Hybrid control

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DISC Course on Modeling and Control of Hybrid Systems

Outline of lecture 3

- Continuation of stability
- Problem B: GAS for a particular class of switching regimes: dwell time restrictions!
- Problem C: Construct a stabilizing switching sequence, a discrete control problem
  - State-dependent switching
  - Time-dependent switching
- Continuous control problem
- Observer design
- Summary

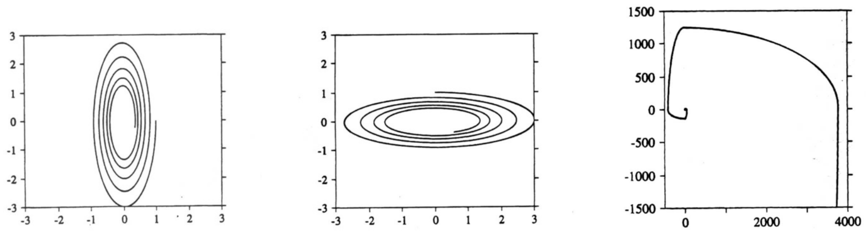
Problem B: Switched systems with dwell time conditions

Switched systems with dwell time conditions

"Dwell time:" duration / how long a system stays ("dwells") in a certain mode

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 x_2 < 0 \\ A_2 x, & \text{if } x_1 x_2 > 0 \end{cases}$$

$$A_1 = \begin{pmatrix} -1 & 10 \\ -100 & -1 \end{pmatrix};\ A_2 = \begin{pmatrix} -1 & 100 \\ -10 & -1 \end{pmatrix}.\ \text{Eigenvalues } = -1 \pm 31.6j$$



"Dwell time too short"

Stability under slow switching for switched linear systems

Consider a switched linear system $\dot{x} = A_i x$, $i = 1, \dots, N$, where each matrix A_i being Hurwitz.

For each $i = 1, \dots, N$ it holds that $\|e^{A_i t} x_0\| \leq c_i e^{-\lambda_i t} \|x_0\|$, where $\lambda_i > 0$

Hence, there also exists $c > 0$ and $\lambda > 0$ such that

$$\|e^{A_i t} x_0\| \leq c e^{-\lambda t} \|x_0\|$$

This implies that if we stay long enough in each mode such that $ce^{-\lambda t} < 1$, we have global asymptotic stability.

For switching signals with minimal dwell time of at least τ_d time units with $\tau_d > \frac{1}{\lambda} \ln c$, the system is (uniformly) GAS

$\longrightarrow$ More elaborate conditions exist, also for nonlinear switched system using Lyapunov functions

E.g. average dwell time assumptions

Switched systems with average dwell time assumptions

$$\dot{x} = f_i(x) \quad i = 1, \dots, N$$

For a switching signal $\sigma : [0, \infty) \to \{1, \dots, N\}$ we denote the number of switchings (discontinuities) in σ in the interval (t, T) by $N_\sigma(t, T)$.

We say that σ has the average dwell time τ_a if there exists a positive number N_0 such that

$$N_\sigma(t, T) \leq N_0 + \frac{T - t}{\tau_a} \quad \text{for all } T \geq t \geq 0$$

- $N_0 = 0$: no switching
- $N_0 = 1$: σ cannot switch twice on interval of length smaller than τ_a: minimal dwell time of τ_a!

Switched systems with average dwell time assumptions

$$\dot{x} = f_i(x) \quad i = 1, \dots, N$$

$$N_\sigma(t, T) \leq N_0 + \frac{T - t}{\tau_a} \quad \text{for all } T \geq t \geq 0$$

Theorem Suppose there exist C^1 functions $V_i : \mathbb{R}^n \to \mathbb{R}_+$, constants $0 < a < b$, $p \in \mathbb{N}$, $\mu \geq 1$ and $\lambda_0 > 0$ such that for all $i = 1, \dots, N$

- $a\|x\|^p \leq V_i(x) \leq b\|x\|^p$ for all x
- $\frac{\partial V_i}{\partial x} f_i(x) \leq -2\lambda_0 V_i(x)$ for all x
- $V_i(x) \leq \mu V_j(x)$ for all i, j and all x

Then the switched system is (uniformly) GAS for all σ with average dwell time

$$\tau_a > \frac{\ln \mu}{2\lambda_0}$$

and N_0 arbitrary.

Question: What happens if $\mu = 1$?

Problem C: Switched control

Introduction

Several "classical" controllers for continuous-time systems are hybrid:

- variable structure control
- sliding mode control
- relay control
- gain scheduling
- bang-bang time-optimal control
- fuzzy control

→ common characteristic: **Switching**

Motivation for switched controllers

Theorem (Brockett's necessary condition)

Consider system

$$\dot{x} = f(x,u) \qquad \text{with } x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ f(0,0) = 0$$

where f is smooth function.
If system is asymptotically stabilizable (around $x = 0$) using continuous feedback law $u = \alpha(x)$, then image of every open neighborhood of $(x,u) = (0,0)$ under f contains open neighborhood of $x = 0$.

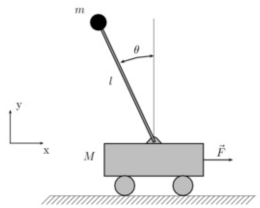
Motivation for switched controllers (cont.)

- For non-holonomic integrator: $$\dot{x} = u \\ \dot{y} = v \\ \dot{z} = xv - yu$$
- Is asymptotically stabilizable (see later)
- Satisfies Brockett's necessary condition?
  - If $f_1 = f_2 = 0$ then $f_3 = 0$
  - Hence, $(0,0,\varepsilon)$ cannot belong to image of f for any $\varepsilon \neq 0$
  - So non-holonomic integrator cannot be stabilized by continuous feedback

→ hybrid control schemes necessary to stabilize it!

Motivation for switched controllers

Inverted pendulum



Simplified equations ($u = F$)

$$\begin{aligned} \ddot{x} &= u \\ J\ddot{\theta} &= mgl\sin\theta - ml\,u\cos\theta \end{aligned}$$

Cannot be globally stabilized by a continuous $u = k(\theta, \dot{\theta})$!!!

Explanation: Equilibria would satisfy $\dot{\theta} = 0$ and

$$H(\theta) := mgl\sin\theta - ml\,k(\theta,0)\cos\theta = 0$$

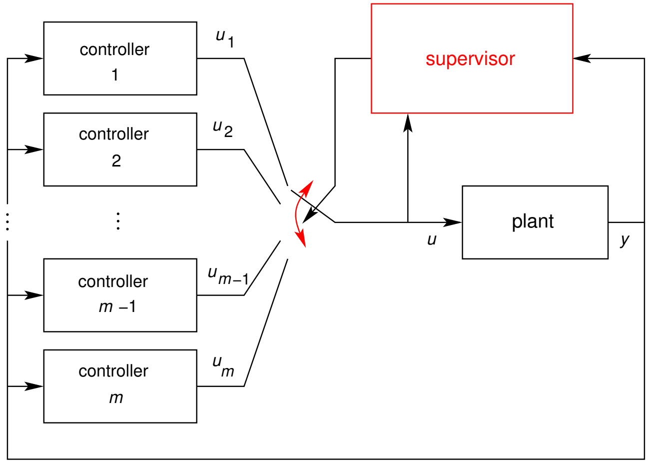
For $H(\pi/2) = mgl$ and $H(3\pi/2) = -mgl$. Thus $H(\theta_0) = 0$ for some $\pi/2 < \theta_0 < 3\pi/2$ due to continuity of H.

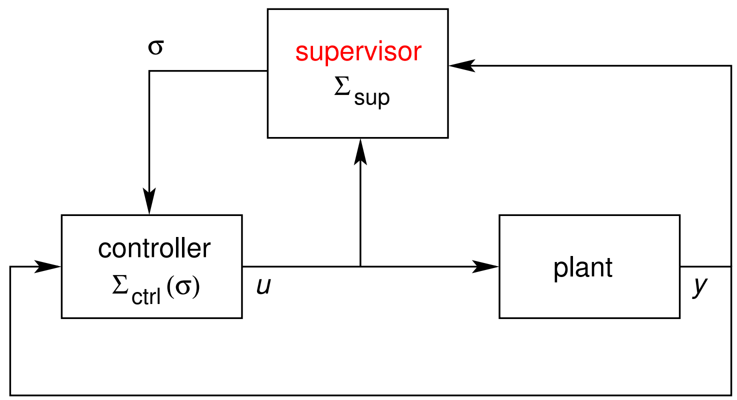
Undesired equilibrium preventing global asymptotic stability!!!

The system can be stabilized though by switching/discontinuous feedback!

`www.youtube.com/watch?v=MWJHcI7UcuE`

Switching control/logic





→ shared controller state variables

Stabiliz. switched linear systems via suitable switching (Pr. C)

$$\dot{x} = A_i x, \quad i \in I := \{1, 2, \dots, N\}$$

Find switching rule σ as function of time / state such that closed-loop is asymptotically stable.

Quadratic stabilization via a single Lyapunov function

Select $\sigma(x) : \mathbb{R}^n \to I := \{1, 2, \dots, N\}$ s.t. closed-loop has single quadratic Lyapunov function $x^T P x$.

One solution: convex combination of A_i is stable

$$A := \sum \alpha_i A_i \quad (\alpha_i \geq 0, \ \sum \alpha_i = 1) \text{ is stable}$$

Select $Q > 0$ and let $P > 0$ be solution of $A^T P + PA = -Q$.

Quadratic stabilization - continued

From $x^T(A^T P + PA)x = -x^T Q x < 0$ it follows that

$$\sum_i \alpha_i [x^T (A_i^T P + PA_i) x] < 0.$$

- For each x there is at least one mode with $x^T(A_i^T P + PA_i)x < 0$ or stronger

$$\bigcup_{i \in I} \{x \mid x^T(A_i^T P + PA_i)x \leq -\frac{1}{N} x^T Q x\} = \mathbb{R}^n$$

- Switching rule:

$$i(x) := \arg\min x^T (A_i^T P + PA_i) x$$

- Leads possibly to sliding modes. Alternative?

Alternative switching rule for quadratic stabilization

- A modified switching rule (based on hysteresis switching logic):

  ⋆ stay in mode i as long as $x^T(A_i^T P + PA_i)x \leq -\frac{1}{2N}x^T Qx$.

  ⋆ when bound reached, switch to a new mode j that satisfies

$$x^T(A_j^T P + PA_j)x \leq -\frac{1}{N}x^T Qx.$$

- There is a lower bound on the duration in each mode!
- no conservatism for 2 modes!

Theorem 1 If there exists a quadratically stabilizing state-dependent switching law for the switched linear system with $N = 2$, then the matrices A_1 and A_2 have a stable convex combination.

Stabilization via multiple Lyapunov functions (Problem C)

Main idea: Find $V_i(x) = x^T P_i x$ that decreases for $\dot{x} = A_i x$ in some region.

Define $\mathscr{X}_i := \{x \mid x^T[A_i^T P_i + P_i A_i]x < 0\}$.

If $\mathscr{X}_1 \cup \mathscr{X}_2 = \mathbb{R}^n$, try to switch to satisfy multiple Lyapunov criterion to guarantee asymptotic stability.

Find P_1 and P_2 such that they satisfy the coupled conditions:

$$x^T(P_1 A_1 + A_1^T P_1)x < 0 \text{ when } x^T(P_1 - P_2)x \geq 0,\ x \neq 0$$

and

$$x^T(P_2 A_2 + A_2^T P_2)x < 0 \text{ when } x^T(P_2 - P_1)x \geq 0,\ x \neq 0.$$

Then $\sigma(t) =$ arg max$\{V_i(x(t)) \mid i = 1,2\}$ stabilizing (V_σ = continuous)

... S-procedure ...

S-procedure There exist $\beta_1, \beta_2 \geq 0$ such that

$$-P_1 A_1 - A_1^T P_1 + \beta_1(P_2 - P_1) > 0$$
$$-P_2 A_2 - A_2^T P_2 + \beta_2(P_1 - P_2) > 0$$

$\sigma(t) =$ arg max$\{V_i(x(t)) \mid i = 1,2\}$ when you can find $\beta_1, \beta_2 \leq 0$

Only find switching sequence (discrete inputs)! What if also continuous inputs are present?

Stabilization of switched linear systems with continuous inputs

Switched linear system with inputs:

$$\dot{x} = A_i x + B_i u,\ i \in I = \{1, \ldots, N\}$$

Now $\sigma : [0, \infty) \to I$ and feedback controllers $u = K_i x$ are to determined.

Case 1: Determine K_i such that closed loop stable under arbitrary switching (assuming **know** mode)!

Case 2: Determine both $\sigma : [0, \infty) \to I$ and K_i

Case 3: σ given as function of state (PWL). Determine K_i

Case 1: Stabiliz. of switched linear system under arb. switching

$$\dot{x} = A_i x + B_i u,\ i \in I = \{1,\ldots,N\}$$

Sufficient condition: find a common *quadratic* Lyapunov function $V(x) = x^T P x$ for some positive definite matrix P and $K_1,\ldots,K_N$.

$$(A_i + B_i K_i)^T P + P(A_i + B_i K_i) < 0 \text{ for all } i = 1,\ldots,N \text{ and } P > 0$$

$$(A_i + B_i K_i)^T P + P(A_i + B_i K_i) < 0 \text{ for all } i = 1,\ldots,N \text{ and } P > 0$$

Pre- and postmultiplying by P^{-1}:

$$P^{-1}(A_i + B_i K_i)^T + (A_i + B_i K_i)P^{-1} < 0 \text{ for all } i = 1,\ldots,N \text{ and } P^{-1} > 0$$

Linear Matrix Inequalities

$$ZA_i + A_i Z + Y_i B_i^T + B_i Y_i < 0 \text{ for all } i = 1,\ldots,N \text{ and } Z > 0,$$

$P^{-1} =: Z$ and $K_i P^{-1} =: Y_i$. Hence, $P = Z^{-1}$ and $K_i = Y_i Z^{-1}$.

Hence, if LMIs feasible, then $u = K_i x$ leads to GUAS "cloop" under arbitrary switching **knowing** the mode as we use $u = K_i x$ when subsystem i is active!

Case 2: Design of switched feedback and switching sequence

$$\dot{x} = A_i x + B_i u,\ i \in I = \{1,2\}$$

Determine $\sigma : [0,\infty) \to I$ and $u = K_i x$, $i = 1,\ldots,N$

Use previous conditions for finding switching sequence

i) Find K_1, K_2 and $\alpha \in [0,1]$ such that $\alpha(A_1 + B_1 K_1) + (1-\alpha)(A_2 + B_2 K_2)$ is stable, i.e.

$$[\alpha(A_1 + B_1 K_1) + (1-\alpha)(A_2 + B_2 K_2)]^T P + P[\alpha(A_1 + B_1 K_1) + (1-\alpha)(A_2 + B_2 K_2)] < 0.$$

For fixed α previous transformation leads to LMIs!

ii) Find $\beta_1 \geq 0$, $\beta_2 \geq 0$, P_1 and P_2 positive definite and gains K_1 and K_2 such that

$$-P_1(A_1 + B_1 K_1) - (A_1 + B_1 K_1)^T P_1 + \beta_1(P_2 - P_1) > 0$$

$$-P_2(A_2 + B_2 K_2) - (A_2 + B_2 K_2)^T P_2 + \beta_2(P_1 - P_2) > 0.$$

Case 3: Design of switched feedback

If switching structure has already been given

$$\dot{x} = A_i x + B_i u, \text{ when } x \in \mathscr{X}_i,$$

$\bigcup_{i=1}^N \mathscr{X}_i = \mathbb{R}^n$ and $\mathscr{X}_i \cap \mathscr{X}_j$ for $i \neq j$ is a (lower-dimensional) boundary.

If $u = K_i x$ when $x \in \mathscr{X}_i$ we obtain closed-loop dynamics

$$\dot{x} = (A_i + B_i K_i)x, \text{ when } x \in \mathscr{X}_i$$

$\longrightarrow V(x) = x^T P x$ $\qquad$ $\mathscr{X}_i \subseteq \{x \mid E_i x \geq 0\}$

Find $K_1,\ldots,K_N$, $P > 0$ and symmetric U_i with nonnegative entries s.t.

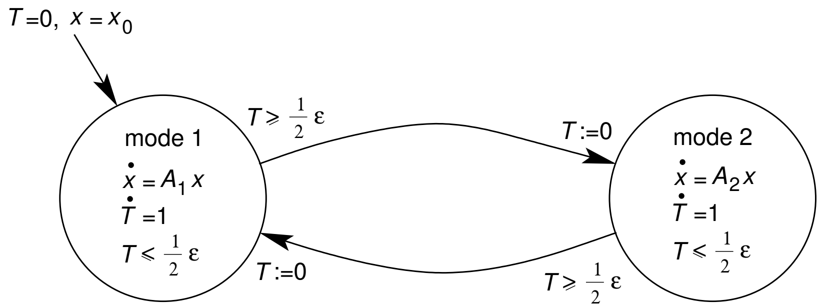
$$(A_i + B_i K_i)^T P + P(A_i + B_i K_i) + E_i^T U_i E_i < 0,\ i = 1,\ldots,N$$

- Extensions via continuous PWQ Lyapunov functions (BMIs!)

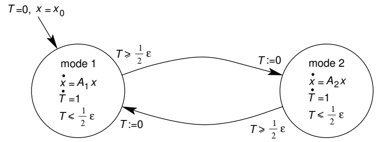
⟶ state-based switching previously ... now ...

Time-controlled switching/pulse width modulation

If dynamical system switches between several subsystems
→ stability properties of total system may be quite different from those of subsystems



Time-controlled switching



- $x(t_0+\frac{1}{2}\varepsilon) = \exp(\frac{1}{2}\varepsilon A_1)x_0 \;=\; x_0 + \frac{\varepsilon}{2}A_1x_0 + \frac{\varepsilon^2}{8}A_1^2x_0 + \cdots$

  $x(t_0+\varepsilon) = (I+\frac{\varepsilon}{2}A_2+\frac{\varepsilon^2}{8}A_2^2+\cdots)(I+\frac{\varepsilon}{2}A_1+\frac{\varepsilon^2}{8}A_1^2+\cdots)x_0$

  $= (I+\varepsilon[\frac{1}{2}A_1+\frac{1}{2}A_2]+\frac{\varepsilon^2}{8}[A_1^2+A_2^2+2A_2A_1]+\cdots)x_0.$

- Compare with

  $$\exp[\varepsilon(\tfrac{1}{2}A_1+\tfrac{1}{2}A_2)] \;=\; I+\varepsilon[\tfrac{1}{2}A_1+\tfrac{1}{2}A_2]+\tfrac{\varepsilon^2}{8}[A_1^2+A_2^2+A_1A_2+A_2A_1]+\cdots$$

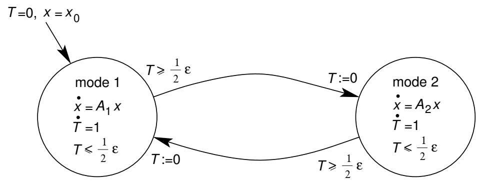
  → same for $\varepsilon \approx 0$

- So for $\varepsilon \to 0$ solution of switched system tends to solution of

  $$\dot{x} \;=\; (\tfrac{1}{2}A_1+\tfrac{1}{2}A_2)x \qquad (\text{“averaged” system})$$

- Possible that A_1 and A_2 are stable, whereas matrix $\frac{1}{2}A_1+\frac{1}{2}A_2$ is unstable, or vice versa.

Example

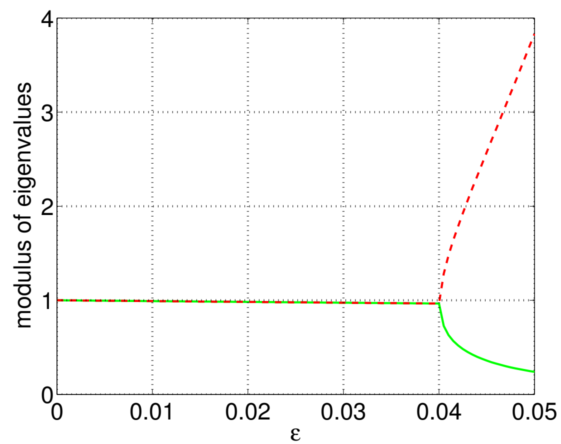


- Consider

$$A_1 \;=\; \begin{bmatrix} -0.5 & 1 \\ 100 & -1 \end{bmatrix}, \quad A_2 \;=\; \begin{bmatrix} -1 & -100 \\ -0.5 & -1 \end{bmatrix}$$

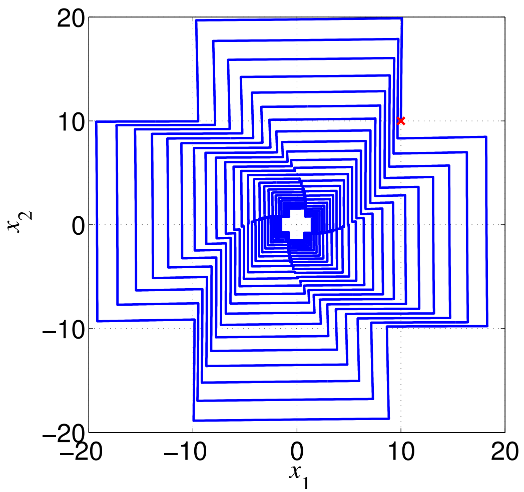
- A_1, A_2 unstable, but matrix $\frac{1}{2}(A_1+A_2)$ is stable
  → switched system should be stable if frequency of switching is sufficiently high
- Minimal switching frequency found by computing eigenvalues of the mapping $\exp(\frac{1}{2}\varepsilon A_1)\exp(\frac{1}{2}\varepsilon A_2)$ (Why?)

Example (cont.)



→ maximal value of ε: 0.04 ($50Hz$)

Example (cont.)



Pulse width modulation

- Assume mode 1 followed during $h\varepsilon$, and mode 2 during $(1-h)\varepsilon$
  → behavior of system is well approximated by system

$$\dot{x} = \big(hA_1 + (1-h)A_2\big)x$$

- Parameter h might be considered as control input
- If h varies, should be on time scale that is much slower than the time scale of switching
- If mode 1 is "power on" and mode 2 is "power off", then h is known as *duty ratio*
- Power electronics: fast switching theoretically provides possibility to regulate power without loss of energy
  → used in power converters (e.g., Boost converter)

Pulse width modulation (PWM)

- System: $\dot{x} = f(x,u), \quad u \in \{0,1\}$
- Duty cycle: Δ (fixed)
- u is switched exactly one time from 1 to 0 in each cycle
- Duty ration α: fraction of duty cycle for which $u = 1$

$$\begin{aligned} u(\tau) &= 1 \quad \text{for } t \leq \tau < t + \alpha\Delta \\ u(\tau) &= 0 \quad \text{for } t + \alpha\Delta \leq \tau < t + \Delta \end{aligned}$$

$$x(t+\Delta) = x(t) + \int_t^{t+\alpha\Delta} f(x(\tau),1)d\tau + \int_{t+\alpha\Delta}^{t+\Delta} f(x(\tau),0)d\tau$$

- Ideal averaged model ($\Delta \to 0$):

$$\dot{x}(t) = \lim_{\Delta\to 0} \frac{x(t+\Delta) - x(t)}{\Delta} = \alpha f(x(t),1) + (1-\alpha)f(x(t),0)$$

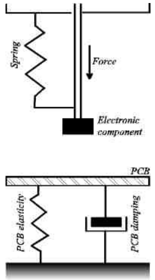
Observer design

Problem statement

- Consider the system:

$$\dot{x} = \begin{cases} A_1 x + Bu, & \text{if } H^T x \le 0 \\ A_2 x + Bu, & \text{if } H^T x > 0 \end{cases}$$
$$y = Cx,$$

Fig. 1. Fast component mounter (courtesy of Assembleon).



Goal: Design an observer that gives the state estimate $\hat{x}$, using only u,y as inputs

Linear case

$$\dot{x} = Ax + Bu \qquad x(0) = x_0$$
$$y = Cx$$

Observer: copy of the system and output injection term

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \qquad \hat{x}(0) = \hat{x}_0$$
$$\hat{y} = C\hat{x}$$

Estimated state $\hat{x}$ and observation error $e := x - \hat{x}$

$$\dot{e} = (A - LC)e$$

GAS ($e(t) \to 0$ when $t \to \infty$), when $A - LC$ Hurwitz or, equivalently

$$P > 0 \text{ and } (A - LC)^T P + P(A - LC) < 0 \text{ has a solution}$$

Note that this is equivalent to (A, C) being detectable (sufficient: observable)

Question: Is this a LMI? Why (not)?
Question: How can we influence the decrease rate of e?

Switched linear systems with known mode ...

$$\dot{x} = A_{\sigma(t)} x + B_{\sigma(t)} u$$
$$y = C_{\sigma(t)} x \qquad \sigma(t) \in \{1, 2, \ldots N\} \text{ known but arbitrary}$$

Observer
$$\dot{\hat{x}} = A_{\sigma(t)} \hat{x} + B_{\sigma(t)} u + L_{\sigma(t)}(y - \hat{y})$$
$$y = C_{\sigma(t)} \hat{x}$$

When mode is known...

$$\dot{x} = A_{\sigma(t)} x + B_{\sigma(t)} u$$
$$y = C_{\sigma(t)} x \qquad \sigma(t) \in \{1, 2, \ldots N\} \text{ known but arbitrary}$$

Observer
$$\dot{\hat{x}} = A_{\sigma(t)} \hat{x} + B_{\sigma(t)} u + L_{\sigma(t)}(y - \hat{y})$$
$$y = C_{\sigma(t)} \hat{x}$$

Observation error $e := x - \hat{x}$

$$\dot{e} = (A_{\sigma(t)} - L_{\sigma(t)} C_{\sigma(t)})e$$

Find common Lyap. function $V(e) = e^T P e$ s.t. $\dot{V} < 0$

$$(A_i - L_i C_i)^T P + P(A_i - L_i C_i) < 0 \text{ and } P > 0$$

Often mode is unknown ...

$$\dot{x} = \begin{cases} A_1 x + Bu, & \text{if } H^T x \leq 0 \\ A_2 x + Bu, & \text{if } H^T x > 0 \end{cases}$$
$$y = Cx,$$

- Proposed observer

$$\dot{\hat{x}} = \begin{cases} A_1\hat{x} + Bu + L_1(y-\hat{y}), & \text{if } H^T\hat{x} \leq 0 \\ A_2\hat{x} + Bu + L_2(y-\hat{y}), & \text{if } H^T\hat{x} > 0 \end{cases}$$
$$\hat{y} = C\hat{x}$$

- observation error $e = x - \hat{x}$

$$\dot{e} = \begin{cases} (A_1 - L_1C)e, & H^T x \leq 0, \quad H^T x - H^T e \leq 0 \\ (A_1 - L_1C)e - \Delta A x, & H^T x > 0, \quad H^T x - H^T e \leq 0 \\ (A_2 - L_2C)e + \Delta A x, & H^T x \leq 0, \quad H^T x - H^T e > 0 \\ (A_2 - L_2C)e, & H^T x > 0, \quad H^T x - H^T e > 0, \end{cases}$$

  where $\Delta A := A_1 - A_2$

We have N^2 modes in error dynamics because of inclusion of mixed modes

Stabilization of error dynamics

Use a Lyapunov function of the form

$$V(e) = e^T P e, \quad P = P^T > 0$$

and demand $\dot{V} \leq -\mu e^T e$, which yields

- $e^T\{(A_1 - L_1C)^T P + P(A_1 - L_1C) + \mu I\}e \leq 0,$

  when $H^T x \leq 0, H^T(x-e) \leq 0,$

- $\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} (A_2 - L_2C)^T P + P(A_2 - L_2C) + \mu I & P\Delta A \\ \Delta A^T P & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0$

  when $H^T x \leq 0, H^T(x-e) \geq 0,$

- $\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} (A_1 - L_1C)^T P + P(A_1 - L_1C) + \mu I & -P\Delta A \\ -\Delta A^T P & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0$

  when $H^T x \geq 0, H^T(x-e) \leq 0,$

- $e^T\{(A_2 - L_2C)^T P + P(A_2 - L_2C) + \mu I\}e \leq 0$

  when $H^T x \geq 0, H^T(x-e) \geq 0$

S-procedure: incorporating regional info

- by requiring
  $$\dot{V} \leq -\mu e^T e \quad \text{everywhere}$$
  global exponential stability of e is achieved
- from $H^T x \leq 0$ and $H^T(x-e) \geq 0$ we have $x^T H H^T (x-e) \leq 0$ or
  $$\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2}HH^T \\ -\frac{1}{2}HH^T & HH^T \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0$$
- from $H^T x \geq 0$ and $H^T(x-e) \geq 0$ we have $x^T H H^T (x-e) \leq 0$ or
  $$\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2}HH^T \\ -\frac{1}{2}HH^T & HH^T \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \geq 0$$
- previous condition can be used to relax requirements on $\dot{V}$ using S-procedure:
  $$x^T S x \geq 0 \Rightarrow x^T T x \geq 0$$

S-procedure

$$\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2}HH^T \\ -\frac{1}{2}HH^T & HH^T \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0$$

should imply

$$\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} (A_2 - L_2C)^T P + P(A_2 - L_2C) + \mu I & P\Delta A \\ \Delta A^T P & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0$$

Hence, it is sufficient to find $\lambda \geq 0$

$$\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} (A_2 - L_2C)^T P + P(A_2 - L_2C) + \mu I & P\Delta A \\ \Delta A^T P & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq \lambda \begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2}HH^T \\ -\frac{1}{2}HH^T & HH^T \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix}$$

S-procedure

$$\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2}HH^T \\ -\frac{1}{2}HH^T & HH^T \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0$$

should imply

$$\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} (A_2-L_2C)^TP+P(A_2-L_2C)+\mu I & P\Delta A \\ \Delta A^TP & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0$$

Hence, it is sufficient to find $\lambda \geq 0$

$$\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} (A_2-L_2C)^TP+P(A_2-L_2C)+\mu I & P\Delta A \\ \Delta A^TP & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq \lambda \begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2}HH^T \\ -\frac{1}{2}HH^T & HH^T \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix}$$

Theorem [Juloski, Heemels, Weiland, IJRNC 2007] If there exist L_1, L_2 and $\lambda \geq 0$, $\mu > 0$ and $P = P^T > 0$ such that

$$\begin{bmatrix} (A_2-L_2C)^TP+P(A_2-L_2C)+\mu I & P\Delta A+\lambda\frac{1}{2}HH^T \\ \Delta A^TP+\lambda\frac{1}{2}HH^T & -\lambda HH^T \end{bmatrix} \leq 0$$

$$\begin{bmatrix} (A_1-L_1C)^TP+P(A_1-L_1C)+\mu I & -P\Delta A+\lambda\frac{1}{2}HH^T \\ -\Delta A^TP+\lambda\frac{1}{2}HH & -\lambda HH^T \end{bmatrix} \leq 0$$

then the error dynamics is exponentially stable.

Question: what happened to the other (non-mixed) modes?

Main result

Theorem If there exist L_1, L_2 and $\lambda \geq 0$, $\mu > 0$ and $P = P^T > 0$ such that

$$\begin{bmatrix} (A_2-L_2C)^TP+P(A_2-L_2C)+\mu I & P\Delta A+\lambda\frac{1}{2}HH^T \\ \Delta A^TP+\lambda\frac{1}{2}HH^T & -\lambda HH^T \end{bmatrix} \leq 0$$

$$\begin{bmatrix} (A_1-L_1C)^TP+P(A_1-L_1C)+\mu I & -P\Delta A+\lambda\frac{1}{2}HH^T \\ -\Delta A^TP+\lambda\frac{1}{2}HH & -\lambda HH^T \end{bmatrix} \leq 0$$

then the error dynamics is exponentially stable.

- Only works for continuous PWL systems

  $$H^Tx = 0 \quad \Rightarrow \quad A_1x = A_2x$$

  which implies that $A_2 = A_1 + GH^T$ and thus

  $$\dot{x} = A_1x + G\max(H^Tx, 0) + Bu$$

- Absolute stability theory / Popov and circle criteria
- Exploiting continuity and common observer gain $L_1 = L_2$ simpler LMIs
- Similar results for discrete-time systems [Juloski, Heemels, Weiland, IJRNC 2007]
- What can you do when system discontinuous (recover mode, make effect x on e small) [Heemels, Weiland, Juloski, HSCC 2007]
- For systems with friction-like characteristics, see [Doris et al, CST 2008], [De Bruijn et al, Automatica, 2008], etc.

Continuous PWA system and common gain

$$\dot{x} = \begin{cases} A_1x+Bu, & \text{if } H^Tx \leq 0 \\ A_2x+Bu, & \text{if } H^Tx > 0 \end{cases}$$
$$y = Cx,$$

- Proposed observer

$$\dot{\hat{x}} = \begin{cases} A_1\hat{x}+Bu+L(y-\hat{y}), & \text{if } H^T\hat{x} \leq 0 \\ A_2\hat{x}+Bu+L(y-\hat{y}), & \text{if } H^T\hat{x} > 0 \end{cases}$$
$$\hat{y} = C\hat{x}$$

- Observation error $e = x - \hat{x}$ and $\Delta A := A_1 - A_2$

$$\dot{e} = \begin{cases} (A_1-LC)e, & H^Tx \leq 0, \; H^Tx-H^Te \leq 0 \\ (A_1-LC)e-\Delta Ax, & H^Tx > 0, \; H^Tx-H^Te \leq 0 \\ (A_2-LC)e+\Delta Ax, & H^Tx \leq 0, \; H^Tx-H^Te > 0 \\ (A_2-LC)e, & H^Tx > 0, \; H^Tx-H^Te > 0, \end{cases}$$

Theorem [Pavlov et al, book 2005] Suppose there exist $P > 0$ and observer gain L such that $(A_i - LC)^TP + P(A_i - LC) < 0 \quad i = 1, 2$, then the error dynamics is exponentially stable

- discrete-time case: [Heemels et al, CDC 2008]

Summary

- Problem B: GAS for a particular class of switching regimes: dwell time restrictions!
- Problem C: Construct a stabilizing switching sequence, a discrete control problem
  - State-dependent switching
    - Find convex combination that is Hurwitz: single LF
    - Multiple LF approach, "max"-switching law
  - Time-dependent switching based on Hurwitz convex combination
- Continuous control problem
  - construct K_i for all σ: common P via LMIs!
  - construct K_i and σ: use top 2 approaches (almost LMI for single LF)!
  - construct K_i given σ (PWL): use conditions from stability analysis (BMIs)!
- Observer design
- No complete systematic controller design (except optimization-based, but own problems)
- Open research area ....
- ... also identification, observer design, etc.: see final chapter for further reading!