OPTIMAL CONTROL OF HYBRID SYSTEMS: *DETERMINISTIC MODELS AND APPLICATIONS*

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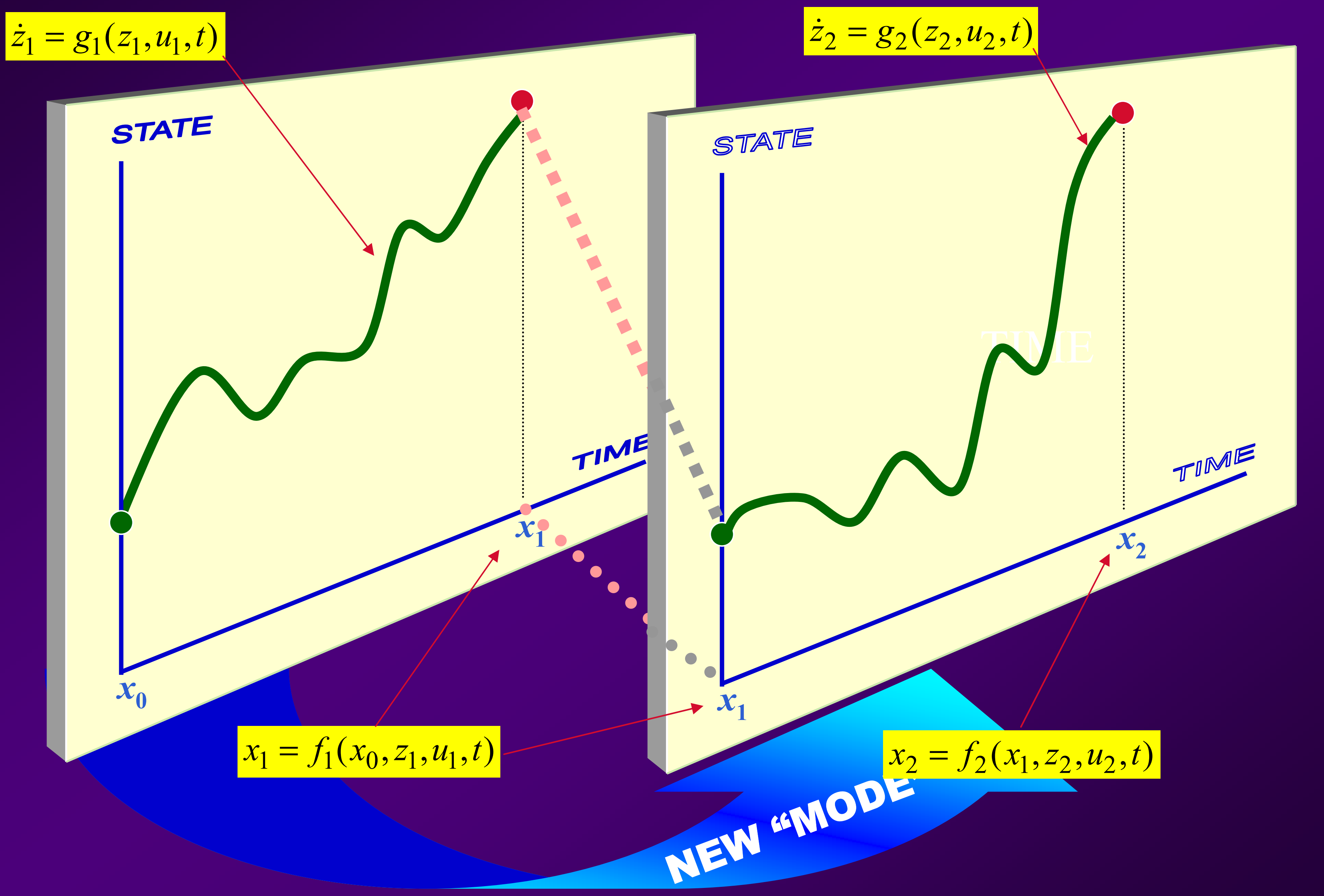
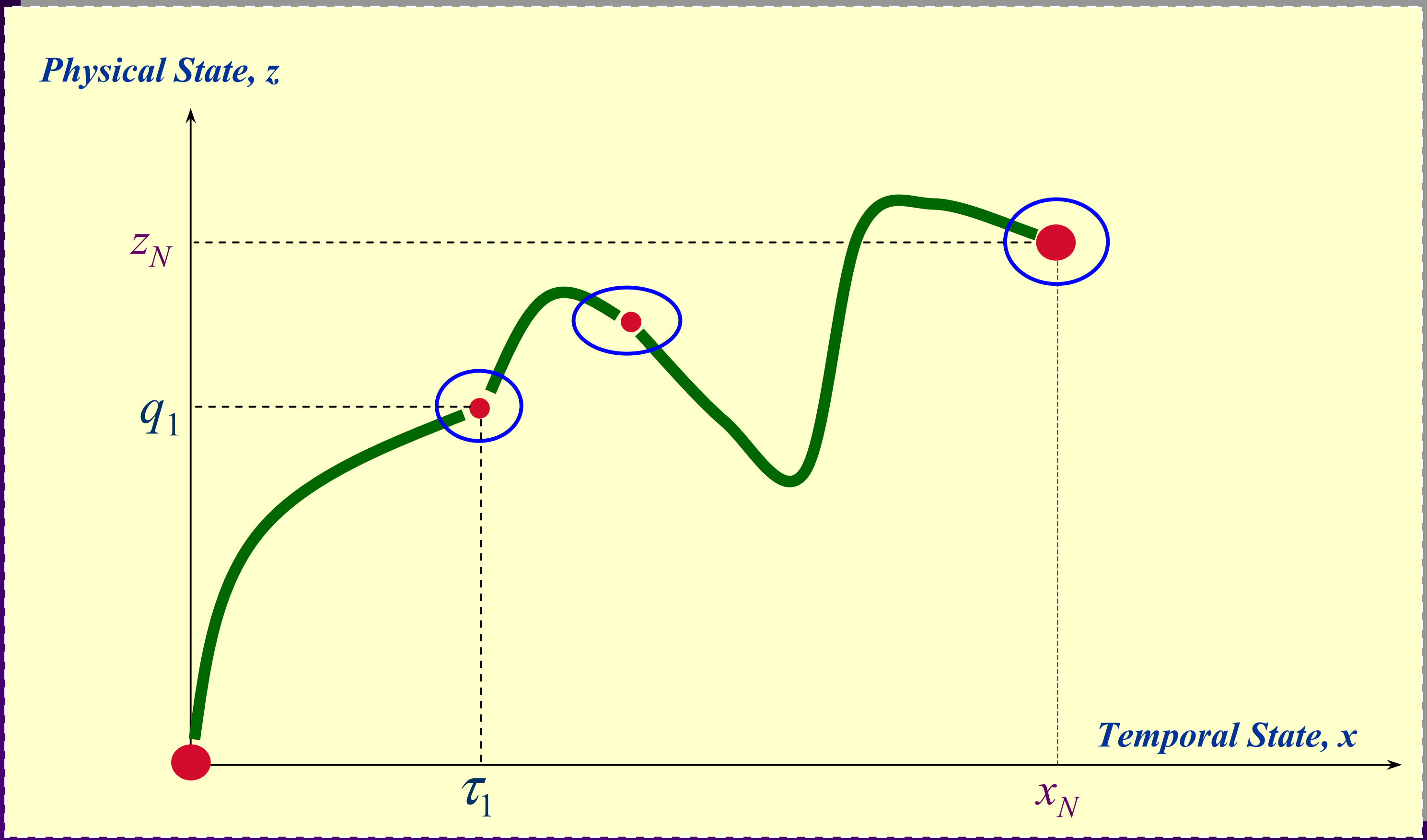
OUTLINE

- Optimal Control Problem Formulation
- Hierarchical Decomposition
- Problems with a *single* event process
- Problems with *asynchronous* event processes
- Complexity reduction: From *exponential* to *linear*
- **Single Stage Manufacturing System**:
  Optimal Control Problem Solution
- Interactive Software Demo

WHAT'S A HYBRID SYSTEM?



OPTIMAL CONTROL PROBLEMS



- Get to desired final physical state z_N in minimum time x_N, subject to N-1 switching events
- Minimize: - deviations from N desired physical states $(z_i - q_i)^2$
  - deviations from target desired times $(x_i - \tau_i)^2$

OPTIMAL CONTROL PROBLEMS

Temporal state

$$\min_{\mathbf{u}} \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} L_i(z_i(t), u_i(t))dt \quad s.t. \quad \begin{cases} \dot{z}_i = g_i(z_i, u_i, t) \\ x_{i+1} = f_i(x_i, u_i, t) \end{cases}$$

Physical state

Time-driven Dynamics

Event-driven Dynamics

- **Maximum principle extensions –** *Piccoli, 1998; Sussman, 1999*
- **Dynamic Programming extensions –** *Hedlund and Rantzer, 1999; Xu and Antsaklis, 2000*
- **Hierarchical decomposition -** *Gokbayrak and Cassandras, 2000; Xu and Antsaklis, 2000*
- **Mixed Integer Programming –** *Bemporad and Morari, 2002*

OPTIMAL CONTROL PROBLEMS

CONTINUED

$$\min_{\mathbf{u}} \sum_{i=1}^{N} \left[\int_{x_{i-1}}^{x_i} L_i(z_i(t), u_i(t))dt + \psi_i(x_i) \right]$$

Cost under $u_i(t)$ over $[x_{i-1}, x_i]$

$$\phi_i(x_i, x_{i-1})$$

Cost of switching time x_i

$$\min_{\mathbf{u}} \sum_{i=1}^{N} \left[\phi_i(x_i, x_{i-1}) + \psi_i(x_i) \right]$$

Let: $s_i = x_i - x_{i-1}$ ← **Time spent at ith operating mode**

Assume: $\phi_i(x_i, x_{i-1}) = \phi_i(s_i)$

HIERARCHICAL DECOMPOSITION

$$\min_{\mathbf{u}} \sum_{i=1}^{N} \left[\phi_i(s_i) + \psi_i(x_i) \right]$$

s.t.

$$\dot{z}_i = g_i(z_i, u_i, t)$$

$$x_{i+1} = f_i(x_i, u_i, t)$$

HIGHER LEVEL PROBLEM:

$$\min_{\mathbf{s}} \sum_{i=1}^{N} \left[\phi_i^*(s_i) + \psi_i(x_i) \right]$$

s.t.

$$x_{i+1} = f_i(x_i, s_i, t)$$

LOWER LEVEL PROBLEMS:

$$\min_{u_i} \phi_i(s_i) = \int_0^{s_i} L_i(z_i(t), u_i(t))dt$$

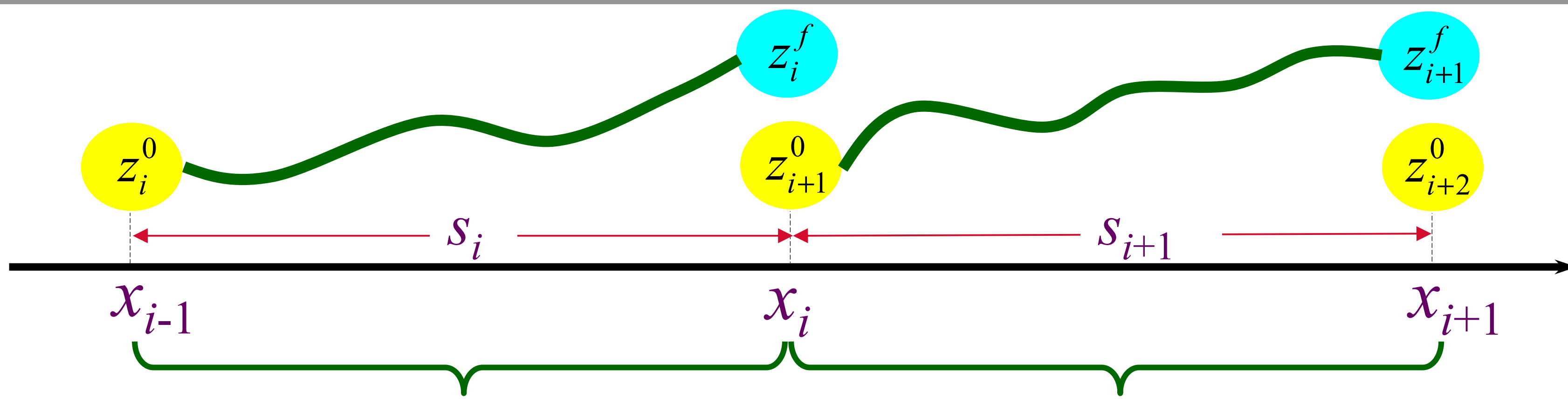
s.t.

$$\dot{z}_i = g_i(z_i, u_i, t)$$

FIXED s_i

HIERARCHICAL DECOMPOSITION

CONTINUED



$$\min_{u_i(z_i^0, z_i^f, s_i)} \phi_i(s_i) \qquad \min_{u_{i+1}(z_{i+1}^0, z_{i+1}^f, s_{i+1})} \phi_{i+1}(s_{i+1})$$

$$u_i^*(z_i^0, z_i^f, s_i)$$

$$\theta_i(z_i^0, z_i^f, s_i) = \min_{u_i} \phi_i(z_i, u_i, s_i)$$

"ROUTINE" OPTIMAL CONTROL PROBLEM!

$$\min_{\mathbf{z}^0, \mathbf{z}^f, \mathbf{s}} \sum_{i=1}^{N} \left[\theta_i(z_i^0, z_i^f, s_i) + \psi_i(x_i) \right] \quad s.t. \quad x_{i+1} = f_i(x_i, u_i, t)$$

<u>REALLY</u> CHALLENGING PROBLEM!

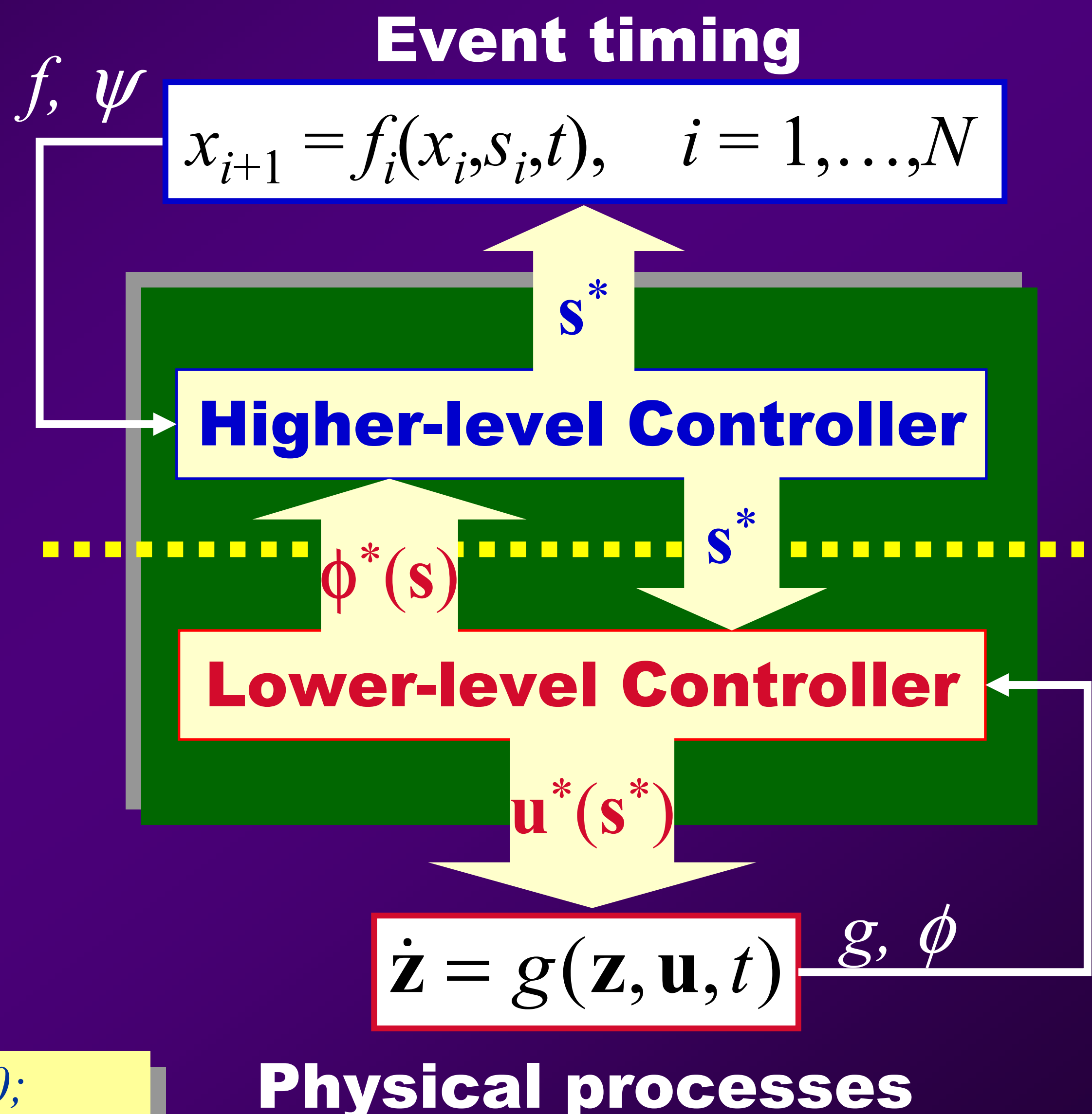
Typical example:

$$x_{i+1} = \max(x_i, a_{i+1}) + s_i(z_i, u_i)$$

HYBRID CONTROLLER STRUCTURE

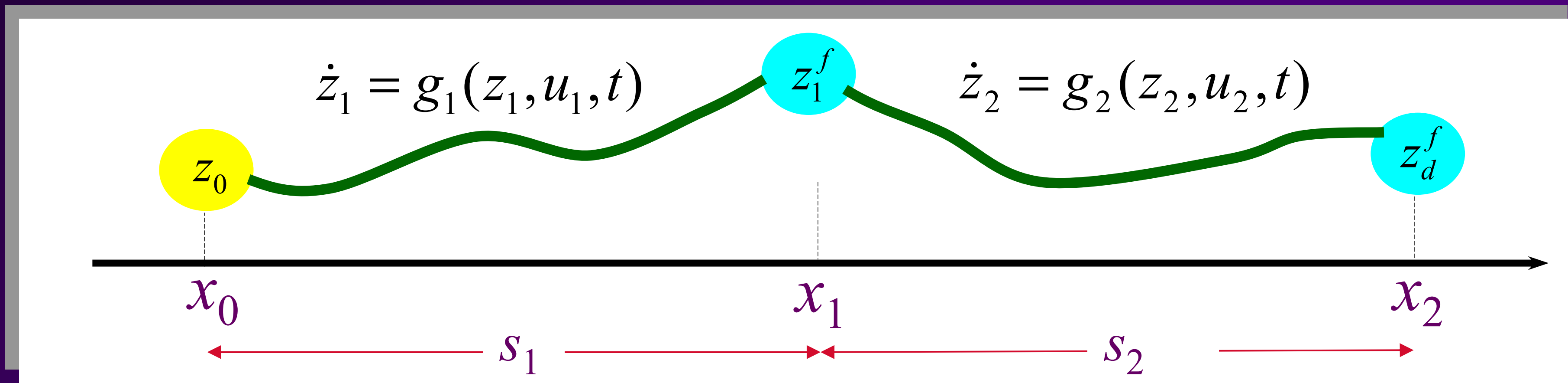
Hybrid controller steps:

- System identification
- Lower-level solution
- Higher-level solution
- Lower-level solution
- Operation...

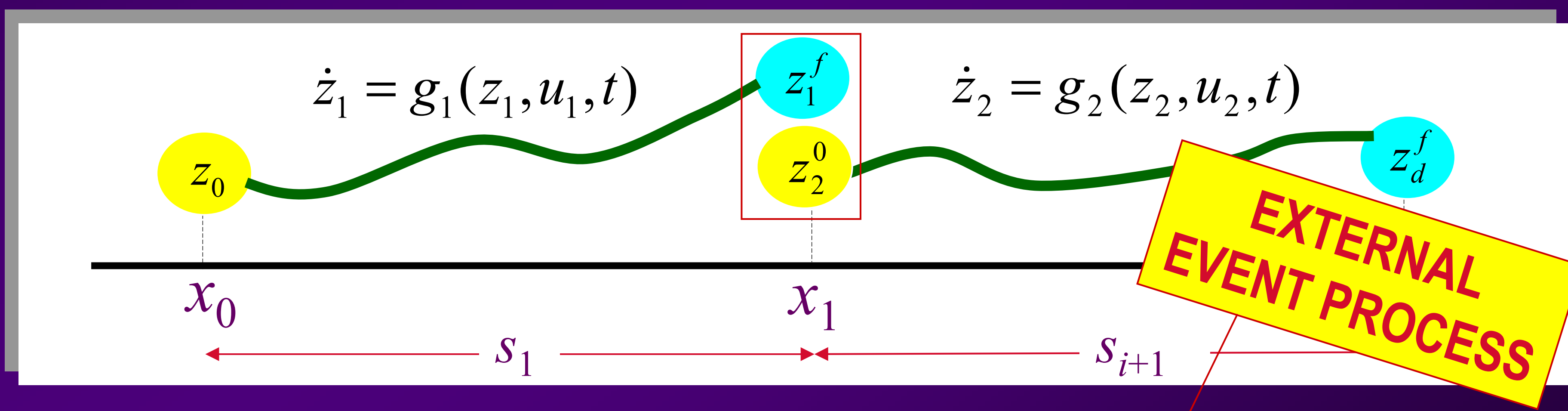


[*Gokbayrak and Cassandras, 2000;*
Xu and Antsaklis, 2000]

TWO TYPES OF PROBLEMS: *SYNCHRONOUS v. ASYNCHRONOUS*

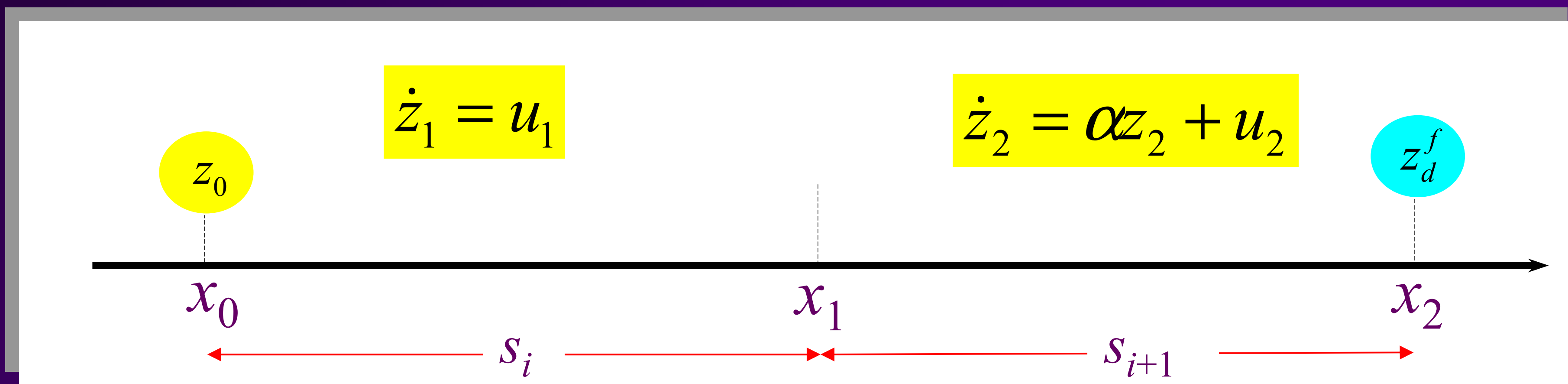


1. A single event process controls switching dynamics: $x_{i+1} = x_i + s_i(z_i, u_i)$



2. Multiple event processes control switching dynamics: $x_{i+1} = \max(x_i, a_{i+1}) + s_i(z_i, u_i)$

TYPE 1 *PROBLEM*: TWO-MODE LINEAR SYSTEM



OBJECTIVE:

$$\min J = \phi_1(z_1, s_1, u_1) + \phi_2(z_2, s_2, u_2) + \psi_2(x_2)$$

$$\phi_1: \int_0^{s_1} \frac{1}{2} r_1 u_1^2(t)\,dt$$

$$\phi_2: \frac{1}{2} h(z_2^f - z_d^f)^2 + \int_0^{s_2} \frac{1}{2} r_2 u_2^2(t)\,dt$$

$$\psi_2: \beta x_2^2$$

LOWER LEVEL PROBLEMS

LQ PROBLEM 1:

$$\min_{u_1} \phi_1(z_1, u_1, s_1) = \frac{1}{2}\int_0^{s_1} r_1 u_1^2(t)\,dt \qquad s.t. \qquad \dot{z}_1 = u_1$$

STANDARD LQ SOLUTION:

$$u_1(t) = u_1 = \frac{z_1^f - z_1^0}{s_1}$$

$$\theta_1(s_1, z_1^0, z_1^f) = \frac{1}{2}\frac{r_1}{s_1}\left(z_1^f - z_1^0\right)^2$$

LQ PROBLEM 2:

$$\min_{u_2} \phi_2(z_2, u_2, s_2) = \frac{1}{2} h(z_2^f - z_d^f)^2 + \frac{1}{2}\int_0^{s_2} r_2 u_2^2(t)dt$$

s.t. $\dot{z}_i = \alpha z_2 + u_2$

STANDARD LQ SOLUTION:

$$u_2(t) = -\frac{2\alpha\left(z_2^f - z_2^0 e^{\alpha s_2}\right)}{e^{-\alpha s_2} - e^{\alpha s_2}} e^{-\alpha t}$$

$$\theta_2(s_2, z_2^0, z_2^f) = \frac{1}{2} h\left(z_2^f - z_d^f\right)^2 + \frac{r_2 \alpha\left(z_2^f - z_2^0 e^{\alpha s_2}\right)^2}{e^{\alpha s_2} - e^{-\alpha s_2}} e^{-\alpha s_2}$$

HIGHER LEVEL PROBLEM

$$\min_{\substack{s_1,s_2,z_1^0,\\ z_2^0,z_1^f,z_2^f}} \left[\frac{1}{2}\frac{r_1}{s_1}(z_1^f - z_1^0)^2 + \frac{1}{2}h(z_2^f - z_d^f)^2 + \frac{\left(z_2^f - z_2^0 e^{\alpha s_2}\right)^2 \alpha r_2}{(e^{2\alpha s_2}-1)} + \beta x_2^2 \right]$$

s.t. $z_1^0 = z_0, \quad z_2^0 = z_1^f, \quad x_2 = s_1 + s_2, \quad s_1, s_2 \geq 0$

$$\min_{\substack{s_1,s_2,\\ z_1^f,z_2^f}} \left[\frac{1}{2}\frac{r_1}{s_1}(z_1^f - z_0)^2 + \frac{1}{2}h(z_2^f - z_d^f)^2 + \frac{\left(z_2^f - z_1^f e^{\alpha s_2}\right)^2 \alpha r_2}{(e^{2\alpha s_2}-1)} + \beta(s_1 + s_2)^2 \right]$$

HIGHER LEVEL PROBLEM

CONTINUED

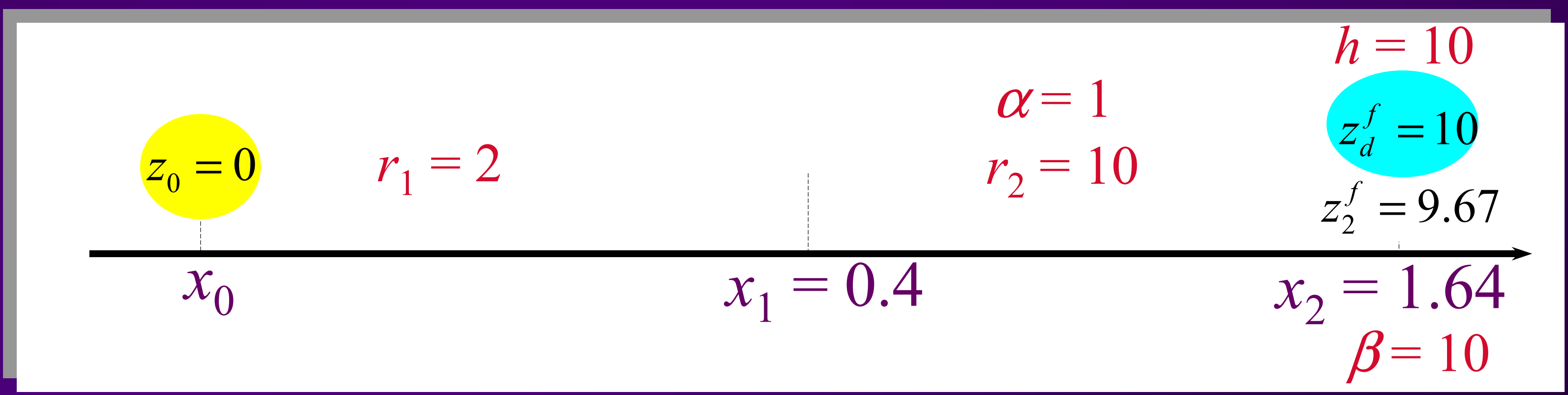
Optimality conditions yield four nonlinear algebraic equations:

$$\frac{r_1}{s_1}(z_1^f - z_0) = \frac{2\left(z_2^f - z_1^f e^{\alpha s_2}\right)\alpha r_2}{\left(e^{\alpha s_2} - e^{-\alpha s_2}\right)}$$

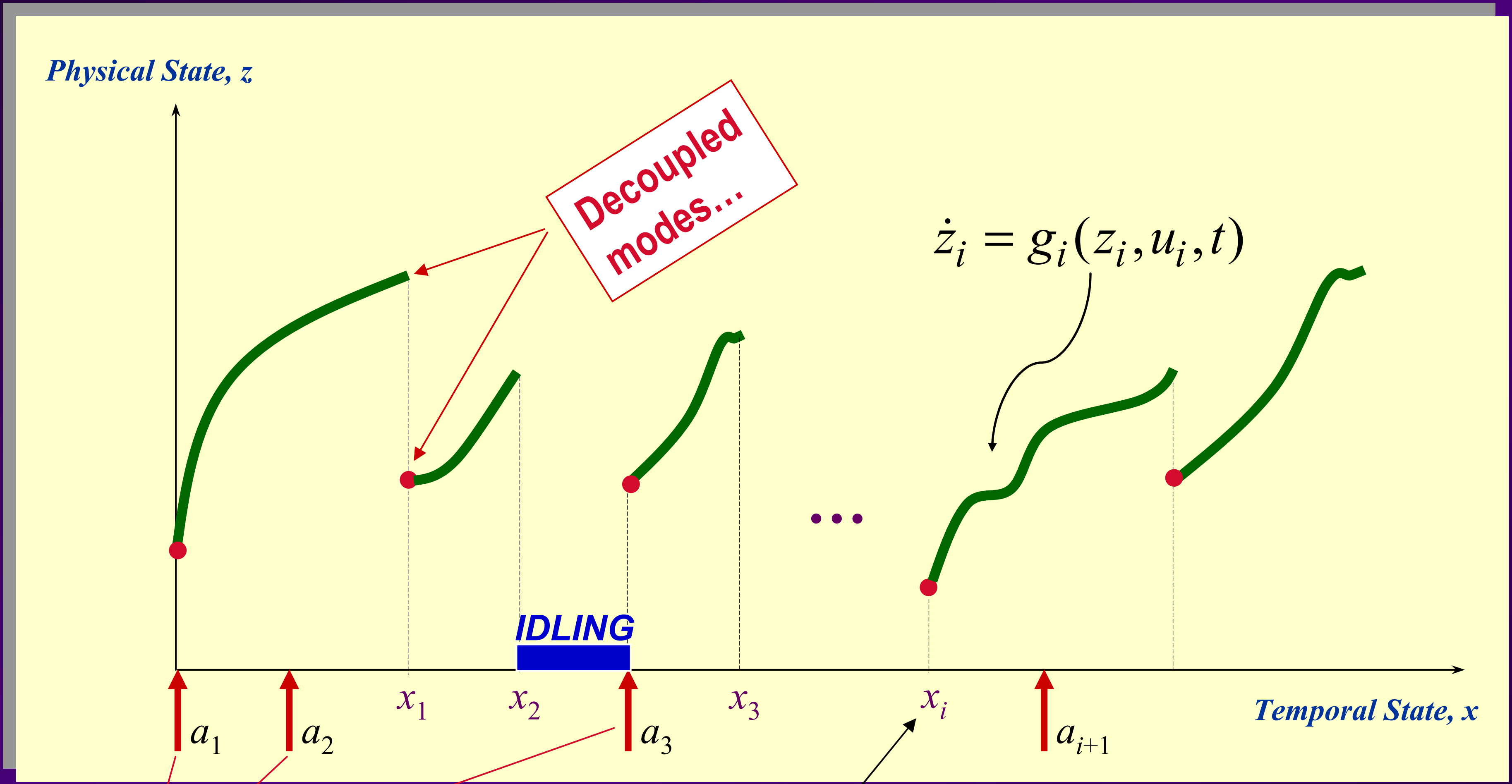
$$h(z_2^f - z_d^f) = -\frac{2\left(z_2^f - z_1^f e^{\alpha s_2}\right)\alpha r_2}{\left(e^{2\alpha s_2} - 1\right)}$$

$$r_1(z_1^f - z_0)^2 = 4\beta s_1^2(s_1 + s_2)$$

$$\beta(s_1 + s_2) = \frac{\alpha^2 r_2\left(z_2^f - z_1^f e^{\alpha s_2}\right)\left(z_1^f e^{\alpha s_2}\right)}{\left(e^{2\alpha s_2} - 1\right)} + \frac{\alpha^2 r_2 e^{2\alpha s_2}\left(z_2^f - z_1^f e^{\alpha s_2}\right)^2}{\left(e^{2\alpha s_2} - 1\right)^2}$$



TYPE 2 PROBLEM: MANUFACTURING SYSTEM



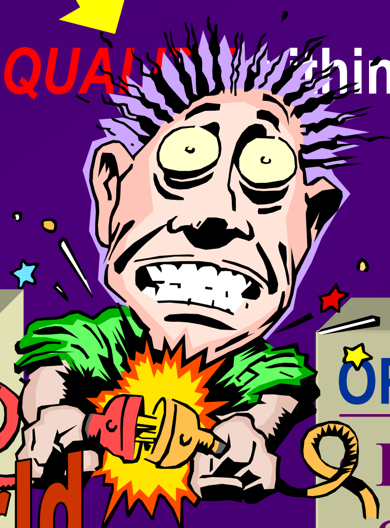
$$x_i = \max\{x_{i-1}, a_i\} + s_i(z_i, u_i)$$

External event process: *i*th mode cannot start before a_{i+1}

HYBRID SYSTEM IN *MANUFACTURING*

Key questions facing manufacturing system integrators:

- How to integrate '*process control*' with '*operations control*' ?
- How to improve product *QUALITY* within reasonable *TIME* ?



PROCESS CONTROL

- Physicists
- Material Scientists
- Chemical Engineers
- ...

Time-Driven World

OPERATIONS CONTROL

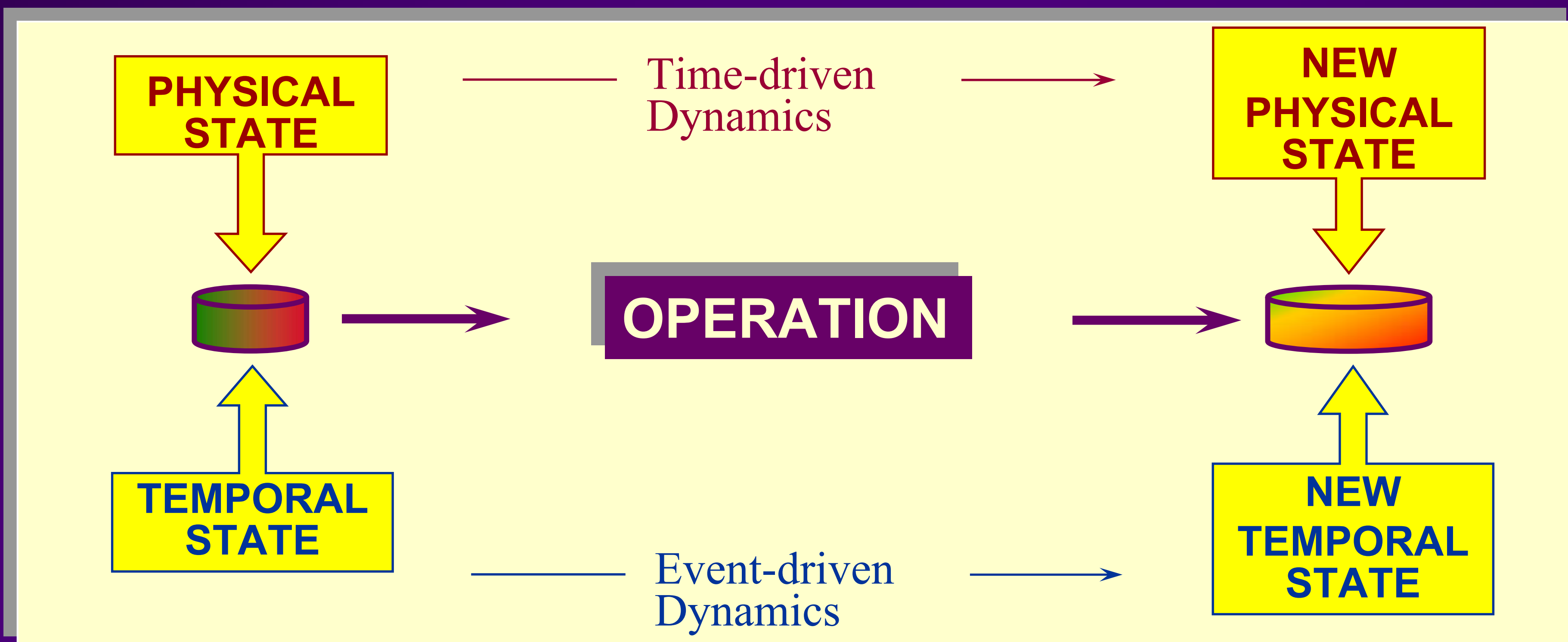
- Industrial Engineers, OR
- Schedulers
- Inventory Control
- ...

Event-Driven World

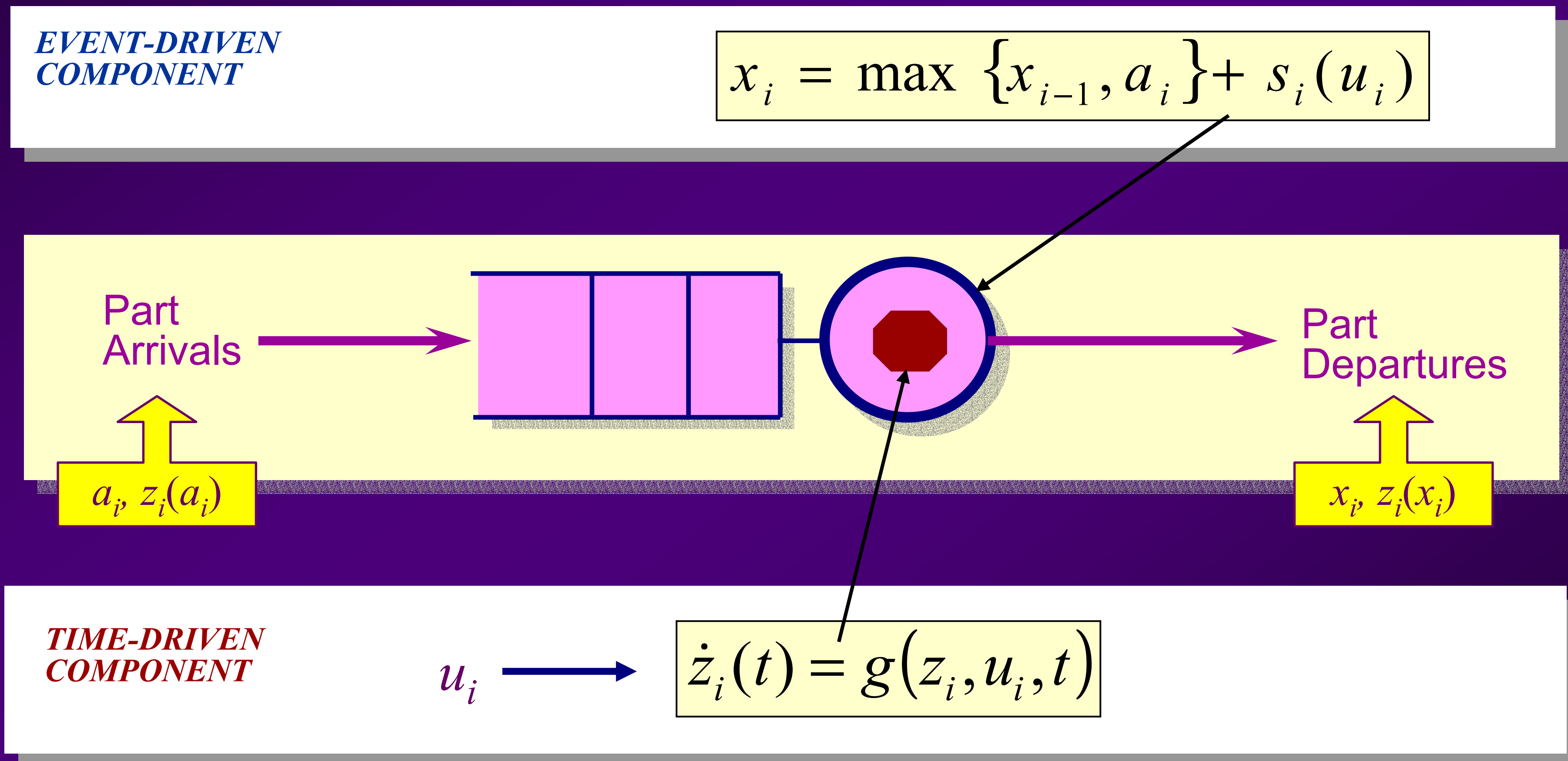
HYBRID SYSTEM IN *MANUFACTURING* CONTINUED

Throughout a manuf. process, each part is characterized by

- A **PHYSICAL** state (e.g., size, temperature, strain)
- A **TEMPORAL** state (e.g., total time in system, total time to due-date)



HYBRID SYSTEM IN *MANUFACTURING* *CONTINUED*



LOWER LEVEL PROBLEM

LQ PROBLEM:

Parameterized by switching times

$$\min_{u_i} \phi_i(z_i, u_i, s_i) = \frac{1}{2} h(z_{fi} - z_{di})^2 + \int_0^{s_i} \frac{1}{2} r u_i^2(t) dt$$

$$s.t. \quad \dot{z}_i = a z_i + b u_i, \quad z_i(0) = \varsigma_i$$

Penalize final state deviation

STANDARD LQ SOLUTION METHOD:

$$\phi_i^*(s_i) = \frac{1}{2} h(z_{fi}^* - z_{di})^2 + \int_0^{s_i} \frac{1}{2} r u_i^*(t) dt$$

HIGHER LEVEL PROBLEM

$$\min_{\mathbf{s}} \sum_{i=1}^{N} \left[\phi_i^*(s_i) + \psi_i(x_i) \right] \quad s.t. \quad x_i = \max\{x_{i-1}, a_i\} + s_i(u_i)$$

Cost of optimal process control over interval [0, s_i]

Cost related to event timing

Given arrival sequence (INPUT)

Processing time (CONTOLLABLE)

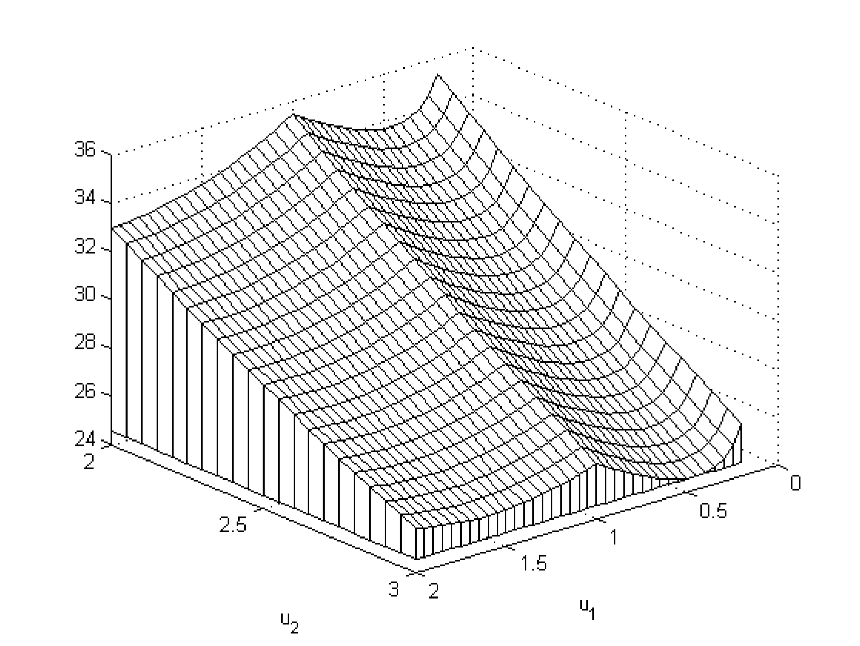
$$\text{EXAMPLE}: \quad \psi_i(x_i) = (x_i - \tau_i)^2$$

HOW DO WE SOLVE THE HIGHER LEVEL PROBLEM?

$$\min_{\mathbf{s}} \sum_{i=1}^{N} \left[\phi_i^*(s_i) + \psi_i(x_i) \right] \quad s.t. \quad x_i = \max\{x_{i-1}, a_i\} + s_i(u_i)$$

Even if these are convex, problem is still NOT convex in s!

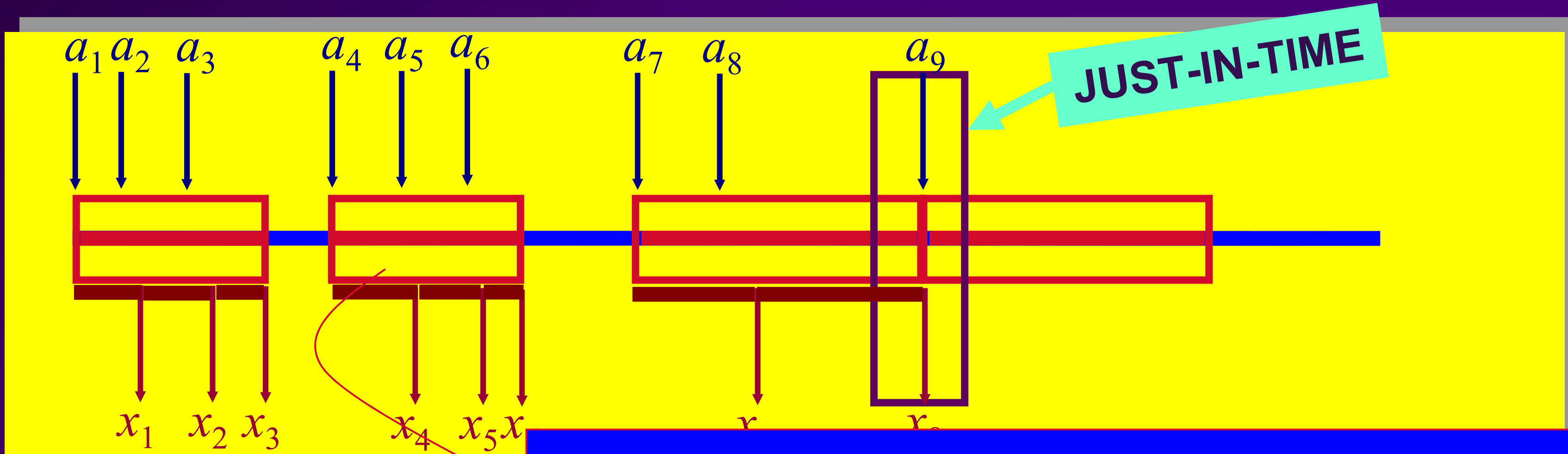
Causes nondifferentiabilities!



Even though problem is **NONDIFFERENTIABLE** and **NONCONVEX**, optimal solution shown to be ***unique***.

[*Cassandras, Pepyne, Wardi, IEEE TAC 2001*]

SOLVING THE HIGHER LEVEL PROBLEM *CONTINUED*



Each "block" corresponds to a ***Constrained Convex Optimization*** problem

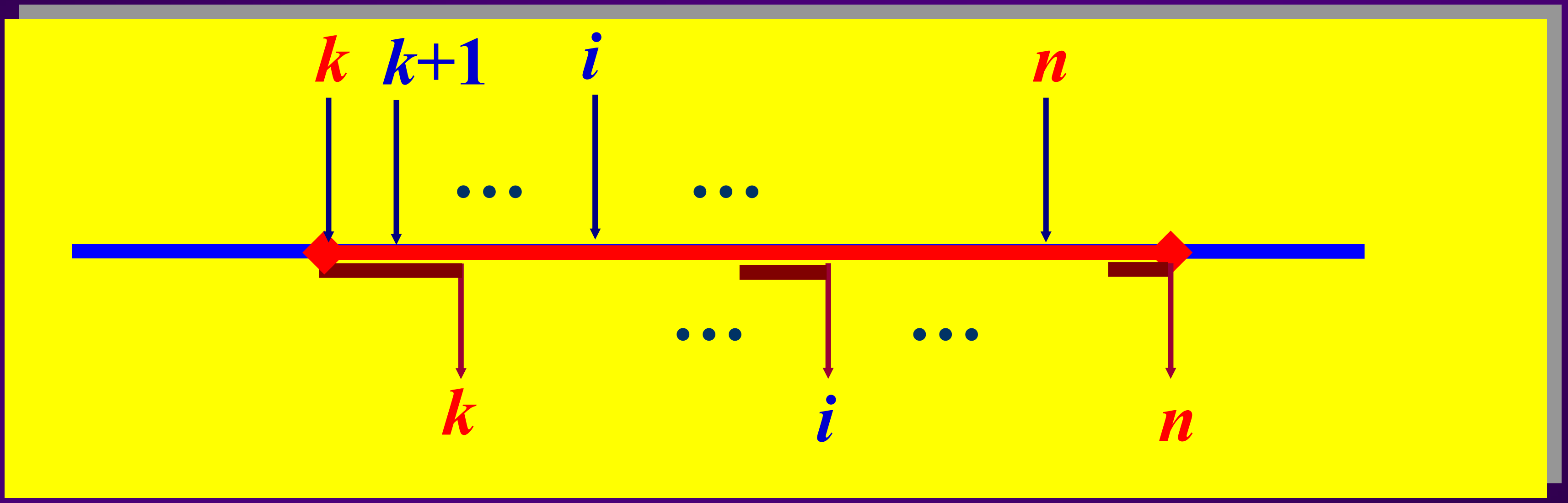
$\Rightarrow$ search over 2^{N-1} possible *Constrained Convex Optimization* problems

BUT algorithms that only need *N Constrained Convex Optimization* problems have been developed $\Rightarrow$ SCALEABILITY

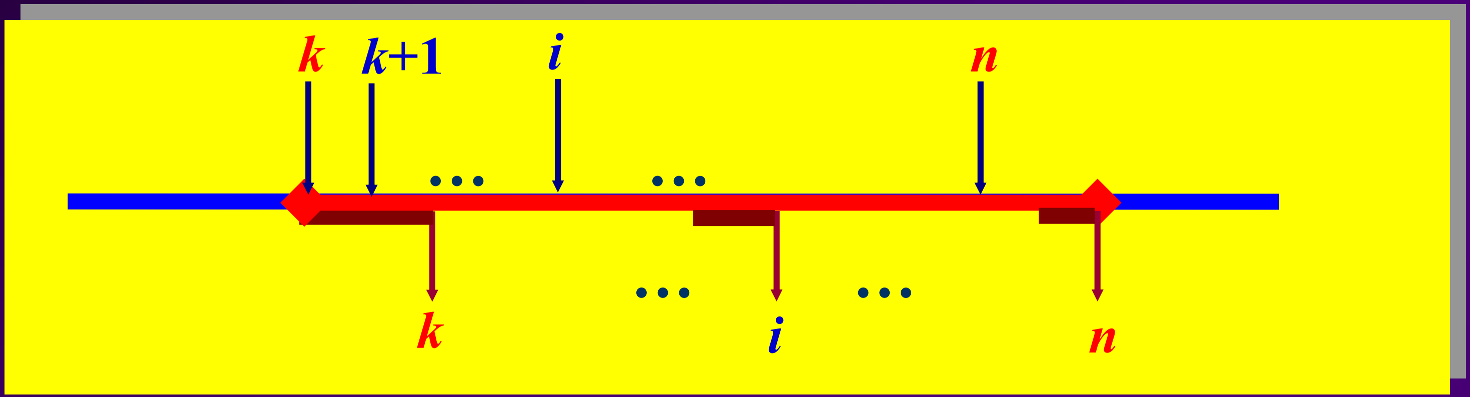
[*Cho, Cassandras, Intl. J. Rob. and Nonlin.Control, 2001*]

THE FORWARD ALGORITHM

Suppose we knew the BUSY PERIOD STRUCTURE of the optimal state trajectory…



What can we say about this single BUSY PERIOD defined by jobs (k, n) ?



FACTS:

$$1.\quad x_i \geq a_{i+1}, \qquad i = k,\ldots,n-1$$

$$2.\quad x_i = a_k + \sum_{j=k}^{i} u_j, \qquad i = k,\ldots,n$$

The optimal controls $u_k,\ldots,u_n$ minimize:

$$J_{k,n}(u_k,\ldots,u_n) = \sum_{i=k}^{n} [\theta_i(u_i) + \psi_i(x_i)]$$

$$= \sum_{i=k}^{n} [\theta_i(u_i) + \psi_i(u_k + \sum_{j=k}^{i} u_j)]$$

The optimal controls $u_k,\ldots,u_n$ are the solution of the nonlinear optimization problem:

$$\min_{u_k,\ldots,u_n} J_{k,n}(u_k,\ldots,u_n) \quad \text{s.t.}$$

$$u_k + \sum_{j=k}^{i} u_j \geq a_{i+1}, \quad i = k,\ldots,n-1;$$

$$u_i \geq 0, \qquad i = k,\ldots,n$$

$\Bigg\} \; Q_{k,n}$

OBVIOUS ALGORITHM:

- Scan over all Busy Period (BP) structures
- For each structure, solve the nonlinear program $Q_{k,n}$
- Check for consistency

QUESTION: How many BP structures are there?

ANSWER: 2^{N-1} ← **TOO MANY!**

An important structural property:

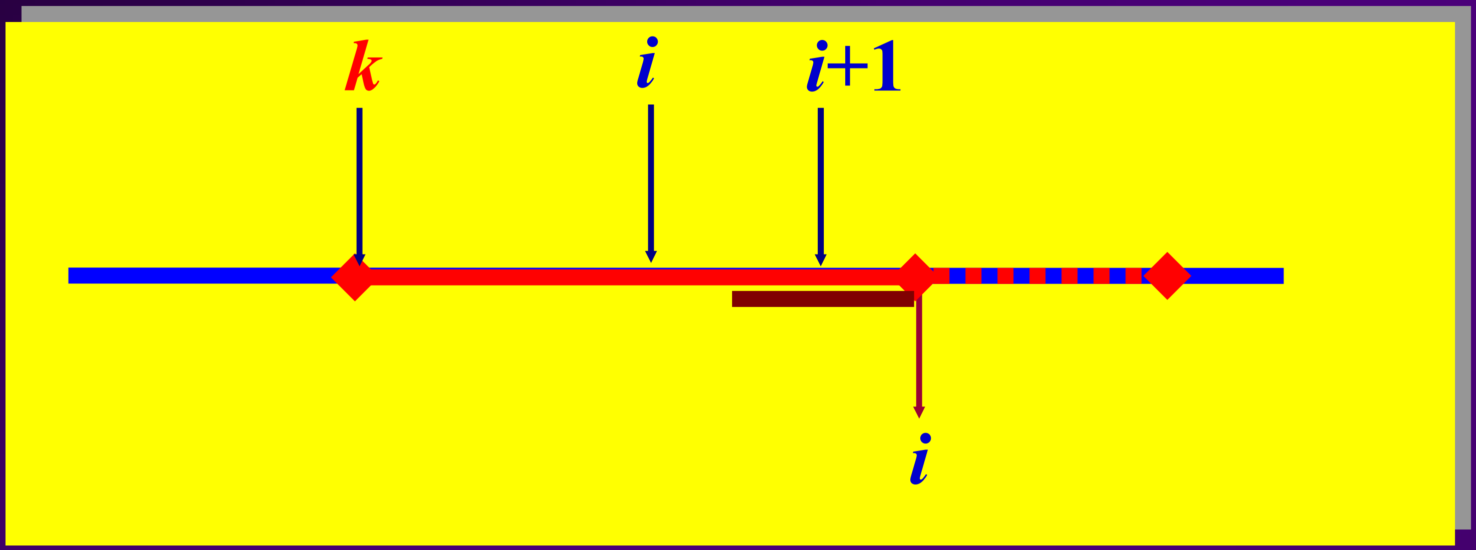
THEOREM: Let $k,\ldots,i$ be contiguous jobs in a BP of the optimal trajectory with completion times x_j, $j = k,\ldots,i$. Then, the BP ends with job i ***if and only if*** $x_i < a_{i+1}$

FORWARD ALGORITHM:

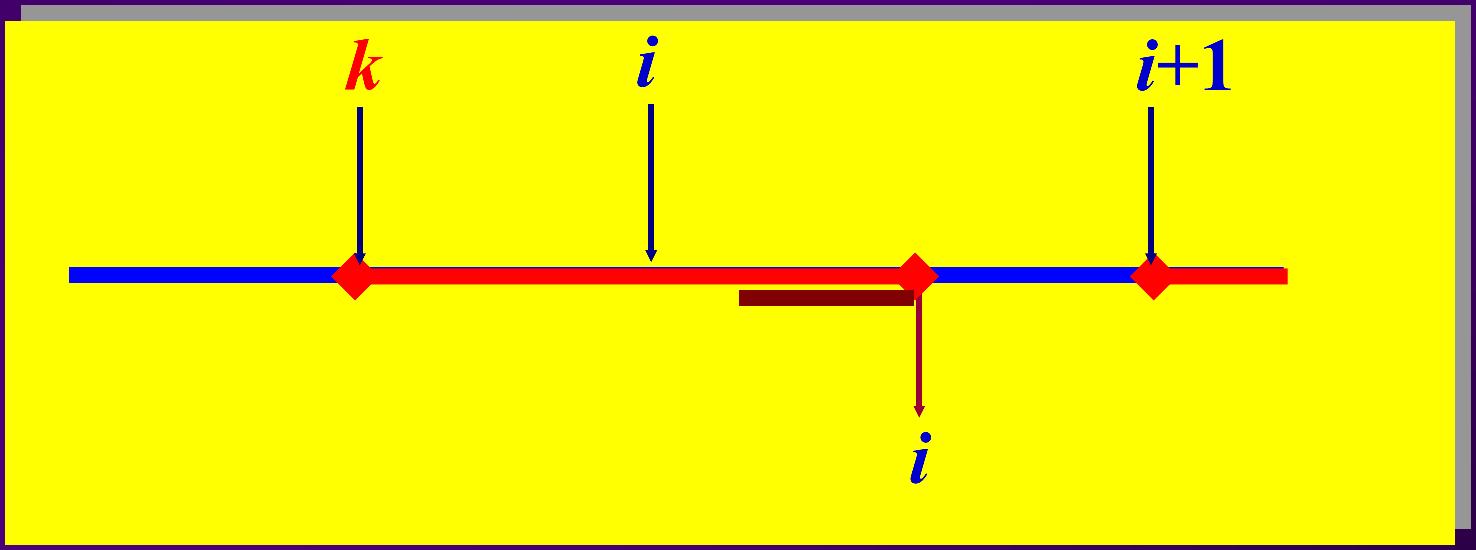
- Knowing that job *k* starts a BP, solve $Q_{k,i}$
- … until a new BP is detected: $x_i < a_{i+1}$

THE FORWARD ALGORITHM

CONTINUED

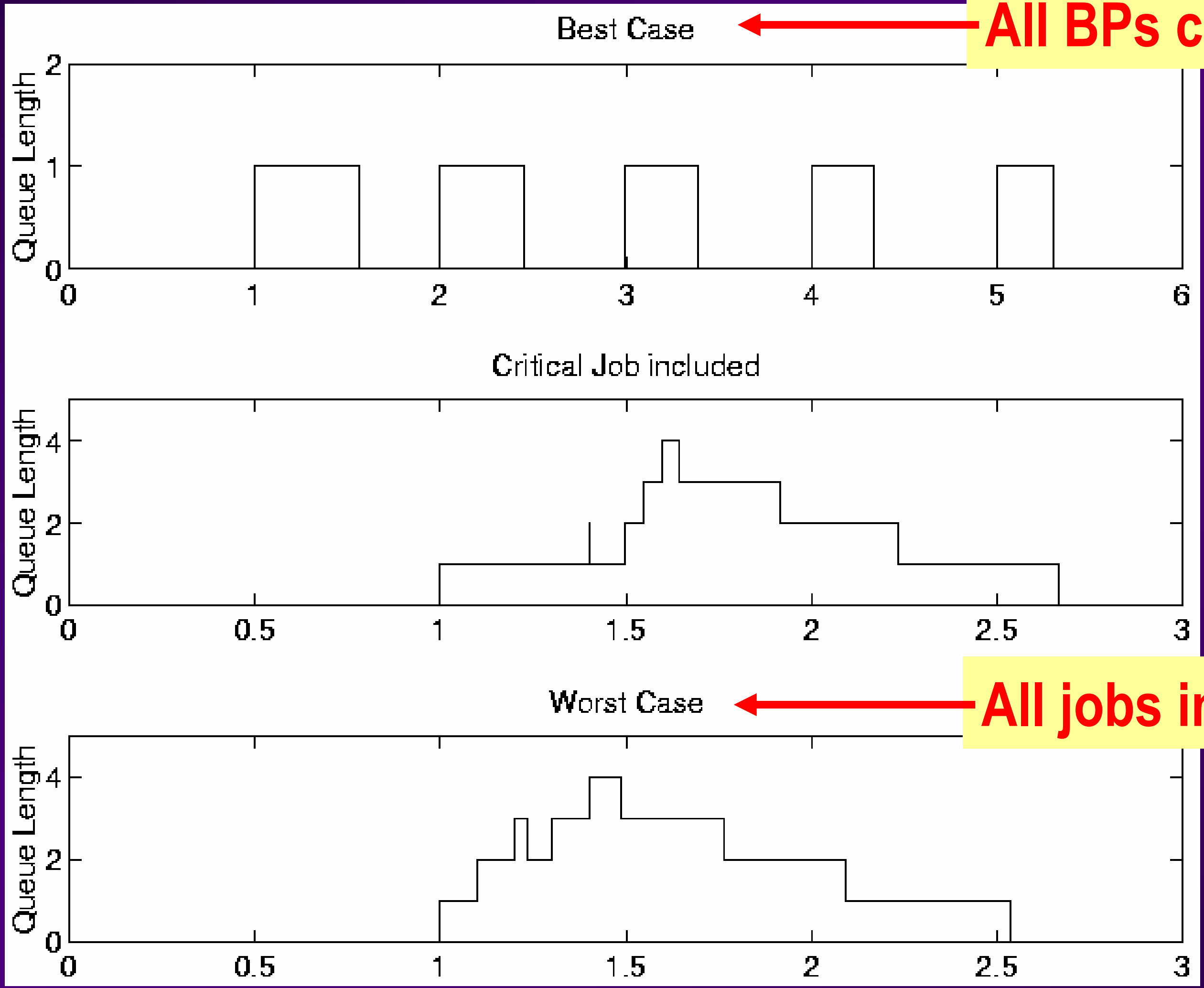


CASE 1



CASE 2

THE FORWARD ALGORITHM - PERFORMANCE



All BPs contain 1 job

All jobs in one BP

http://vita.bu.edu/cgc/hybrid