LMI Relaxations in Robust Control ...

... How to Reduce Conservatism?

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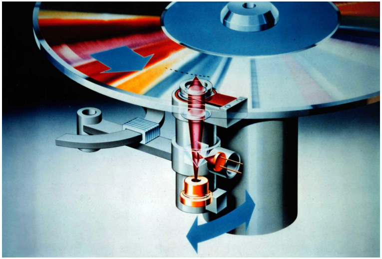

TU Delft

The Watershed

The great watershed in optimization is not between linearity and nonlinearity, but between convexity and non-convexity.

Rockafellar

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Rockafellar

... convex constraints with self-concordant barriers ...

Nesterov & Nemirovski

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Nesterov & Nemirovski

LMI-Problem: Infimize $c^T x$ over

$$W(x) = W_0 + x_1 W_1 + \cdots + x_n W_n \prec 0$$

Negative Definite

Exact Convexification in Control

Disturbance Attenuation

Controlled system

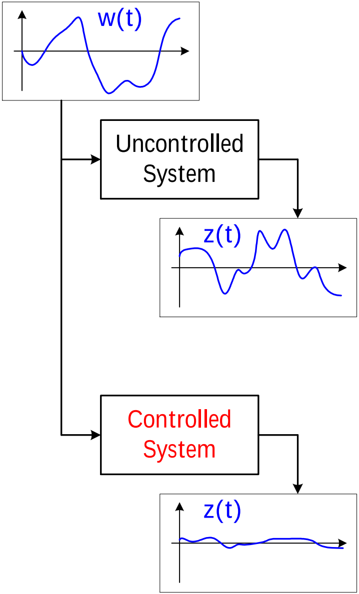
$$\dot{x} = Ax + Bw$$
$$z = Cx + Dw$$

with transfer matrix

$$T(s) = C(sI - A)^{-1}B + D$$

Attenuation level = Infimal γ with

$\sigma_{\max}(T(i\omega)) < \gamma$ for all $\omega \in \mathbb{R}$

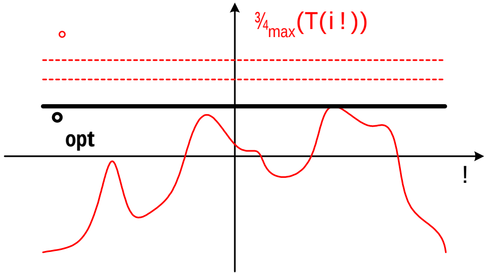


A 40-Year Old Surprise

Non-convex for T

$\downarrow$

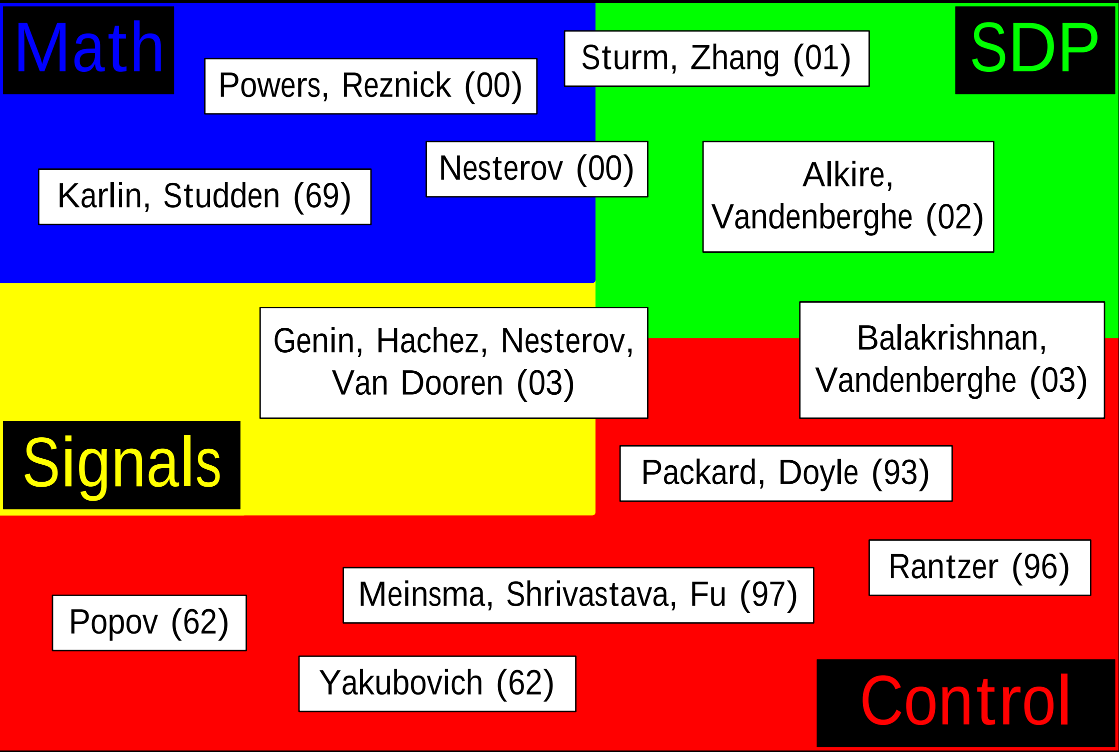
Convex for A, B, C, D



$$\sigma_{\max}(T(i\omega)) < \gamma \quad \text{for all} \quad \omega \in \mathbb{R}$$

$$\Longleftrightarrow$$

$$\text{exists } X \text{ with } \begin{bmatrix} A^T X + X A & XB & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \prec 0.$$


Math
SDP
Signals
Control
Powers, Reznick (00)
Sturm, Zhang (01)
Karlin, Studden (69)
Nesterov (00)
Alkire, Vandenberghe (02)
Genin, Hachez, Nesterov, Van Dooren (03)
Balakrishnan, Vandenberghe (03)
Packard, Doyle (93)
Rantzer (96)
Meinsma, Shrivastava, Fu (97)
Popov (62)
Yakubovich (62)

A 20-Year Old Convexification Technique

State-feedback synthesis:

$$\boxed{\begin{aligned}\dot{x} &= Ax + Bu + Gw\\ z &= Hx\end{aligned}} \quad \xrightarrow{u = Fx} \quad \boxed{\begin{aligned}\dot{x} &= (A + BF)x + Gw\\ z &= Hx\end{aligned}}$$

$$\begin{bmatrix} (A+BF)^T X + X(A+BF) & XG & H^T \\ G^T X & -\gamma I & 0 \\ H & 0 & -\gamma I \end{bmatrix} \prec 0$$

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$$\begin{bmatrix} (A+BF)^T X + X(A+BF) & XG & H^T \\ G^T X & -\gamma I & 0 \\ H & 0 & -\gamma I \end{bmatrix} \prec 0$$

$$\updownarrow$$

$$\boxed{\begin{aligned} Y &= X^{-1} \\ K &= FX^{-1} \end{aligned}} \quad \longleftrightarrow \quad \begin{bmatrix} (AY+BK)^T + (AY+BK) & G & YH^T \\ G^T & -\gamma I & 0 \\ HY & 0 & -\gamma I \end{bmatrix} \prec 0$$

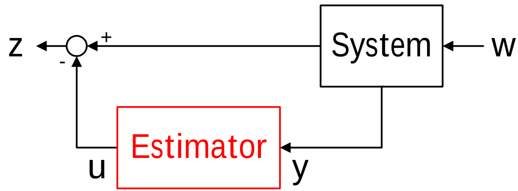
Extensions of Exact Convexification

Output Estimation

Shaked, de Souza (95)

Geromel (99)

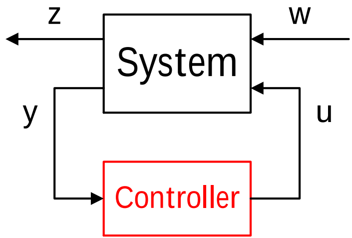
Palhares, Peres (00)



Output Feedback

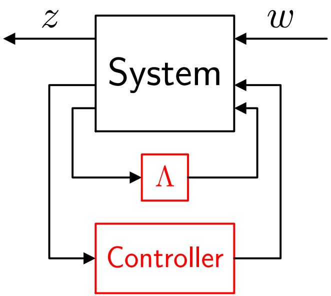
Masubuchi, Ohara, Suda (98)

Scherer, Gahinet, Chilali (97)



Parametric-Dynamic Synthesis

- Interconnection affine in finite-dimensional parameter Λ.
- Constraints on Λ: LMI's
- Performance spec: LMI's

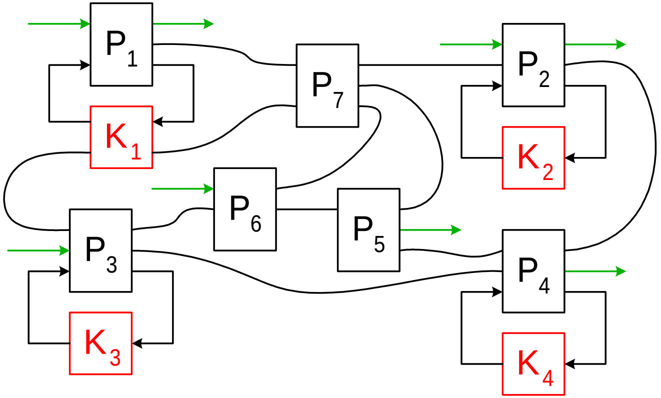


Find Λ and controller satisfying performance spec

$\longrightarrow$ Standard LMI-problem - efficiently tractable.

Rotstein, Sideris (94), Sznaier (94-98), Scherer (00-02)

Structure



Design of decentralized controller components

Objectives on distributed input-output channels

Why Does it Fail?

H_∞-control by structured state-feedback:

$$F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}.$$

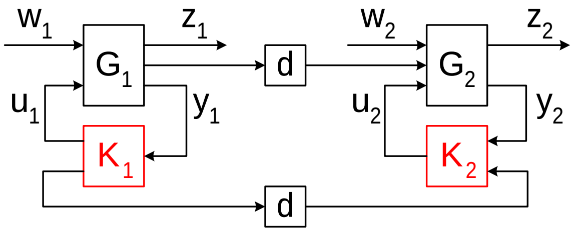
Usual transformation:

$$K = \begin{bmatrix} K_1 & K_{12} \\ K_{21} & K_2 \end{bmatrix} = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} X^{-1}$$

Non-convex constraints on K $\rightarrow$ Correct structure of F.

The Path via Infinite Dimensions

Example: Chain Structures



One-sided delayed information exchange.

Examples:
- Platooning or formation flight-control
- Networked systems

Why distinct from general problem?

Structured Youla-Kucera Parameterization

Control channel $\begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix}$. Controller $\begin{bmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{bmatrix}$.

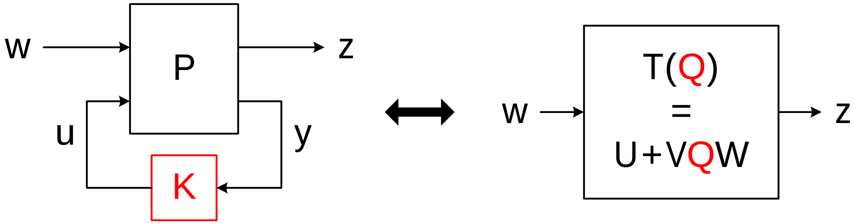
Open-loop stable: Youla-Kucera transformation $K \to Q$ is

$$\underbrace{\begin{bmatrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{bmatrix}}_{Q} = \underbrace{\begin{bmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{bmatrix}}_{K} \left\{ I - \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{bmatrix} \right\}^{-1}$$

Triangular structure preserved !

Goodwin et al (99), Voulgaris et al (00,03)

Effect of Youla-Kucera Parameterization

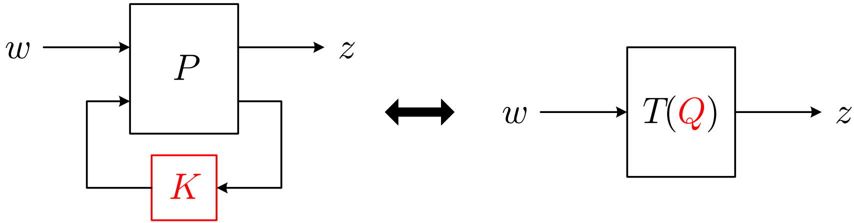


Affine dependence of $T(Q)$ on structured YK parameter:

$$T(Q) = U + \begin{bmatrix} V_{11} & V_{12} \end{bmatrix} \begin{bmatrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix} =$$

$$= U + \begin{bmatrix} V_{11} & V_{12} \end{bmatrix} \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} W_{11} + V_{12} Q_{22} W_{21}.$$

Structured Model-Matching



Infinite-dimensional optimization problem:

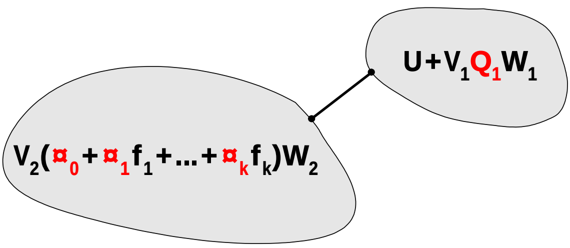
$$\inf_{Q \text{ stable}} \|T(Q)\|_\infty = \inf_{Q_1,\, Q_2 \text{ stable}} \|U + V_1 Q_1 W_1 + V_2 Q_2 W_2\|_\infty$$

Computation of Upper Bounds

$f_0(s), f_1(s), \ldots, f_k(s)$ with **dense span** in RH_∞:

$$\inf_{Q_1 \text{ stable}, \Lambda_j} \left\|U + V_1 Q_1 W_1 + V_2 \left(\sum_{j=0}^{k} \Lambda_j f_j \right) W_2 \right\|_\infty .$$

Numerical solution: Parametric-Dynamic Synthesis



Computation of Lower Bounds

$$F_0 + F_1 \frac{1}{s} + F_2 \frac{1}{s^2} + \cdots$$
$$\downarrow$$
$$\begin{bmatrix} F_0 & 0 & 0 \\ F_1 & F_0 & 0 \\ F_2 & F_1 & F_0 \end{bmatrix}$$

Transfer matrix inequality

$$\|U + V_1 Q_1 W_1 + V_2 Q_2 W_2\|_\infty < \gamma$$

implies **truncated Toeplitz matrix** inequality

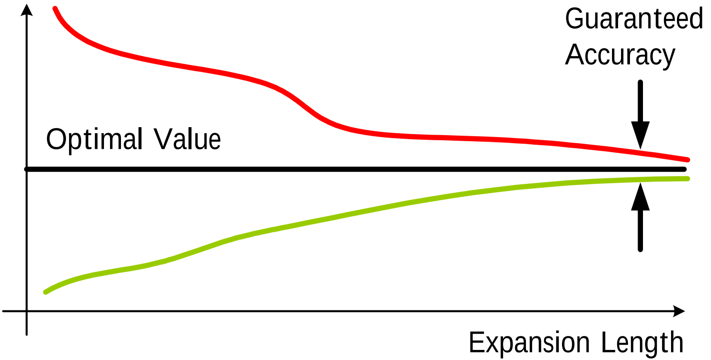
$$\left\| M_U^l + M_{V_1}^l M_{Q_1}^l M_{W_1}^l + M_{V_2}^l M_{Q_2}^l M_{W_2}^l \right\| < \gamma.$$

Structured Toeplitz matrix problem solvable with LMI's.

Nontrivial: Prove of convergence.

Scherer (99)

Summary



Techniques applicable to structure in performance.

Price tag: **Controller order high**.

Polynomial Optimization

Parametric Uncertainties

Guarantee that

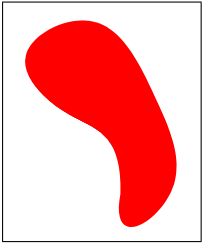
$$\dot{x} = A(\delta)x \ \ \textbf{stable} \text{ for all } \ \delta \in \boldsymbol{\delta}.$$

Assumptions

$A(\delta)$ **rational** in δ, parameter set $\boldsymbol{\delta}$ **semi-algebraic**.

With polynomials $g_1(\delta), \ldots, g_k(\delta)$:

$$\boldsymbol{\delta} = \{\delta \in \mathbb{R}^m : \ g_1(\delta) \leq 0, \ldots, g_k(\delta) \leq 0\}$$



Real Structured Singular Value

Data: Matrix M and block-diagonal structure

$$\Delta(\delta) = \begin{bmatrix} \delta_1 I & & \\ & \ddots & \\ & & \delta_m I \end{bmatrix} \quad \text{with} \quad \delta \in \boldsymbol{\delta} \subset \mathbb{R}^m.$$

Check whether

$$\det(I - M\Delta(\delta)) > 0 \quad \text{for all} \quad \delta \in \boldsymbol{\delta}.$$

Positivity of **higher-order polynomial** in $\delta \in \boldsymbol{\delta}$.

Polynomial Optimization

With multivariable polynomials $g_0(x)$, $g_1(x), \ldots, g_k(x)$:

$$p_{\text{opt}} := \inf_{g_1(x)\leq 0,\ldots,g_k(x)\leq 0} g_0(x).$$

Upper bounds ... standard nonlinear optimization.

How to compute **lower** bounds?

Polynomial Optimization

Primal: $p_{\text{opt}} := \inf\{g_0(x) : \; g_1(x) \leq 0, \ldots, g_k(x) \leq 0\}$.

Lower Bound: Lagrange Dual Relaxation

$d_{\text{opt}} =$ Maximal t such that exist $y_1 \geq 0, \ldots, y_k \geq 0$ with

$$g_0(x) + y_1 g_1(x) + \cdots + y_k g_k(x) - t \geq 0 \quad \forall\, x \in \mathbb{R}^m.$$

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- When does equality $p_{\text{opt}} = d_{\text{opt}}$ hold?
- Can we compute d_{opt}? Sometimes ...

The Quadratic Case

Indefinite quadratic $g_j : \mathbb{R}^m \to \mathbb{R}$ can be represented as

$$g_j(x) = \begin{bmatrix} 1 \\ x \end{bmatrix}^T P_j \begin{bmatrix} 1 \\ x \end{bmatrix} \quad \text{with } \textbf{unique} \quad P_j = P_j^T.$$

$$g_0(x) + y_1 g_1(x) + \cdots + y_k g_k(x) - t \geq 0 \quad \forall\, x \in \mathbb{R}^m$$

$$\Longleftrightarrow$$

$$P_0 + y_1 P_1 + \cdots + y_k P_k - t \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \succcurlyeq 0.$$

d_{opt} computable with LMI's - **Shor Relaxation (87)**.

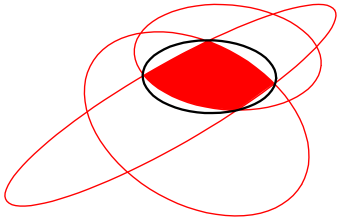
Application: S-Procedure

Problem: Verify whether

$$g_1(x) \leq 0, \ldots, g_k(x) \leq 0 \text{ implies } g_0(x) \geq 0.$$

Geometric Problems

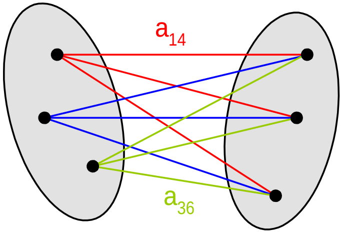
Ellipsoidal covering ...



At heart of computing SSV upper bounds for full blocks ...

Boyd, El Ghaoui, Feron, Balakrishan (94)

Application: Max-Cut



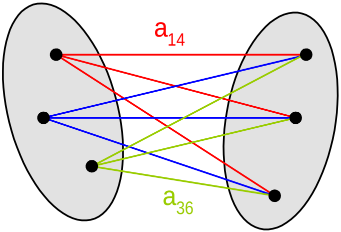
Find graph partition maximizing total weight of linking arcs.

With non-negative weight matrix A translated into

$$p_{\text{opt}} = \min \left\{ x^T A x - \sum a_{jk} : \ \pm(x_j^2 - 1) \leq 0 \right\} \leq 0.$$

Application: Max-Cut

Find graph partition maximizing total weight of linking arcs.



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$$p_{\text{opt}} = \min\left\{x^T A x - \sum a_{jk} : \ \pm(x_j^2 - 1) \leq 0\right\} \leq 0.$$

Goemans, Williamson (95): $d_{\text{opt}} \leq p_{\text{opt}} \leq 0.879\, d_{\text{opt}}$

LMI's in combinatorial optimization: Laurent, Rendl (02)

How to Reduce the Relaxation Gap ?

A Simple Idea for Improvement

Introduce redundant variables: Replace $g_j(x) \leq 0$ with

$$y_j(x) g_j(x) \leq 0 \quad \text{where} \quad y_j(x) = \begin{bmatrix} 1 \\ x \end{bmatrix}^T Y_j \begin{bmatrix} 1 \\ x \end{bmatrix}, \;\; Y_j > 0.$$

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Higher Order Relaxation

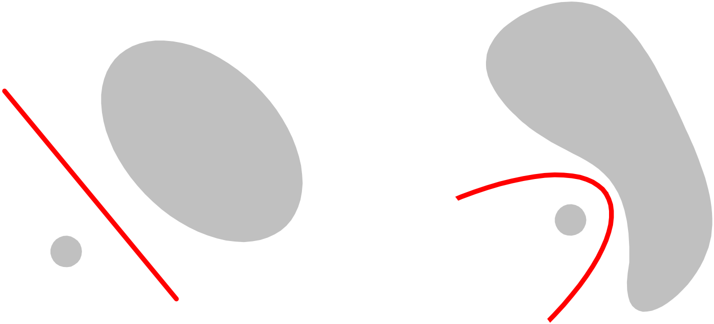
Maximize t for which exist $y_1(x), \ldots, y_k(x)$ with

$$g_0(x) + y_1(x) g_1(x) + \cdots + y_k(x) g_k(x) - t \geq 0 \quad \forall\, x \in \mathbb{R}^m.$$

Sherali & Adams (90), Lovász & Schrijver (91), ...

Geometric Interpretation

Convex: Separation by linear functionals



Non-convex: Separation by quadratic functionals

Verifying Positivity of Polynomials ?

With $z(x) = (1 \; x_1 \; \cdots \; x_m \; x_1^2 \; x_1 x_2 \; \cdots)$ represent

$$p(x) = z(x)^T M z(x) \quad \text{for all} \quad M \in \boldsymbol{M}.$$

$\boldsymbol{M}$ is easily determined **affine space**.

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Gram Matrix Method

Exists $M \in \boldsymbol{M}$ with $M \succcurlyeq 0$? (LMI-feasibility)

- **Yes**: Have verified $p(x) \geq 0$ for all $x \in \mathbb{R}^m$.

Carsten Scherer

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Gram Matrix Method

Exists $M \in \boldsymbol{M}$ with $M \succcurlyeq 0$? (LMI-feasibility)

- **Yes**: Have verified $p(x) \geq 0$ for all $x \in \mathbb{R}^m$.
- **No**: Does **not disprove** $p(x) \geq 0$ for all $x \in \mathbb{R}^m$!

Background

Gram Matrix Method: Is p a **sum-of-squares (sos)**:

$$p(x) = p_1(x)^2 + \cdots + p_r(x)^2\,?$$

Choi, Lam, Reznick (95) - Powers, Wörman (98)

In control: Chesi, Garulli, Tesi, Vicino (00) - Parrilo (00)

Not all non-negative polynomials are **sos**.

Hilbert (1888), Motzkin (1969)

Are they **sos** of **rational** functions? Yes.

Artin (1927)

Family of Higher Order Relaxations

Primal: $p_{\text{opt}} := \inf\{g_0(x) : \; g_1(x) \leq 0, \ldots, g_k(x) \leq 0\}$.

SOS-Relaxation

$d_{\text{opt}} =$ Maximal t for which exist sos $y_1(x), \ldots, y_k(x)$:

$$g_0(x) + y_1(x)g_1(x) + \cdots + y_k(x)g_k(x) - t \;\text{ is sos.}$$

$\longrightarrow$ Computing lower bound d_{opt} is LMI-problem

$\longrightarrow$ Reduce $p_{\text{opt}} - d_{\text{opt}}$ by **increasing length** of

$$z(x) = (1 \; x_1 \; \cdots \; x_m \; x_1^2 \; x_1x_2 \; \cdots)$$

Family of Higher Order Relaxations

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$$g_0(x) + y_1(x)g_1(x) + \cdots + y_k(x)g_k(x) - t \; \text{ is sos.}$$

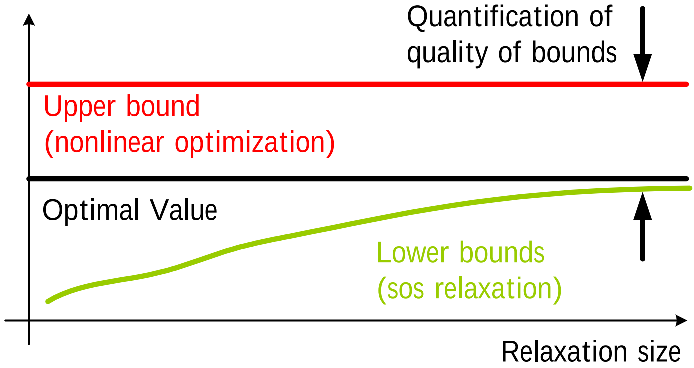
Despite non-convexity

If $\{x \in \mathbb{R}^m : \; g_1(x) \leq 0\}$ is compact then $p_{\text{opt}} = d_{\text{opt}}$.

Putinar (93), Jacobi, Prestel (00)

Freely available code: Lasserre (01) and Henrion, Lasserre (02)

Summary



Price tag: **Exponential explosion** of problem size.

Surprisingly effective for small-sized problems.

Parrilo, Sturmfels (01)

Robustness Analysis and Synthesis

Robust Stability Analysis ...

Guarantee that

$$x_{t+1} = A(\delta)x_t \ \ \textbf{stable} \text{ for all } \ \delta \in \boldsymbol{\delta}.$$

Robust Stability Analysis ...

Guarantee that

$$x_{t+1} = A(\delta)x_t \;\; \textbf{stable} \text{ for all } \;\; \delta \in \boldsymbol{\delta}.$$

Search Lyapunov function $V(x) = x^T X(\delta) x$.

Test whether there exist continuous $X(\delta) \succ 0$ with

$$A(\delta)^T X(\delta) A(\delta) - X(\delta) \prec 0 \;\; \text{for all} \;\; \delta \in \boldsymbol{\delta}.$$

Robust Stability Analysis ...

Guarantee that

$$x_{t+1} = A(\delta)x_t \;\; \textbf{stable} \text{ for all } \;\; \delta \in \boldsymbol{\delta}.$$

Search Lyapunov function $V(x) = x^T X(\delta) x$.

if nominally stable

Test whether there exist continuous $X(\delta)$ with

$$A(\delta)^T X(\delta) A(\delta) - X(\delta) \prec 0 \;\; \text{for all} \;\; \delta \in \boldsymbol{\delta}.$$

... Reduced to Finite Dimensions ...

Restriction to finite-dimensional subspace:

Search in span of $\{X_1(\delta), \ldots, X_n(\delta)\}$.

Find $x_1, \ldots, x_n$ such that

$$A(\delta)^T \left(\sum_{j=1}^{n} x_j X_j(\delta) \right) A(\delta) - \sum_{j=1}^{n} x_j X_j(\delta) \prec 0$$

for all $\delta \in \boldsymbol{\delta}$.

... Subsumed to Robust LMI Problem

$W(x)$ affine, $F(\delta)$ nonlinear.

Infimize $c^T x$ over the **robust LMI-constraint**

$$\begin{bmatrix} F(\delta) \\ I \end{bmatrix}^T W(x) \begin{bmatrix} F(\delta) \\ I \end{bmatrix} \prec 0 \quad \text{for all} \quad \delta \in \boldsymbol{\delta}.$$

Looks pretty special ...

... is very general !

Robust Synthesis

State-feedback synthesis:

$$x_{t+1} = A(\delta)x_t + B(\delta)u_t \xrightarrow{u_t = Fx_t} x_{t+1} = [A(\delta) + B(\delta)F]x_t$$

Usual Transformation for **general** $X(\delta)$:

$$[A(\delta) + B(\delta)F]X(\delta) \xrightarrow{K(\delta) = FX(\delta)} A(\delta)X(\delta) + B(\delta)K(\delta)$$

Gain is **parameter-dependent**: $F(\delta) = K(\delta)X(\delta)^{-1}$.

Robust Synthesis

State-feedback synthesis:

$$x_{t+1} = A(\delta)x_t + B(\delta)u_t \xrightarrow{u_t = Fx_t} x_{t+1} = [A(\delta) + B(\delta)F]x_t$$

Usual Transformation for **constant** X:

$$[A(\delta) + B(\delta)F]X \xrightarrow{K = FX} A(\delta)X(\delta) + B(\delta)K$$

Introduces conservatism.

Redundant Variables

$x_{t+1} = Ax_t$ **stable** iff exists $X \succ 0$ with

$$\begin{bmatrix} I & A \end{bmatrix} \begin{bmatrix} -X & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} I \\ A^T \end{bmatrix} = AXA^T - X \prec 0.$$

Equivalent reformulation: Exists X and G with

$$\begin{bmatrix} -X & 0 \\ 0 & X \end{bmatrix} + \begin{bmatrix} A \\ -I \end{bmatrix} G \begin{bmatrix} I & 0 \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} G^T \begin{bmatrix} -I & A^T \end{bmatrix} \prec 0.$$

de Oliveira, Bernussou, Geromel (99)

Peaucelle, Arzelier, Bachelier, Bernussou (00), Daafouz et al. (02)

Robust Synthesis

State-feedback synthesis:

$$x_{t+1} = A(\delta)x_t + B(\delta)u_t \quad \xrightarrow{u_t = F x_t} \quad x_{t+1} = [A(\delta) + B(\delta)F]x_t$$

Search F, $X(\delta)$, $G(\delta)$ with

$$\begin{bmatrix} -X(\delta) & 0 \\ 0 & X(\delta) \end{bmatrix} + \text{sym}\begin{bmatrix} [A(\delta) + B(\delta)F]G(\delta) & 0 \\ -G(\delta) & 0 \end{bmatrix} \prec 0.$$

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Choose **constant** G. Transform $K = FG$.

Robust Synthesis

State-feedback synthesis:

$$x_{t+1} = A(\delta)x_t + B(\delta)u_t \quad \xrightarrow{u_t = Fx_t} \quad x_{t+1} = [A(\delta) + B(\delta)F]x_t$$

Search K, $X(\delta)$, G with

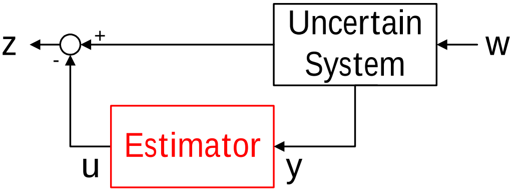
$$\begin{bmatrix} -X(\delta) & 0 \\ 0 & X(\delta) \end{bmatrix} + \text{sym} \begin{bmatrix} A(\delta)G + B(\delta)K & 0 \\ -G & 0 \end{bmatrix} \prec 0.$$

Results in robust LMI problem for **general** $X(\delta)$.

Often less conservative than **quadratic stability**.

Where Does it Apply?

Robust Estimation

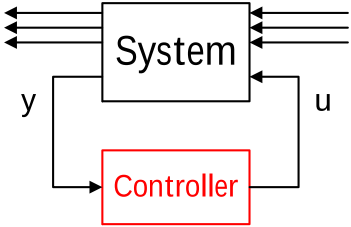


Palhares, Peres (00)

Barbosa, de Souza, Trofino (02)

...

Multi-Objective Control



de Oliveira, Geromel, Bernussou (99)

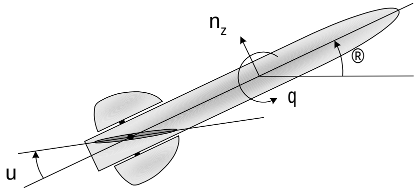
Apkarian, Pellanda, Tuan (02)

...

Robust output-feedback ? ... No !

LPV Analysis and Synthesis

High-Performance Aircraft System



u: Control input

α: Measurable parameter

n_z: Tracked output

Nonlinear system description with aerodynamic effects:

$$\dot{\alpha} = K M \left[\left(a_n \alpha^2 + b_n \alpha + c_n (2 - M/3) \right) \alpha + d_n \delta \right] + q$$

$$\dot{q} = M^2 \left[\left(a_m \alpha^2 + b_m \alpha - c_m (7 - 8M/3) \right) \alpha + d_m \delta \right]$$

$$n_z = M^2 \left[\left(a_n \alpha^2 + b_n \alpha + c_n (2 - M/3) \right) \alpha + d_n \delta \right]$$

Main Idea

Rewrite as linear parameter-varying system

$$\dot\alpha = K\pi_1\left[\left(a_n{\pi_2}^2 + b_n\pi_2 + c_n(2-\pi_1/3)\right)\alpha + d_n\delta\right] + q$$

$$\dot q = {\pi_1}^2\left[\left(a_m{\pi_2}^2 + b_m\pi_2 - c_m(7-8\pi_1/3)\right)\alpha + d_m\delta\right]$$

$$n_z = {\pi_1}^2\left[\left(a_n{\pi_2}^2 + b_n\pi_2 + c_n(2-\pi_1/3)\right)\alpha + d_n\delta\right]$$

with bounds $2 \leq \pi_1(t) \leq 4$ and $-20 \leq \pi_2(t) \leq 20$.

Good control of linear parameter-varying system

$\longrightarrow$ Good control of nonlinear system

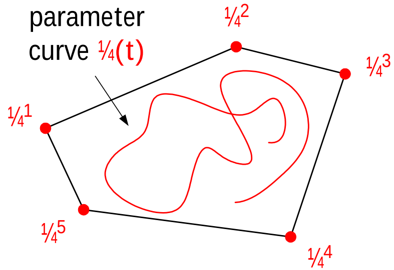
Linear Parameter Varying Systems

LPV system

$$\dot{x}(t) = A(\pi(t))x(t)$$

with value & velocity bounds

$$\pi(t) \in \Pi, \quad \dot{\pi}(t) \in \Phi.$$



Uniform exponential stability if exists $X(\pi)$ with

$$\sum_j \partial_j X(\pi)\phi_j + A(\pi)^T X(\pi) + X(\pi)A(\pi) \prec 0$$

for all $\delta = (\pi, \phi) \in \Pi \times \Phi$.

Performance

L_2-gain smaller than γ ... for LPV system

$$\dot{x}(t) = A(\pi(t))x(t) + B(\pi(t))w(t)$$
$$z(t) = C(\pi(t))x(t) + D(\pi(t))w(t)$$

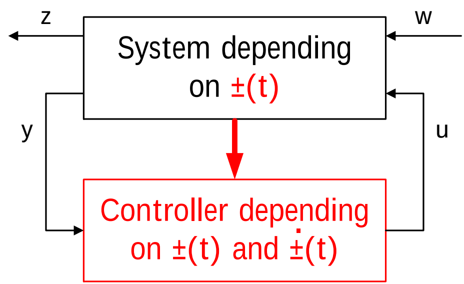
Check existence of $X(\pi)$ with

$$\left[\begin{array}{cc} I & 0 \\ A(\pi) & B(\pi) \\ \hline 0 & I \\ C(\pi) & D(\pi) \end{array}\right]^T \left[\begin{array}{cc|cc} \sum_j \partial_j X(\pi)\phi_j & X(\pi) & 0 & 0 \\ X(\pi) & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A(\pi) & B(\pi) \\ \hline 0 & I \\ C(\pi) & D(\pi) \end{array}\right] \prec 0$$

for all $\delta = (\pi, \phi) \in \Pi \times \Phi$.

LPV Output-Feedback Synthesis

Controller fed with on-line measurements of $\delta(t)$, $\dot{\delta}(t)$



Can minimize L_2-gain.

$\delta(t)$ = Component of state $\longrightarrow$ Nonlinear controller

Wu, Yang, Packard, Becker (95), Apkarian, Adams (98), ...

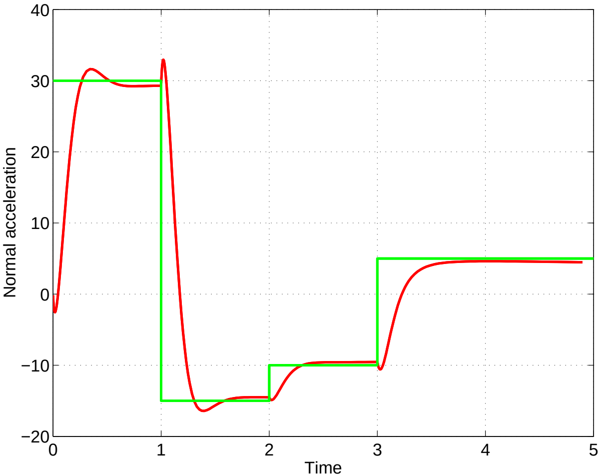
Application to Aircraft Model

$M(t)$ decreases in 5 seconds from 4 to 2.

Normal acceleration

Reference

Response



How to Solve Robust LMI Problems ?

Robust LMI Problem

Infimize $c^T x$ over the robust LMI-constraint

$$\begin{bmatrix} F(\delta) \\ I \end{bmatrix}^T W(x) \begin{bmatrix} F(\delta) \\ I \end{bmatrix} \prec 0 \quad \text{for all} \quad \delta \in \boldsymbol{\delta}.$$

Many parameters, general nonlinear $F(\delta)$:

Probabilistic techniques. No guarantees with certainty.

Khargonekar, Tikku (96), Tempo et al. (97), Polyak, Tempo (01)

Remainder: Few parameters, rational $F(\delta)$.

Linear Fractional Representation

Let $F(\delta)$ be **rational** in δ without pole at zero.

Can **construct** A, B, C, D such that

$$F(\delta) = D + C\Delta(\delta)\left(I - A\Delta(\delta)\right)^{-1} B$$

with

$$\Delta(\delta) = \begin{bmatrix} \delta_1 I_1 & & 0 \\ & \ddots & \\ 0 & & \delta_k I_k \end{bmatrix}.$$

Covers higher-order polynomials ...

Full-Block S-Procedure

Exact reformulation of robust LMI problem:

Infimize $c^T x$ over all x, P with

$$\begin{bmatrix} \Delta(\delta) \\ I \end{bmatrix}^T P \begin{bmatrix} \Delta(\delta) \\ I \end{bmatrix} \succ 0 \quad \text{for all} \quad \delta \in \boldsymbol{\delta}$$

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T P \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T W(x) \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \prec 0.$$

Iwasaki, Shibata (01), Scherer (01)

Numerical Implementation

Trouble: No description of set of **all** multipliers P with

$$L(\delta, P) := \begin{bmatrix} \Delta(\delta) \\ I \end{bmatrix}^T P \begin{bmatrix} \Delta(\delta) \\ I \end{bmatrix} \succ 0 \quad \text{for all} \quad \delta \in \boldsymbol{\delta}.$$

Parameter set finitely generated: $\boldsymbol{\delta} = \text{con}\{\delta_1, \ldots, \delta_N\}$

Need to guarantee:

$$L(\lambda_1 \delta_1 + \cdots + \lambda_N \delta_N, P) \succ 0 \quad \text{for } \lambda \in \text{standard simplex.}$$

Numerical Implementation

Trouble: No description of set of **all** multipliers P with

$$L(\delta, P) := \begin{bmatrix} \Delta(\delta) \\ I \end{bmatrix}^T P \begin{bmatrix} \Delta(\delta) \\ I \end{bmatrix} \succ 0 \text{ for all } \delta \in \boldsymbol{\delta}.$$

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Need to guarantee:

$$L(\lambda_1 \delta_1 + \cdots + \lambda_N \delta_N, P) \succ 0 \text{ for } \lambda \in \text{standard simplex.}$$

Sufficient: Coefficients positive definite.

A Novel Path to Exactness

Replace by **sufficient** LMI condition:

$$[\lambda_1 + \cdots + \lambda_N]^d L(\lambda_1 \delta_1 + \cdots + \lambda_N \delta_N, P)$$

has positive definite coefficients.

New Generalization of Polya's Theorem

There exists d such that LMI condition is **necessary**.

Relaxation family to compute **upper bounds**: $d = 0, 1, \ldots$

Guaranteed to be **asymptotically exact**.

Test Exactness on the Way

Let U be Lagrange optimal Lagrange multiplier for

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T P \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T W(x) \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \prec 0.$$

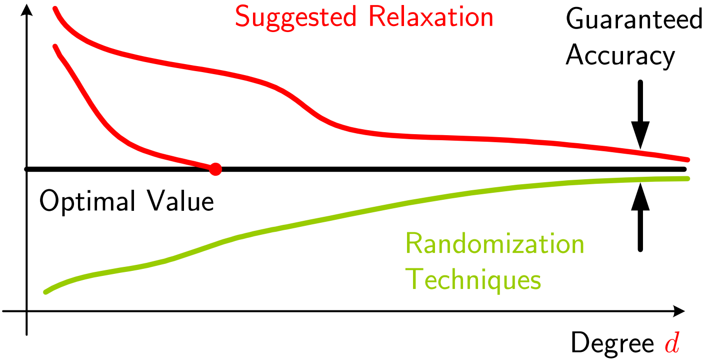
Relaxation is **not conservative** if

$$\begin{bmatrix} I & -\Delta_0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} U = 0 \text{ has a solution } \Delta_0 \in \boldsymbol{\Delta}.$$

Easily verifiable in practice.

Central path-following techniques **guarantee** success !

Summary



Price tag: **Exponential explosion** of problem size.

Only available technique to arbitrarily reduce conservatism.

Conclusions

Tried to highlight

... paths for (arbitrarily) reducing conservatism

... yet unsurmountable difficulties

Exists strong need to

... exploit **system theoretic structure** in computations

... construct **efficient recursive relaxation** schemes