2^T, \quad C_K = -B_2^TV^{-T}.$$

DGKF Solution of H_∞ Problem

Exist γ-suboptimal controller if and only if

- The unique solution P of the ARE

$$AP + PA^T + P(\frac{1}{\gamma^2}C_1^TC_1 - C_2^TC_2)P + B_1B_1^T = 0$$

$$\text{with stable} \quad A + P(\frac{1}{\gamma^2}C_1^TC_1 - C_2^TC_2)$$

exists and satisfies $P \geq 0$.

- The unique solution Q of the ARE

$$A^TQ + QA + Q(\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T)Q + C_1^TC_1 = 0$$

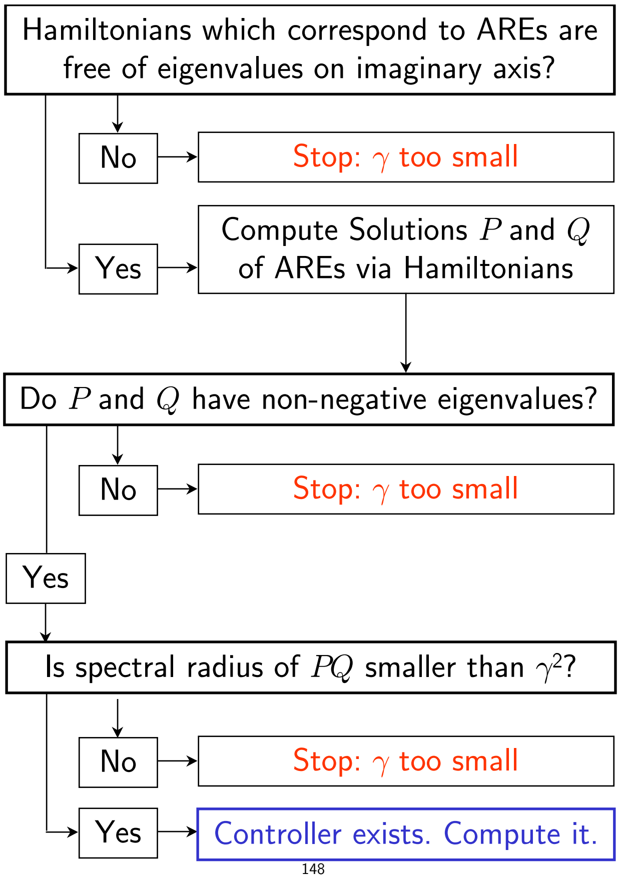
$$\text{with stable} \quad A + (\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T)Q$$

exists and satisfies $Q \geq 0$.

- P and Q satisfy coupling condition

$$\rho(PQ) < \gamma^2.$$

Test Conditions



Hamiltonians which correspond to AREs are free of eigenvalues on imaginary axis?

No → Stop: γ too small

Yes → Compute Solutions P and Q of AREs via Hamiltonians

Do P and Q have non-negative eigenvalues?

No → Stop: γ too small

Yes

Is spectral radius of PQ smaller than γ^2?

No → Stop: γ too small

Yes → Controller exists. Compute it.

DGKF: How to Compute Controller?

Compute close to optimal γ by bisection.

With P, Q and $Z = (I - \frac{1}{\gamma^2}PQ)^{-1}$, **one** γ-suboptimal controller is

$$\begin{aligned} A_K &= A + \frac{1}{\gamma^2}B_1B_1^TQ - B_2B_2^TQ - ZPC_2^TC_2 \\ B_K &= ZPC_2^T \\ C_K &= -B_2^TQ. \end{aligned}$$

Important Remark

If γ is close to optimal value, if often happens that

$$I - \frac{1}{\gamma^2}QP \quad \text{is close to singular.}$$

Then inversion to compute Z is ill-conditioned.

Leads to a fast pole of the controller.

Hence: Don't get too close to optimal value.

Required Hypotheses for ARE Solution

Command `hinfsyn` requires following propeties:

1. (A, B_2) stabilizable, (A, C_2) detectable.
2. D_{12} full column rank, D_{21} full row rank.
3. Ranks of
$$\begin{pmatrix} A - i\omega I & B_2 \\ C_1 & D_{12} \end{pmatrix}, \quad \begin{pmatrix} A - i\omega I & B_1 \\ C_2 & D_{21} \end{pmatrix}$$
do not drop for $\omega \in \mathbb{R}$.

Enforce Hypotheses 2 and 3

With $\epsilon > 0$, solve problem for

$$\begin{pmatrix} C_1 & D_{12} \end{pmatrix} \to \begin{pmatrix} C_1 & D_{12} \\ \epsilon I & 0 \\ 0 & \epsilon I \end{pmatrix}, \quad \begin{pmatrix} B_1 \\ D_{21} \end{pmatrix} \to \begin{pmatrix} B_1 & \epsilon I & 0 \\ D_{21} & 0 & \epsilon I \end{pmatrix}$$

$$D_{11} \to \begin{pmatrix} D_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For small $\epsilon > 0$, original and perturbed problem equivalent.

Example

`himat` example in μ-tools. Generalized plant with

8 states
4 disturbance inputs/controlled outputs
2 control inputs/measured outputs

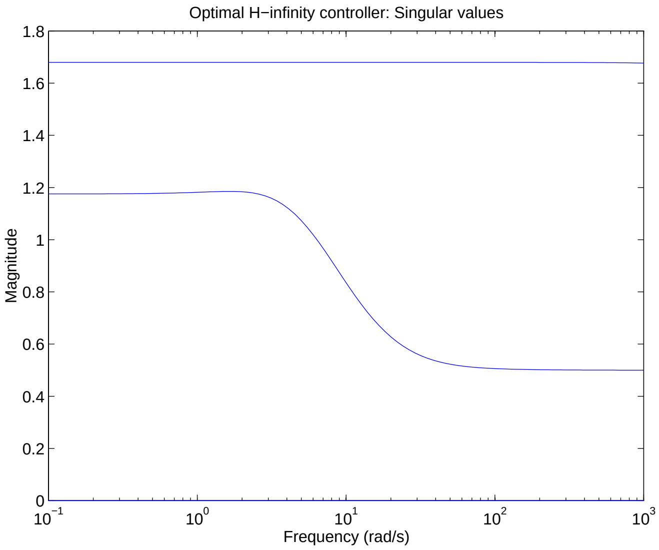
Analyse closed-loop with loop-shaping controller.

Typical Run of `hinfsyn`

```
[k,g]=hinfsyn(sys,nm,nc,0,10,0.01);

Test bounds:     0.0000 <  gamma  <=     10.0000

 gamma  hamx_eig  xinf_eig  hamy_eig   yinf_eig    nrho_xy    p/f
 10.000  2.3e-002  1.2e-007  2.3e-002   0.0e+000   0.0223     p
  5.000  2.3e-002  1.3e-007  2.3e-002   0.0e+000   0.0908     p
  2.500  2.3e-002  1.3e-007  2.3e-002   0.0e+000   0.3913     p
  1.250  2.3e-002  1.5e-007  2.3e-002   0.0e+000   2.2947#    f
  1.875  2.3e-002  1.3e-007  2.3e-002  -3.0e-014   0.7572     p
  1.562  2.3e-002  1.4e-007  2.3e-002   0.0e+000   1.1977#    f
  1.719  2.3e-002  1.4e-007  2.3e-002   0.0e+000   0.9373     p
  1.641  2.3e-002  1.4e-007  2.3e-002  -3.0e-014   1.0547#    f
  1.680  2.3e-002  1.4e-007  2.3e-002   0.0e+000   0.9932     p
  1.660  2.3e-002  1.4e-007  2.3e-002   0.0e+000   1.0232#    f
  1.670  2.3e-002  1.4e-007  2.3e-002   0.0e+000   1.0080#    f

 Gamma value achieved:     1.6797
```

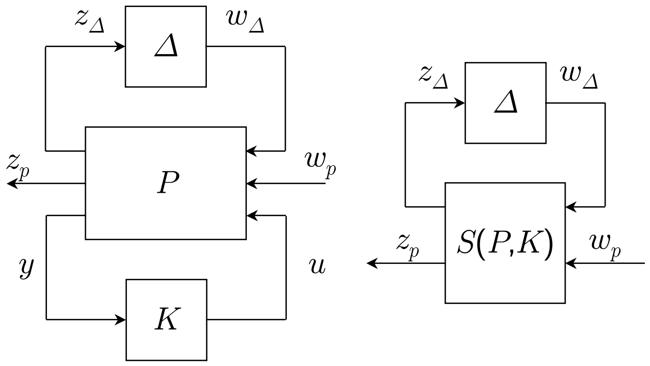
Result of H_∞ Optimization

Analyse closed-loop with optimal H_∞-controller.



Typical: Largest singular value is **flat**.

General Setup for Robust Controller Design



Uncertainy and Performance Channels

Find controller K that stabilizes P and achieves robust performance.

Uncertainty weightings and performance weightings supposed to be absorbed in P.

Structure set for uncertainty is $\boldsymbol{\Delta_c}$.

SSV Synthesis

Define **extended** value set

$$\boldsymbol{\Delta_e} := \left\{ \begin{pmatrix} \Delta_c & 0 \\ 0 & \hat{\Delta}_c \end{pmatrix} : \ \Delta_c \in \boldsymbol{\Delta_c}, \ \hat{\Delta}_c \in \mathbb{C}^{p_2 \times q_2}, \ \|\hat{\Delta}_c\| < 1 \right\}.$$

SSV Design Problem
Find controller K that stabilizes P and achieves

$$\mu_{\boldsymbol{\Delta_e}}(S(P,K)(i\omega)) \leq 1 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}.$$

No general solution available.

Hard to devise algorithms since not even the the SSV is easily computable.

Remedy: Try to reduce computable upper bound below 1. Are on the safe side!

Recall Structure of Scalings

Structure of Δ_c in $\boldsymbol{\Delta_c}$:

$$\Delta_c = \begin{pmatrix} p_1 I & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & p_{n_r} I & & & & & & \\ & & & \delta_1 I & & & & & \\ & & & & \ddots & & & & \\ & & & & & \delta_{n_c} I & & & \\ & & & & & & \Delta_1 & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & \Delta_{n_f} \end{pmatrix}.$$

Structure of D in $\boldsymbol{D}$:

$$D = \begin{pmatrix} P_1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & P_{n_r} & & & & & & \\ & & & D_1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & D_{n_c} & & & \\ & & & & & & d_1 I & & \\ & & & & & & & \ddots & \\ & & & & & & & & d_{n_f} I \end{pmatrix}$$

where

- P_j, D_j Hermitian positive definite,
- d_j positive real scalars.

Remark. Scalings **neglect** that p_j is real!

Upper Bound Synthesis

To **extended** value set $\boldsymbol{\Delta}_e$ correspond **extended** scalings:

$$\boldsymbol{D}_e := \left\{ \begin{pmatrix} D & 0 \\ 0 & I \end{pmatrix} > 0 \mid D \in \boldsymbol{D} \right\}.$$

Recall that

$$\mu_{\boldsymbol{\Delta_e}}(S(P,K)(i\omega)) \leq \inf_{D \in \boldsymbol{D_e}} \|D^{-1} S(P,K)(i\omega) D\|.$$

Upper Bound Design Problem
Find controller K that stabilizes P and achieves

$$\inf_{D \in \boldsymbol{D}_e} \|D^{-1} S(P,K)(i\omega) D\| < 1 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}.$$

Inequality equivalent to:

There exists function $D(\omega) \in \boldsymbol{D}_e$ such that

$$\|D(\omega)^{-1} S(P,K)(i\omega) D(\omega)\| < 1 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}.$$

Upper Bound Synthesis

Find controller K that stabilizes P and for which there exists a function $D(\omega) \in \boldsymbol{D}_e$ such that

$$\sup_{\omega \in \mathbb{R} \cup \{\infty\}} \|D(\omega)^{-1} S(P,K)(i\omega) D(\omega)\| < 1.$$

Minimize

$$\sup_{\omega \in \mathbb{R} \cup \{\infty\}} \|D(\omega)^{-1} S(P,K)(i\omega) D(\omega)\|$$

over all controllers K that stabilize P and
over all functions $D(\omega) \in \boldsymbol{D}_e$.

If minimum smaller than 1: Done.

Have found controller that achieves robust performance.

If minimum larger than 1: Procedure fails.

May or may not exist controller that achieves RP.

Scalings-Controller Iteration

Minimizing

$$\sup_{\omega \in \mathbb{R} \cup \{\infty\}} \|D(\omega)^{-1} S(P,K)(i\omega) D(\omega)\|$$

over K and $D(\omega)$ **together** still not possible.

Instead, iterate the two steps:

Fix scaling function $D(\omega)$, minimize over K.
Fix controller K, minimize over $D(\omega)$.

Procedure is called D/K-iteration.

In each step, value of

$$\sup_{\omega \in \mathbb{R} \cup \{\infty\}} \|D(\omega)^{-1} S(P,K)(i\omega) D(\omega)\|$$

is reduced. Hence it converges.

Stopping criterion: If value < 1.

Details of D/K-Iteration: Start

Set $D_1(\omega) = I$.

Minimize

$$\sup_{\omega \in \mathbb{R} \cup \{\infty\}} \|D_1(\omega)^{-1} S(P,K)(i\omega) D_1(\omega)\| = \|S(P,K)\|_\infty$$

over all K that stabilize P.

H_∞ problem. Minimum $\approx \gamma_1$, controller K_1.

After k steps: Have found $D_k(\omega)$, K_k with

$$\sup_{\omega \in \mathbb{R} \cup \{\infty\}} \|D_k(\omega)^{-1} S(P,K_k)(i\omega) D_k(\omega)\| = \gamma_k.$$

Optimization over Scaling

SSV analysis problem: At each $\omega \in \mathbb{R} \cup \{\infty\}$, calculate

$$\inf_{D \in \boldsymbol{D}_e} \|D^{-1} S(P, K_k)(i\omega) D\|$$

and an (almost) optimal scaling $D_{k+1}(\omega)$.

Observe:

$$\|D_{k+1}(\omega)^{-1} S(P, K_k)(i\omega) D_{k+1}(\omega)\| \leq$$
$$\leq \|D_k(\omega)^{-1} S(P, K_k)(i\omega) D_k(\omega)\| \leq \gamma_k$$

and hence

$$\sup_{\omega \in \mathbb{R} \cup \{\infty\}} \|D_{k+1}(\omega)^{-1} S(P, K_k)(i\omega) D_{k+1}(\omega)\| \leq \gamma_k.$$

Optimization over Controller

Cannot optimize over K with arbitrary $D_{k+1}(\omega)$.

Find real rational proper stable $\hat{D}(s)$ with proper stable $\hat{D}(s)^{-1}$ such that

$$\|D_{k+1}(\omega) - \hat{D}(i\omega)\| \approx 0 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}.$$

Find (almost) optimal K_{k+1} for

$$\inf_{K \text{ stabilizes } P} \|\hat{D}^{-1} S(P, K) \hat{D}\|_\infty.$$

Is standard H_∞ problem!

For good scalings approximation

$$\gamma_{k+1} = \sup_{\omega \in \mathbb{R} \cup \{\infty\}} \|D_{k+1}(\omega)^{-1} S(P, K_{k+1})(i\omega) D_{k+1}(\omega)\| \leq$$
$$\leq \sup_{\omega \in \mathbb{R} \cup \{\infty\}} \|D_{k+1}(\omega)^{-1} S(P, K_k)(i\omega) D_{k+1}(\omega)\| \leq \gamma_k$$

and hence value is improved. Iterate.

Comments

- The μ-tools support the fitting of $D(\omega)$ with rational transfer matrices $\hat{D}(i\omega)$.

- Value of the upper bound is not increased in each step.

  Stop if it cannot be significantly decreased.

  There is no guarantee for optimality whatsoever!

- Order of $\hat{D}$ is provided by user.

  Controller K that (approximately) minimizes

  $$\|\hat{D}^{-1} S(P, K) \hat{D}\|_\infty$$

  is given by

  $$2 \times \text{order}(\hat{D}) + \text{order}(P).$$

  Keep the order of $\hat{D}$ small!

- If order of controller too large, perform model reduction.

  Chapters 7/9 in Zhou, Doyle, Glover,

  Chapter 11 in Skogestad, Postlethwaite.

Example

Continue `himat` example from μ tools.

One full block uncertainty: 2×2

One performance block: 2×2

Extended block structure

$$\boldsymbol{\Delta}_e := \left\{ \Delta_c = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \mid \Delta_k \in \mathbb{C}^{2\times 2}, \|\Delta_c\| < 1 \right\}.$$

Extended scalings

$$\boldsymbol{D}_e := \left\{ D = \begin{pmatrix} d\, I_2 & 0 \\ 0 & I_2 \end{pmatrix} : d \in \mathbb{R},\ d > 0 \right\}.$$

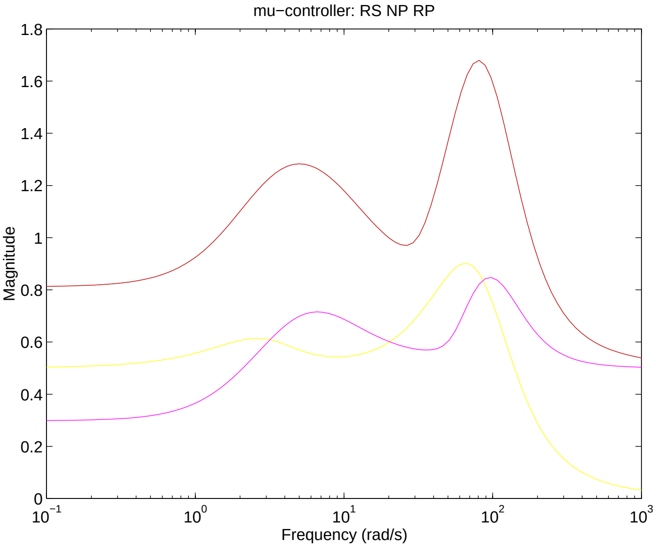
Have performed first step: Unscaled H_∞ design.

SSV Analysis for H_∞ Controller

Robust stability: yellow

Nominal performance: violet

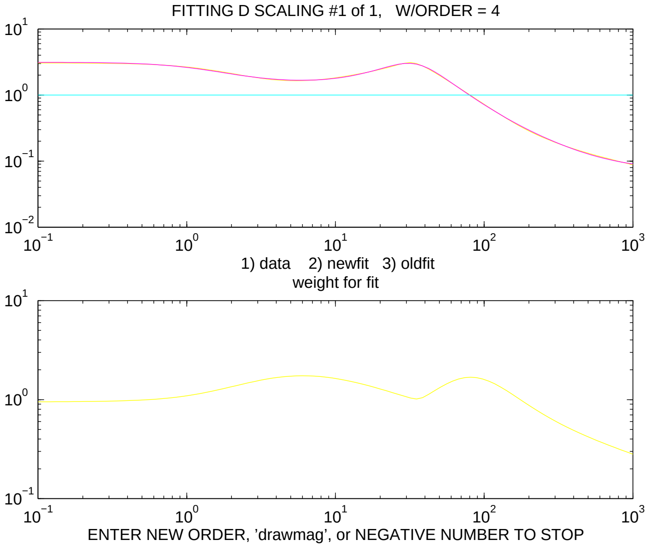
Robust performance: red



Fit Scalings

Frequency by frequency scalings: yellow

Rational fit: violet



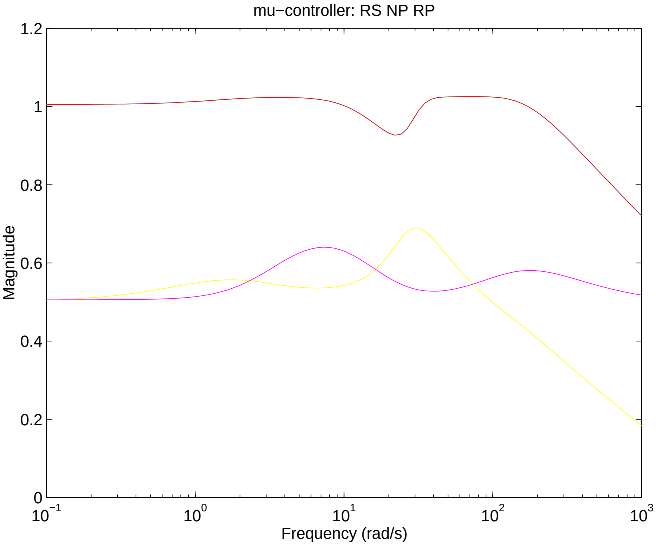
Perfect fit.

H_∞ Design and SSV Analysis

Robust stability: yellow

Nominal performance: violet

Robust performance: red

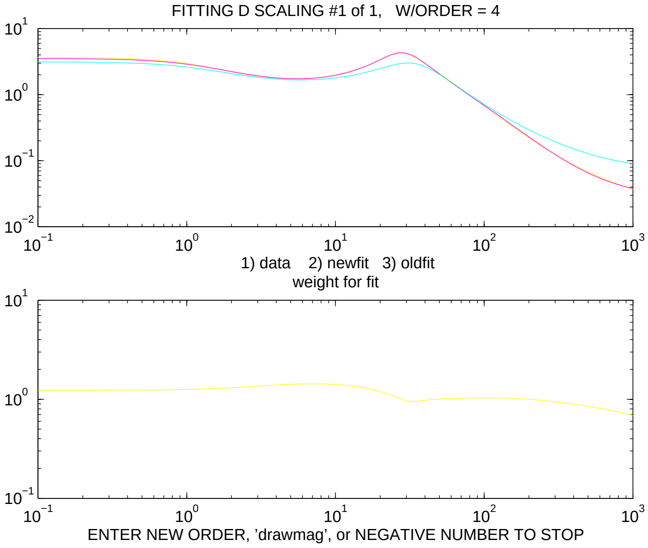


Robust performance not achieved. Continue.

Fit Scalings

Frequency by frequency scalings: yellow

Rational fit: violet

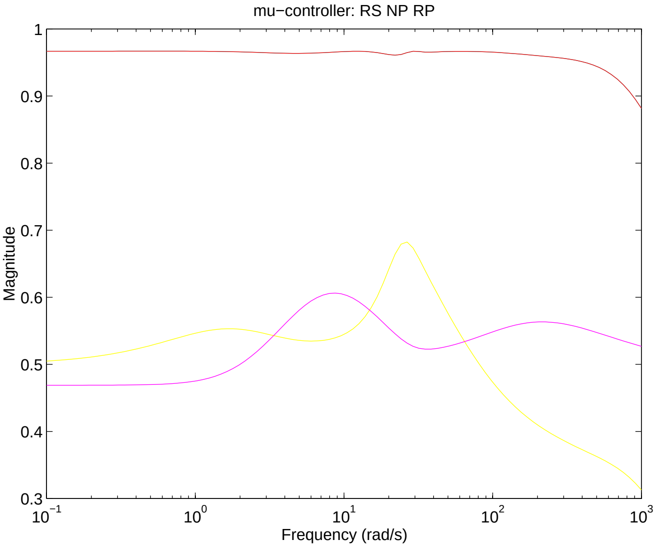


H_∞ Design and SSV Analysis

Robust stability: yellow

Nominal performance: violet

Robust performance: red



Robust performance achieved. Stop.

Summary

Controller Analysis with Structured Singular Value

- Sound computations
- Can test robust stability and robust performance
- Allowed Uncertainties - linear time-invariant: parametric or dynamic, full or repeated
- Delicate: parametric uncertainties.

Controller Synthesis

- General solution for H_∞ problem
- Robust performance design via D/K iteration:
  Theory: No optimality guarantee. Alternatives?
  Practice: Works fine on many examples
- Linear controllers only

Outlook on LMIs

Linear Matrix Inequalities Techniques

- More flexible choice of performance specs:
  positive real, H_2, amplitude constraints
- Allows extension to multi-objective design:
  various performance specs on different channels
- Extension of class of uncertainties:
  Time-varying parametric, non-linear
  Description with Integral Quadratic Constraints
- Can use more scalings - lesser conservatism
- Refined alternatives to D/K iteration
- Can perform systematic gain-scheduling for
  non-linear controller design problems