Stability of switched systems

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DISC Course on Modeling and Control of Hybrid Systems

Outline of lecture

- Switched systems
- Recall: stability of smooth systems
- 3 Different problems
- Summary

Switched systems

$$\dot{x} = f_\sigma(x)$$

$\{f_1(x), f_2(x), \ldots, f_N(x)\}$ family of smooth vector fields from $\mathbb{R}^n$ to $\mathbb{R}^n$

Switching signal $\sigma : [0,\infty) \to \{1,2,\ldots,N\}$ piecewise constant function of time

- Function of time t: $\sigma(t)$
- Function of state $x(t)$: $\sigma(x)$
- Combinations: $\sigma(t,x)$

No generalized solutions concepts including chattering, infinitely fast switching, sliding motions → include as additional modes!

Although we get back to this!

Switched linear systems

Switched linear system $\quad \dot{x} = A_\sigma x$

Piecewise linear system
Switching is only state-dependent

$$\dot{x} = A_i x \text{ when } x \in \mathscr{X}_i,$$

where $\mathscr{X}_i \subseteq \mathbb{R}^n$ are polyhedra (given by a finite number of inequalities ($a_k^\top x \geq b_k$, $k = 1,\ldots,K$)

- Well-posedness: cells form partitioning of $\mathbb{R}^n$ (necessary condition only)

$$\cup_{i=1}^n \mathscr{X}_i = \mathbb{R}^n \text{ and interior}(\mathscr{X}_i) \cap \text{interior}(\mathscr{X}_j) = \emptyset$$

- **Piecewise affine (PWA) systems**

$$\dot{x} = A_i x + a_i, \text{ when } E_i x \geq e_i,\ i \in I := \{1,\ldots,N\}.$$

Problem formulation

• Global asymptotic stability (GAS) of a system with state x:

Something like .... $\lim_{t\longrightarrow\infty} x(t) = 0$ for all initial states x_0.

GUAS: global uniform asymptotic stability: uniform in σ

Problem A : Find conditions for which the switched system is GAS for any switching signal (GUAS) ("robust stability").

Problem B : Show that the switched system is GAS for a given switching strategy or a class of switching strategies.

Problem C : Construct a switching signal that makes the switched system GAS (i.e. a stabilization problem).

$\longrightarrow$ Problem C will be treated in the lecture on Hybrid Control.

Formal definitions for problem A

$$\dot{x} = f_\sigma(x)$$

$$\{f_1(x), f_2(x), \ldots, f_N(x)\} \text{ family of smooth vector fields from } \mathbb{R}^n \text{ to } \mathbb{R}^n$$

• 0 should be an equilibrium of the switched system under arbitrary switching, implying that $f_i(0) = 0$ for all $i = 1, \ldots, N$.

• Lyapunov stability of the origin

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x_0,\|x_0\|\leq\delta}\ \forall_{\sigma:\mathbb{R}_+\to\{1,\ldots,N\}}\ \forall_{t\geq 0}\ \|x_{x_0,\sigma}(t)\| \leq \varepsilon$$

• Global attractivity

$$\forall_{x_0\in\mathbb{R}^n}\ \forall_{\sigma:\mathbb{R}_+\to\{1,\ldots,N\}}\ x_{x_0,\sigma}(t) \to 0$$

• Global uniform attractivity

$$\forall_{R>0}\ \forall_{\varepsilon>0}\ \exists_{T>0}\ \forall_{x_0,\|x_0\|\leq R}\ \forall_{\sigma:\mathbb{R}_+\to\{1,\ldots,N\}}\ \forall_{t>T}\ \|x_{x_0,\sigma}(t)\| \leq \varepsilon$$

• Global Uniform Asymptotic Stability (GUAS): Lyapunov stability and global uniform attractivity

Back to basics: Lyapunov theory for stability of continuous systems

$f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous!

Theorem 1 Let $x = 0$ be an equilibrium of $\dot{x} = f(x)$ (i.e. $f(0) = 0$) and let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

- $V(x) \to \infty$ as $\|x\| \to \infty$ (V is *radially unbounded*);
- $V(0) = 0$ and $V(x) > 0$, if $x \neq 0$ (i.e. V is positive definite); and
- $\dot{V}(x) = L_f V(x) := \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x} f(x) < 0$ for all $x \neq 0$.

Then $x = 0$ is GAS.

Converse theorem: If $x = 0$ is a GAS equilibrium of $\dot{x} = f(x)$ (i.e. $f(0) = 0$), then there exists a Lyapunov function V

Stability of linear systems

Consider the linear system $\dot{x} = Ax$ and consider a quadratic LF $V(x) = x^\top Px$ with P symmetric ($P = P^\top$) positive definite, i.e.

- $x^\top Px > 0$ for all $x \neq 0$ and P symmetric
- all eigenvalues of P are positive
- all leading principal minors $det P_{JJ} > 0$ for all $J = \{1, \ldots, j\}$ for $j = 1, \ldots, n$.
- $P = H^\top H$ for an invertible matrix H

• Q is called negative definite, if $-Q$ is positive definite.

$$\dot{V}(x) = \dot{x}^\top Px + x^\top P\dot{x} = x^\top [A^\top P + PA]x < 0 \text{ for all } x \neq 0$$

Hence, $A^\top P + PA$ should be a negative definite matrix: $A^\top P + PA < 0$

Stability of linear systems - continued

Theorem 2 Equivalent:

- $\dot{x} = Ax$ is GAS;
- A is a Hurwitz matrix (all eigenvalues in open left half plane)
- there is a quadratic Lyapunov function $V(x) = x^\top Px$ for some positive definite matrix P such that the Lyapunov inequality $A^\top P + PA < 0$ holds.

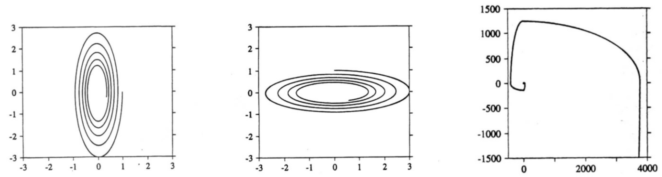
Moreover, for every Hurwitz A and for any $Q > 0$ there is a $P > 0$ such that the following Lyapunov equality holds

$$A^\top P + PA = -Q$$

True hybrid problem: Combining stable dynamics → stable?

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 x_2 < 0 \\ A_2 x, & \text{if } x_1 x_2 > 0 \end{cases}$$

$$A_1 = \begin{pmatrix} -1 & 10 \\ -100 & -1 \end{pmatrix};\ A_2 = \begin{pmatrix} -1 & 100 \\ -10 & -1 \end{pmatrix}. \text{ Eigenvalues } = -1 \pm 31.6j$$



Problem A: When is switched system UGAS for any switching signal?

Also for constant switching signals $\sigma(t) = i$ for all t

⇓

$\dot{x} = f_i(x)$ should be GAS

⇓

There is a radially unbounded Lyapunov function for each i!

Common Lyapunov function approach

• Try to find one shared Lyapunov function that decreases along any of the submodels:

A C^1-function $V : \mathbb{R}^n \to \mathbb{R}$ is called a *common Lyapunov function* for $\dot{x} = f_\sigma(x)$ with $\sigma \in \{1, \ldots, N\}$ if

$$\dot{V}(x) = L_{f_i} V(x) = \frac{\partial V}{\partial x} f_i(x) < 0, \text{ when } x \neq 0 \text{ and for all } i = 1, \ldots, N.$$

Theorem 3 If all the smooth submodels share a positive definite radially unbounded common Lyapunov function, then the switched system is GUAS.

Question: What about sliding modes?

A converse theorem

Necessary and sufficient condition:

Theorem 4 If the switched system is GUAS, then all f_i share a positive definite radially unbounded common Lyapunov function.

Hence, no conservatism in result!

Switched linear systems: a common quadratic LF approach

Stability of switched linear systems of the form

$$\dot{x} = A_\sigma x, \quad \sigma \in \{1, \ldots, N\}$$

Common LF of quadratic type $V(x) = x^\top Px$ for positive definite P?

$$\dot{V}(x) = L_{f_i}V(x) := \frac{\partial V}{\partial x} f_i(x) = x^\top [PA_i + A_i^\top P]x < 0 \text{ for all } x \neq 0 \text{ and } i$$

Hence, we obtain **linear matrix inequalities** (LMIs)

$$A_i^\top P + PA_i < 0 \text{ for all } i = 1, \ldots, N \text{ and } P > 0$$

• LMIs can be efficiently solved (SEDUMI/YALMIP)!

Converse quadratic LF theorem?

Asymptotic stability of switched linear system $\dot{x} = A_\sigma x \Rightarrow$ existence common quadratic Lyapunov function???

The answer is negative

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}; \, A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix},$$

which is GUAS, but no common quadratic LF by infeasibility condition

However, there is a common LF that is homogeneous of degree 2:

$$V(x) = \max_{i=1,2,\ldots,k} (l_i^\top x)^2$$

Conditions implying existence common quadratic LF

Theorem 5 If the matrices $\{A_1, \ldots, A_N\}$ commute pairwise

$$\text{for all } i, j, \text{ it holds that } A_iA_j = A_jA_i.$$

and are all Hurwitz, then there exists a common quadratic Lyapunov function $P = P_N$, that can be found from solving the following set of Lyapunov equalities successively:

$$\begin{aligned} A_1^\top P_1 + P_1A_1 &= -I \\ A_2^\top P_2 + P_2A_2 &= -P_1 \\ A_3^\top P_3 + P_3A_3 &= -P_2 \\ \vdots \quad &\quad \vdots \\ A_N^\top P_N + P_NA_N &= -P_{N-1}. \end{aligned}$$

More involved conditions exist (cf. references in lecture notes!)

Problem B: Is the switched system GAS for given switching strategies?

Piecewise smooth systems

Piecewise smooth systems: $\dot{x} = \begin{cases} f_1(x), & \text{when } x \in \mathscr{X}_1 \\ f_2(x), & \text{when } x \in \mathscr{X}_2 \\ \vdots & \\ f_N(x), & \text{when } x \in \mathscr{X}_N \end{cases}$

Sufficient: existence of common Lyapunov function

This might be conservative ... let's see if we can find relaxations ...

Relaxation 1: Decrease of Lyapunov function only in active region

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 x_2 \leq 0 \\ A_2 x, & \text{if } x_1 x_2 > 0, \end{cases} \text{ with } A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix};\ A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}$$

- No common quadratic LF.
- However, for $V(x) = x_1^2 + x_2^2$ it holds that $\dot{V} < 0$ along the nonzero solutions of the switched system, which implies GAS.

Relaxation w.r.t. common LF approach: Indeed, we only need

- $\dot{V}(x) = L_{A_1 x} V(x) = \frac{\partial V(x)}{\partial x} A_1 x < 0$ if $x_1 x_2 \leq 0$
- $\dot{V}(x) = L_{A_2 x} V(x) = \frac{\partial V(x)}{\partial x} A_2 x < 0$ if $x_1 x_2 > 0$.

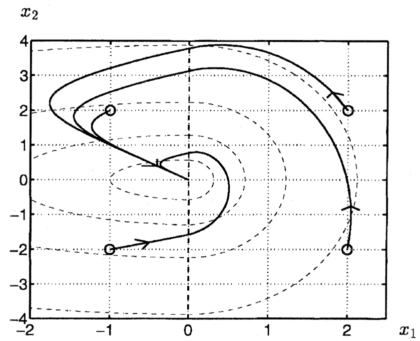
Hence, general set-up:

Find V pos. def. s.t. $L_{f_i} V(x)$ is only negative, where $\dot{x} = f_i(x)$ can be active.

Relaxation 2: Multiple Lyapunov functions

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 \leq 0 \\ A_2 x, & \text{if } x_1 > 0, \end{cases} \text{ where } A_1 = \begin{pmatrix} -5 & -4 \\ -1 & -2 \end{pmatrix};\ A_2 = \begin{pmatrix} -2 & -4 \\ 20 & -2 \end{pmatrix}.$$

- No common LF



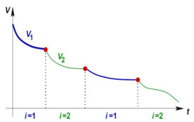
Relaxation 2: multiple Lyapunov function

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 \le 0 \\ A_2 x, & \text{if } x_1 > 0, \end{cases} \quad \text{where } A_1 = \begin{pmatrix} -5 & -4 \\ -1 & -2 \end{pmatrix};\ A_2 = \begin{pmatrix} -2 & -4 \\ 20 & -2 \end{pmatrix}.$$

No common LF → However, one can use two quadratic LF $V_i(x) = x^\top P_i x$ with

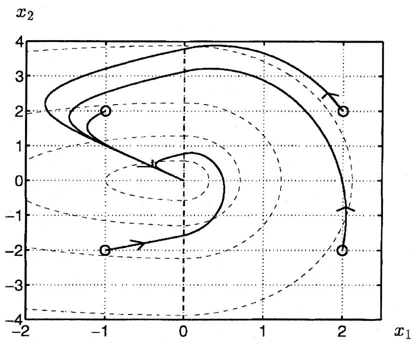
$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix};\ P_2 = \begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix}.$$

V_i is LF for $\dot{x} = A_i x$ (i.e. $A_i^\top P_i + P_i A_i < 0$ and $P_i > 0$, $i = 1, 2$)



$$V(x) = \begin{cases} V_1(x) = x_1^2 + 3x_2^2, & \text{when } i = 1 \text{ is active subsystem, i.e. } x_1 \le 0 \\ V_2(x) = 10x_1^2 + 3x_2^2, & \text{when } i = 2 \text{ is active subsystem, i.e. } x_1 > 0 \end{cases}$$

which is a continuous piecewise quadratic Lyapunov function.



Relaxation 3: Lyapunov function positive definite

$$V(x) = \begin{cases} V_1(x) = x_1^2 + 3x_2^2, & \text{when } i = 1 \text{ is active subsystem, i.e. } x_1 \le 0 \\ V_2(x) = 10x_1^2 + 3x_2^2, & \text{when } i = 2 \text{ is active subsystem, i.e. } x_1 > 0 \end{cases}$$

$V(x)$ must be positive definite, but since $V(x) = V_i(x)$ when $x \in \mathscr{X}_i$ the constituting functions $V_i(x)$ should only be positive when $x \in \mathscr{X}_i$.

Hence,

$$x \in \mathscr{X}_i, x \ne 0 \Rightarrow V_i(x) > 0$$

and possibly $V_i(x) < 0$ when $x \notin \mathscr{X}_i$.

The three relaxations for piecewise linear systems

$$\dot{x} = A_i x \text{ if } x \in \mathscr{X}_i$$

There are several relaxations possible w.r.t. common quadratic LF:

- One can require that the derivative $L_{A_i x} V(x)$ of $V(x) = x^\top P_i x$ is only negative in the region where the subsystem is active.
- One can use multiple Lyapunov functions, say $V_i(x) = x^\top P_i x$, for each submodel and "connect them" in a suitable way.
- One can require that the Lyapunov function $V_i(x) = x^\top P_i x$ is only positive definite in its active region.

Two "regional conditions:"

- $x \in \mathscr{X}_i$ and $x \ne 0$ should imply $x^\top [A_i^\top P_i + P_i A_i] x < 0$
- $x \in \mathscr{X}_i$ and $x \ne 0$ should imply $x^\top P_i x > 0$

How to cope with this? S-procedure!

S-procedure

Aim: $V(x) = x^\top Px$, $P > 0$ s.t. $x^\top [A_i^\top P + PA_i]x < 0$ for $0 \neq x \in \mathscr{X}_i$.

Find: $S_i(x)$ based on $\mathscr{X}_i$ in the sense that $S_i(x) \geq 0$ when $x \in \mathscr{X}_i$

Next: search for $\beta \geq 0$ satisfying

$$x^\top A_i^\top Px + x^\top PA_i x + \beta S_i(x) < 0 \text{ for all } x$$

Result: Since $S_i(x)$ might be negative outside $\mathscr{X}_i$, less conservative than $A_i^\top P + PA_i < 0$.

Computationally interesting: $S_i(x) = x^\top S_i x$, then LMI:

find $\beta_i \geq 0$ and $P > 0$ such that

$$A_i^\top P + PA_i + \beta_i S_i < 0.$$

Example and extension

Consider the region $\mathscr{X}_i = \{x \in \mathbb{R}^2 \mid E_i x \geq 0\}$ where $E_i = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$

The function $S_i(x) = \beta_i x^\top E_i^\top E_i x$ with $\beta_i \geq 0$ satisfies $S_i(x) \geq 0$ when $x \in \mathscr{X}_i$

Reason: we are multiplying a row vector and column vector with nonnegative elements ...

More general: take U_i a matrix with nonnegative elements, e.g.

$$U_i = \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}$$

then $\tilde{S}_i(x) = x^\top E_i^\top U_i E_i x \geq 0$ when $x \in \mathscr{X}_i$

Reason: we are multiplying a row vector $x^\top E_i$ and a column vector $U_i E_i x$ with nonnegative elements ...

Hence, to guarantee $x^\top [A_i^\top P_i + P_i A_i]x < 0$ when $x \in \mathscr{X}_i$, it suffices to find a matrix U_i with nonnegative elements such that

$$A_i^\top P_i + P_i A_i + E_i^\top U_i E_i < 0.$$

Second relaxation: multiple Lyapunov functions

Aim: $V(x) = V_i(x) = x^\top P_i x$ when $x \in \mathscr{X}_i$ where $V_i(x)$ is such that for each dynamics $\dot{x} = A_i x$

- $\dot{V}_i < 0$ when $x \in \mathscr{X}_i$
- V is continuous over the boundary[a].

$$\dot{x} = A_i x, \text{ when } E_i x \geq 0, \ i \in I.$$

Assumption:

- cells partition state space, no sliding modes
- Let the switching planes

  $$\mathscr{X}_i \cap \mathscr{X}_j \subseteq \ker H_{ij} = \{x \mid H_{ij} x = 0\} = \operatorname{im} Z_{ij} = \{Z_{ij} v \mid v \in \mathbb{R}^{m_{ij}}\}$$

  for certain matrices H_{ij} and Z_{ij}.

[a] Not needed in discrete-time case!

Second relaxation - continued

Observation for first relaxation: Take $S_i = E_i^\top U_i E_i$ with U_i nonnegative entries

$$A_i^\top P_i + P_i A_i + E_i^\top U_i E_i < 0$$

To guarantee continuity of the *piecewise quadratic Lyapunov function*

$$V(x) = x^\top P_i x, \text{ when } x \in \mathscr{X}_i, \text{i.e.}$$

$$x^\top P_i x = x^\top P_j x \text{ for all } x \in \mathscr{X}_i \cap \mathscr{X}_j \subseteq \operatorname{im} Z_{ij}.$$

Hence as any $x \in \mathscr{X}_i \cap \mathscr{X}_j$ can be written as $x = Z_{ij} v$ for some v it suffices to have

$$v^\top Z_{ij}^\top P_i Z_{ij} v = v^\top Z_{ij}^\top P_j Z_{ij} v$$

or

$$Z_{ij}^\top [P_i - P_j] Z_{ij} = 0.$$

$$\mathscr{S} := \{(i,j) \in \{1,\ldots,N\} \times \{1,\ldots,N\} \mid i \neq j \text{ and } \mathscr{X}_i \cap \mathscr{X}_j \neq \{0\}\}.$$

Third relaxation and assembling the stuff!

Observation $x^\top P_i x > 0$ only when $0 \neq x \in \mathscr{X}_i$

Theorem
If one can find symmetric matrices W_i and U_i with nonnegative entries and such that P_i satisfy

$$A_i^\top P_i + P_i A_i + E_i^\top U_i E_i < 0,\ i = 1,\dots,N \qquad \text{(decreasing in region where active)}$$

and

$$P_i - E_i^\top W_i E_i > 0,\ i = 1,\dots,N \qquad \text{(positive in region where active)}$$

and

$$Z_{ij}^\top [P_i - P_j] Z_{ij} = 0,\ (i,j) \in \mathscr{S}. \qquad \text{(continuity)}$$

then every ordinary trajectory (without sliding modes) tends to zero exponentially.

• Linear matrix inequalities!!!

• Note that one does not have to take W_i and U_i full matrices. One could reduce the solution space by only allowing diagonal matrices.

Alternative for warranting continuity for Relaxation 2

Assumption:

- there exist matrices F_i such that $F_i x = F_j x$ for all $x \in \mathscr{X}_i \cap \mathscr{X}_j$.

To guarantee continuity of the *piecewise quadratic Lyapunov function*

$$V(x) = x^\top P_i x, \text{ when } x \in \mathscr{X}_i, i.e.$$

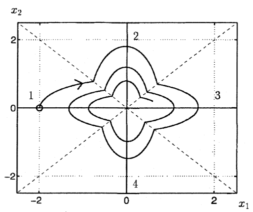
$$x^\top P_i x = x^\top P_j x \text{ for all } x \in \mathscr{X}_i \cap \mathscr{X}_j.$$

one can impose:

- $P_i = P_j$ (gives quadratic Lyapunov function)
- $P_i = P_j + F_{ij}^\top T_{ij} + T_{ij} F_{ij}$ where $F_{ij} x = 0$ when $x \in \mathscr{X}_i \cap \mathscr{X}_j$ (e.g. $F_{ij} = F_i - F_j$)
- $P_i = F_i^\top T F_i$, $i \in I$ for some symmetric matrix T.

Example of the “Flower system”

$$A_1 = A_3 = \begin{pmatrix} -0.1 & 1 \\ -5 & -0.1 \end{pmatrix};\ A_2 = A_4 = \begin{pmatrix} -0.1 & 5 \\ -1 & -0.1 \end{pmatrix}.$$

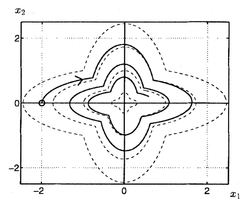


$$E_1 = -E_3 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix};\ E_2 = -E_4 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$H_{12} = H_{21} = H_{34} = H_{43} = [1\ 1]$$
$$H_{23} = H_{32} = H_{41} = H_{14} = [-1\ 1]$$
$$Z_{12} = Z_{21} = Z_{34} = Z_{43} = [-1\ 1]^\top$$
$$Z_{23} = Z_{32} = Z_{41} = Z_{14} = [1\ 1]^\top$$

Flower system - continued

$$P_1 = P_3 = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix};\ P_2 = P_4 = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}.$$



For alternative continuity guarantee, one could use $F_i := [E_i^\top I]^\top$

Summary

- Stability of submodels $\not\Rightarrow$ stability! “Hybrid problem”
- Problem A: GUAS for arbitrary switchings:
  - common LF approach
  - converse theorem
  - piecewise linear: common quadratic LF (only sufficient!)
- Problem B: GAS for specific switchings
  - State dependent switching: 3 relaxations w.r.t. common quadratic LF
    - decrease of LF only in active region
    - multiple LF (continuous over boundary)
    - LF only positive in active region
  - Piecewise linear systems:
    - S-procedure nice tool to get LMI
    - Piecewise quadratic LF