QUANTIZED SYSTEMS AND CONTROL

Daniel Liberzon

Coordinated Science Laboratory and
Dept. of Electrical & Computer Eng.,
Univ. of Illinois at Urbana-Champaign

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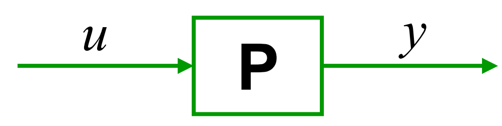
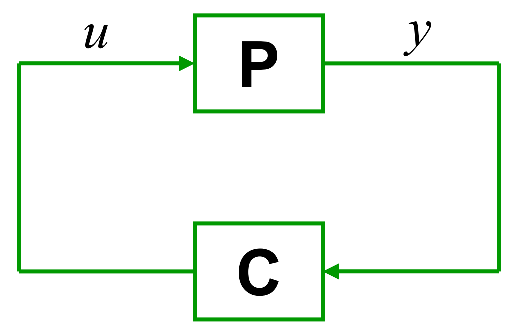
HYBRID CONTROL

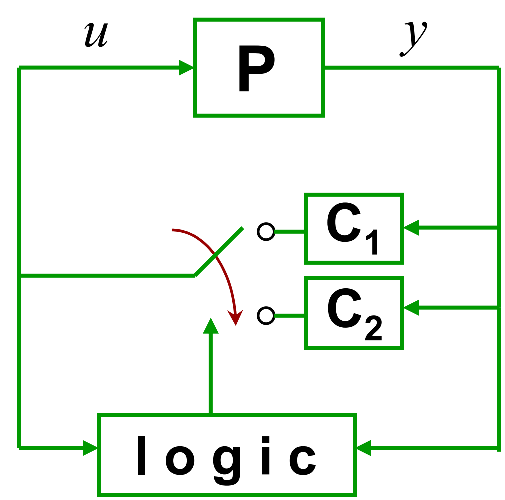
Plant:



Classical continuous feedback paradigm:



But logical decisions are often necessary:



The closed-loop system is hybrid

REASONS for SWITCHING

- Nature of the control problem
- Sensor or actuator limitations
- Large modeling uncertainty
- Combinations of the above

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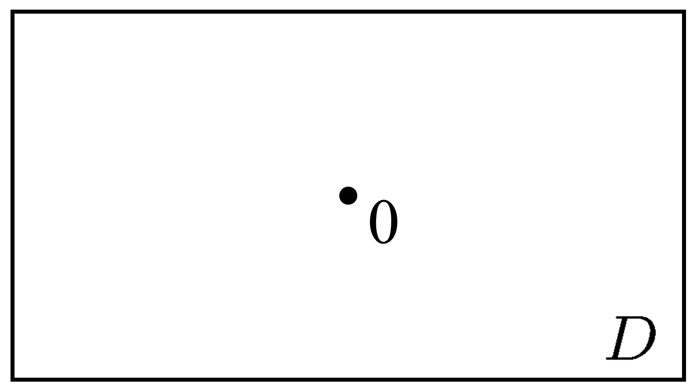
CONSTRAINED CONTROL

$$\dot{x} = f(x, u)$$



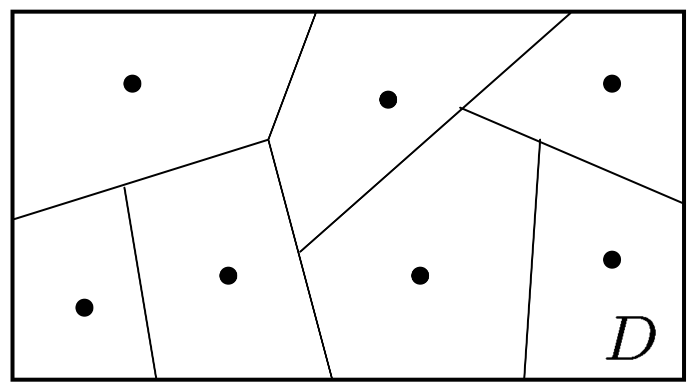
Control objectives: stabilize to 0 or to a desired set containing 0, exit D through a specified facet, etc.

Constraint: $u \in \{u_1, \dots, u_N\}$, N – given

control commands

LIMITED INFORMATION SCENARIO

$$\dot{x} = f(x, u)$$



$\mathcal{W} = \{W_1, \dots, W_N\}$ – partition of D

$\mathcal{Q} = \{q_1, \dots, q_N\}$ – points in D, $q_i \in W_i$

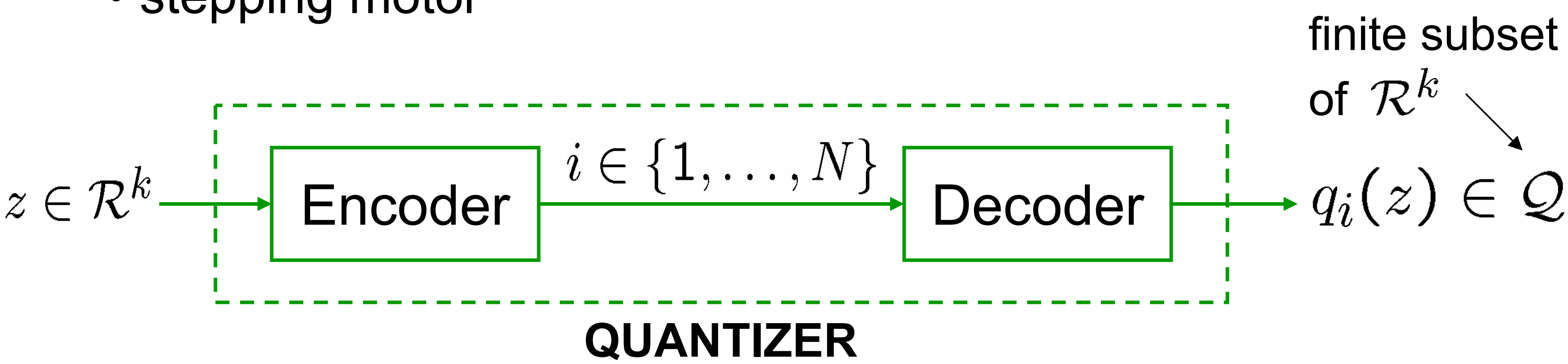
Quantizer/encoder:

$q : D \to \{1, \dots, N\}$, $q(x) = q_i$ for $x \in W_i$

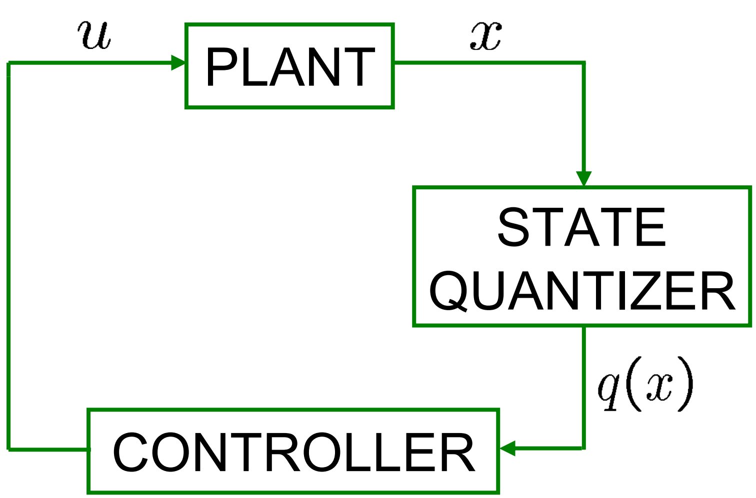
Control: $u = k(q(x))$

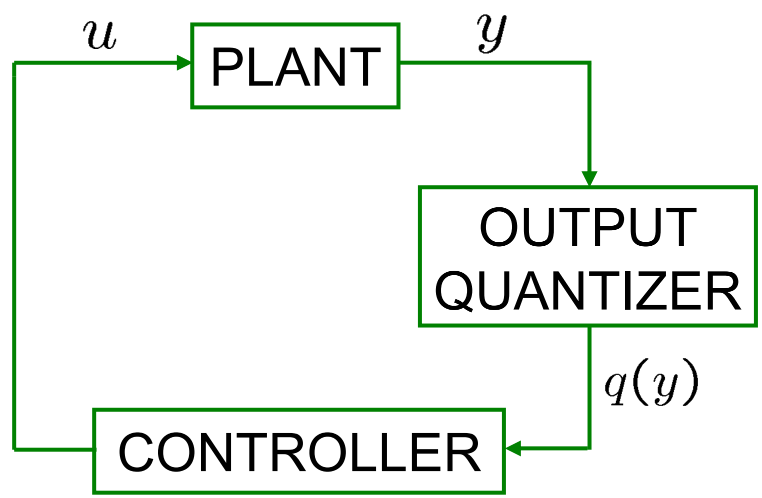
MOTIVATION

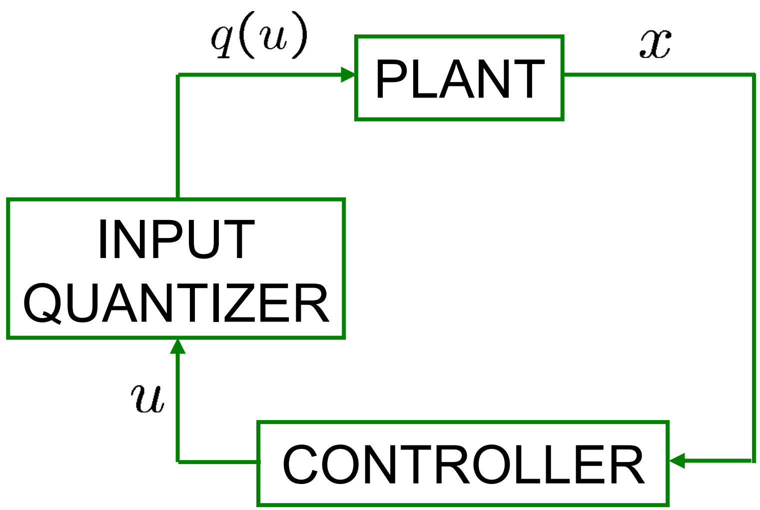
- Limited communication capacity
  - many systems sharing network cable or wireless medium
  - microsystems with many sensors/actuators on one chip
- Need to minimize information transmission (security)
- Event-driven actuators
  - PWM amplifier
  - manual car transmission
  - stepping motor

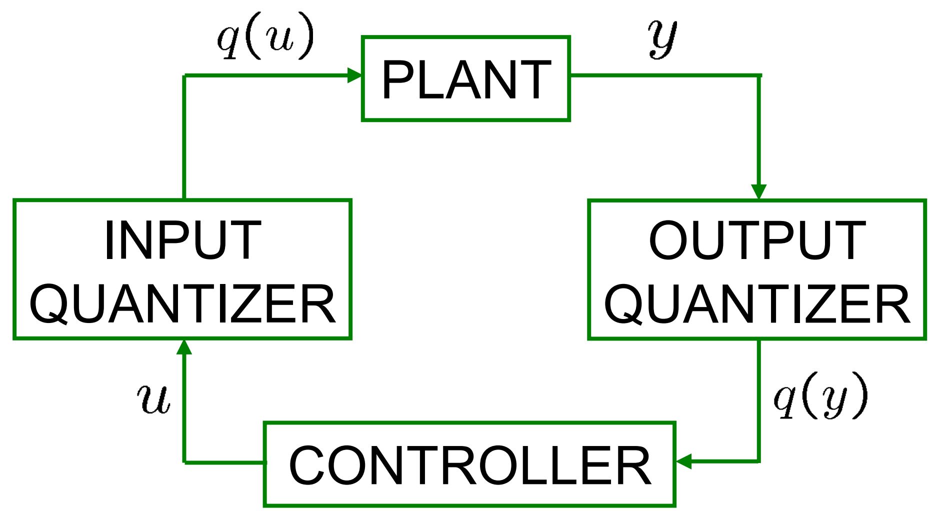


QUANTIZED CONTROL ARCHITECTURES



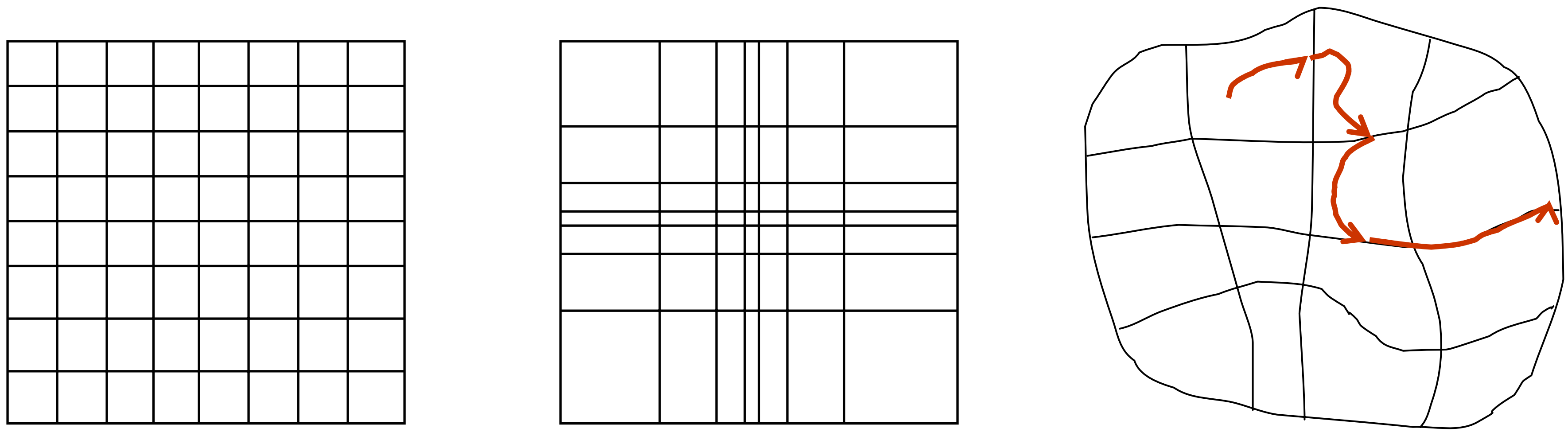






QUANTIZER GEOMETRY

$\mathcal{R}^k$ is partitioned into quantization regions

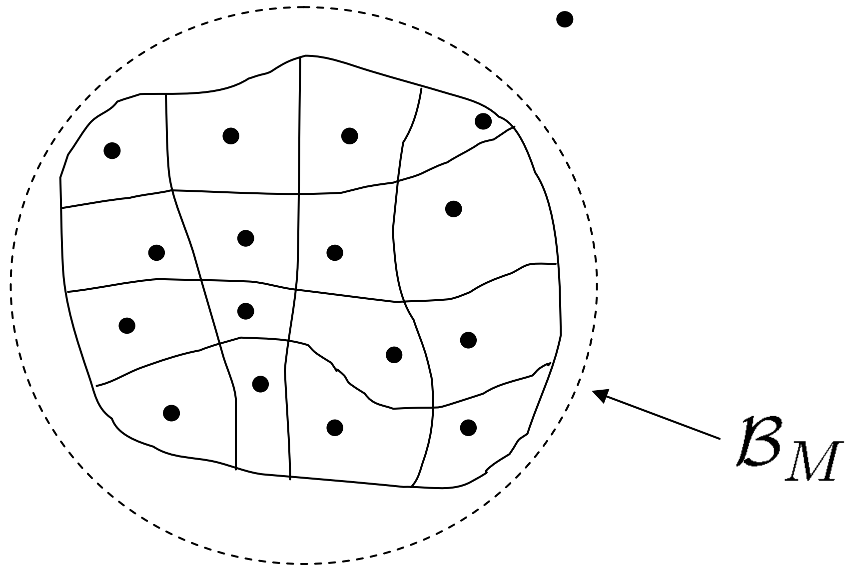


uniform logarithmic arbitrary

Dynamics change at boundaries => hybrid closed-loop system

Chattering on the boundaries is possible (sliding mode)

QUANTIZATION ERROR and RANGE



Assume $\exists M, \Delta > 0$ such that:

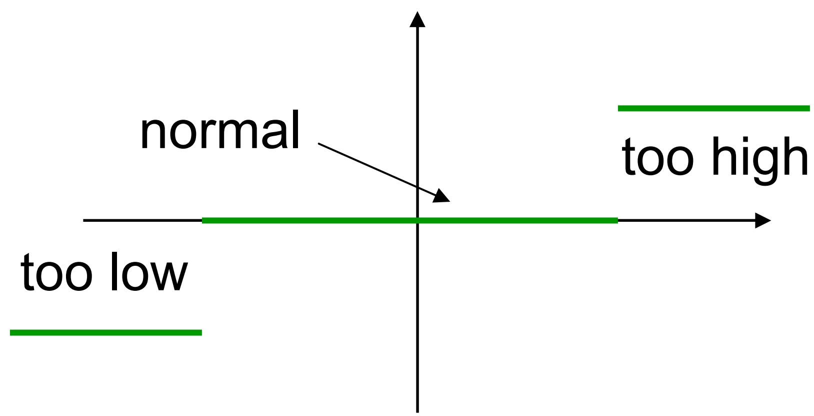
1. $|z| \leq M \Rightarrow |q(z) - z| \leq \Delta$
2. $|z| > M \Rightarrow |q(z)| > M - \Delta$

M is the range, Δ is the quantization error bound

For $|z| > M$, the quantizer saturates

EXAMPLES of QUANTIZERS

- A/D conversion
- Temperature sensor



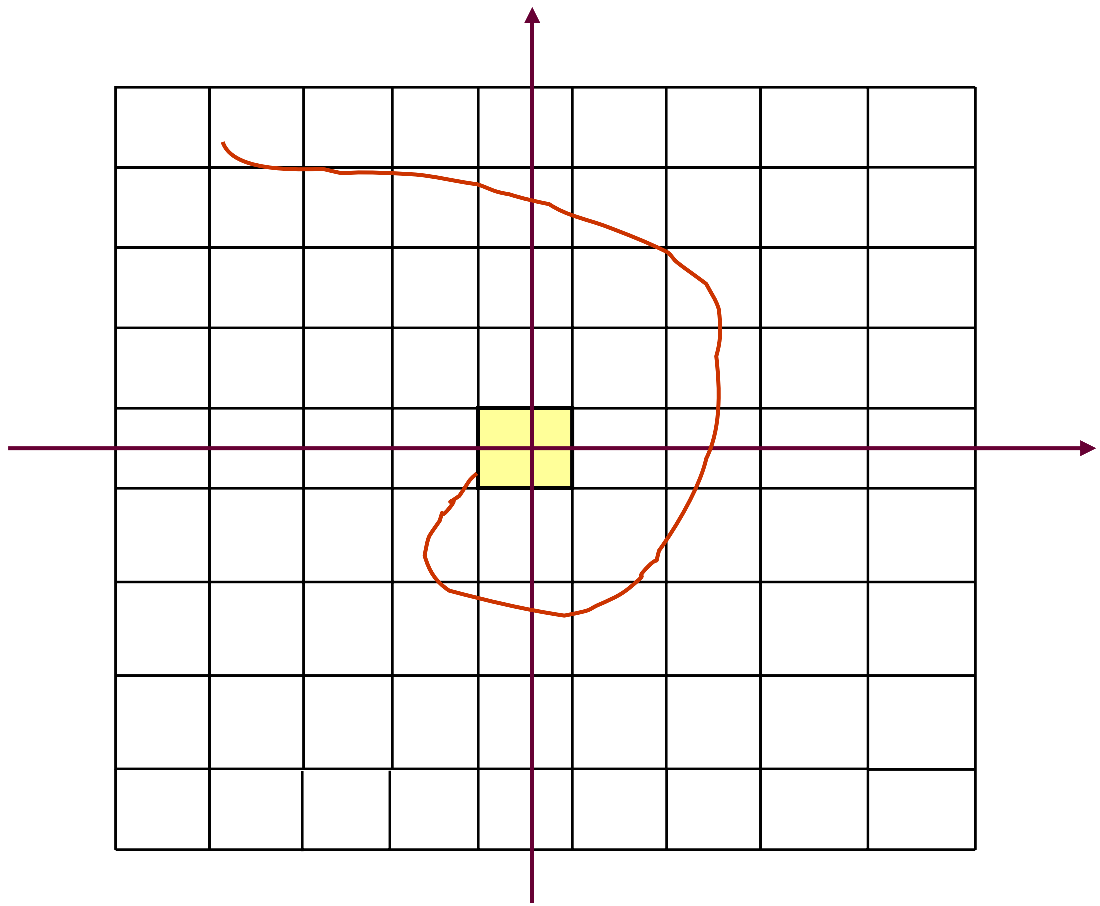
- Camera with zoom

  Tracking a golf ball

- Coding and decoding

OBSTRUCTION to STABILIZATION

Assume:
M, Δ fixed



Asymptotic stabilization is usually lost

BASIC QUESTIONS

- What can we say about a given quantized system?
- How can we design the “best” quantizer for stability?
- What can we do with very coarse quantization?
- What are the difficulties for nonlinear systems?

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STATE QUANTIZATION: LINEAR SYSTEMS

$$\dot{x} = Ax + Bu$$

$\exists K\colon\ \dot{x} = (A + BK)x$ is asymptotically stable

9 Lyapunov function $V = x^T P x$:

$$(A + BK)^T P + P(A + BK) = -I$$

Quantized control law: $u = Kq(x) = K(x + e)$

where $e = q(x) - x$ is quantization error

Closed-loop system: $\dot{x} = (A + BK)x + BKe$

$$\dot{V} = -|x|^2 + 2x^T PBKe \le -|x|^2 + 2|x|\|PBK\||e|$$

$$= -|x|\Big[|x| - 2\|PBK\||e|\Big] < 0 \text{ if } \boxed{|x| > 2\|PBK\||e|}$$

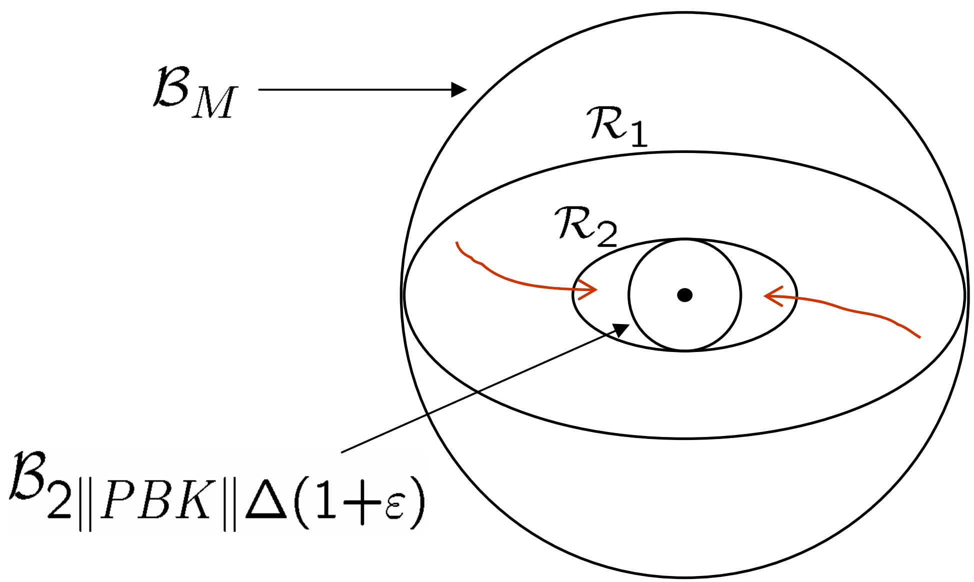
LINEAR SYSTEMS (continued)

Previous slide: $|x| > 2\|PBK\||e| \Rightarrow \dot{V} < 0$

Recall: $|x| \le M \Rightarrow |e| \le \Delta$

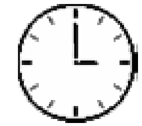
Combine: $\|PBK\|\Delta < |x| \le M \Rightarrow \dot{V} < 0$

$$\boxed{\|PBK\|\Delta(1+\varepsilon) \le |x| \le M \Rightarrow \dot{V} \le -\gamma < 0}$$



Lemma: solutions that start in $\mathcal{R}_1$ enter $\mathcal{R}_2$ in finite time T

NONLINEAR SYSTEMS



For linear systems, we saw that if

$u = Kx$ gives $\dot{V} < 0 \ \forall x \neq 0$ then

$u = K(x+e)$ automatically gives

$\dot{V} < 0$ when $|x| > 2\|PBK\||e|$

This is robustness to measurement errors

For nonlinear systems, GAS $\not\Rightarrow$ such robustness

$$\dot{x} = f(x, k(x) + e)$$

To have the same result, need to assume $\exists \rho \in \mathcal{K}$:

$$\boxed{\dot{V} < 0 \text{ when } |x| > \rho(|e|)}$$

This is input-to-state stability (ISS) for measurement errors!

SUMMARY: PERTURBATION APPROACH

1. Design $u = k(x)$ ignoring constraint
2. View $u = k(q(x))$ as approximation
3. Prove that this u still solves the problem

Issue: $u = k(x) \;\mapsto\; u = k(q(x)) = k(x + e)$

error

Need $k(x)$ to be ISS w.r.t. measurement errors

INPUT QUANTIZATION

$$\boxed{\dot{x} = Ax + Bu}$$

Control law: $u = q(Kx) = Kx + e$

where $e = q(Kx) - Kx$

Closed-loop system: $\dot{x} = (A + BK)x + Be$

Analysis – same as before

$$\boxed{\dot{x} = f(x, u)}$$

Control law: $u = q(k(x)) = k(x) + e$

where $e = q(k(x)) - k(x)$

Closed-loop system: $\dot{x} = f(x, k(x) + e)$

Need ISS with respect to actuator errors

OUTPUT QUANTIZATION

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}}$$

Control law: $\begin{aligned} \dot{\hat{x}} &= (A + LC)\hat{x} + Bu - Lq(y) \\ u &= K\hat{x} \end{aligned}$

Closed-loop system:

$$\begin{pmatrix} \dot{x} \\ \dot{x} - \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A + BK & -BK \\ 0 & A + LC \end{pmatrix} \begin{pmatrix} x \\ x - \hat{x} \end{pmatrix} + L \begin{pmatrix} 0 \\ q(y) - y \end{pmatrix}$$

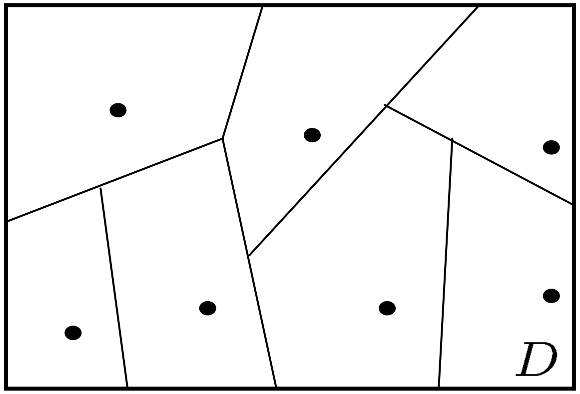
Analysis – same as before (need a bound on initial state)

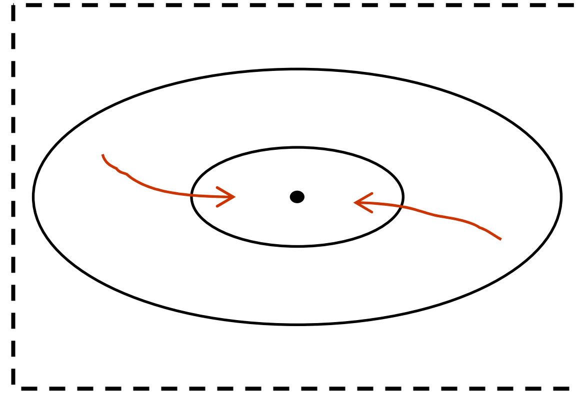
Can also treat input and state/output quantization together

BASIC QUESTIONS

- What can we say about a given quantized system?
- How can we design the “best” quantizer for stability?
- What can we do with very coarse quantization?
- What are the difficulties for nonlinear systems?

LOCATIONAL OPTIMIZATION: NAIVE APPROACH





$$\mathcal{W} = \{W_1, \ldots, W_N\}$$

$$\mathcal{Q} = \{q_1, \ldots, q_N\}$$

$$q(x) = q_i \text{ for } x \in W_i$$

$$\Delta \geq |e| = |q(x) - x|$$

Smaller Δ => smaller $|x(\infty)|$

Also true for nonlinear systems ISS w.r.t. measurement errors

This leads to the problem:

$$\boxed{H(\mathcal{Q}, \mathcal{W}) := \max_{i=1,\ldots,N} \max_{x \in W_i} |x - q_i| \rightarrow \min_{\mathcal{Q},\mathcal{W}}}$$

Compare: mailboxes in a city, cellular base stations in a region

MULTICENTER PROBLEM

$$H(\mathcal{Q},\mathcal{W}) := \max_{i=1,\ldots,N} \max_{x\in W_i} |x-q_i| \;\to\; \min_{\mathcal{Q},\mathcal{W}}$$

Critical points of $H(\mathcal{Q},\mathcal{W})$ satisfy

1. $\mathcal{W}$ is the Voronoi partition $\mathcal{V}(\mathcal{Q})$:

$$x \in W_i \Leftrightarrow |x-q_i| \le |x-q_j| \;\forall j$$

2. Each q_i is the Chebyshev center $q^*(W_i)$ (solution of the 1-center problem). This is the center of enclosing sphere of smallest radius

Lloyd algorithm: $\mathcal{Q} \mapsto \mathcal{Q}^*(\mathcal{W})$

$\mathcal{W} \mapsto \mathcal{V}(\mathcal{Q})$

iterate

LOCATIONAL OPTIMIZATION: REFINED APPROACH

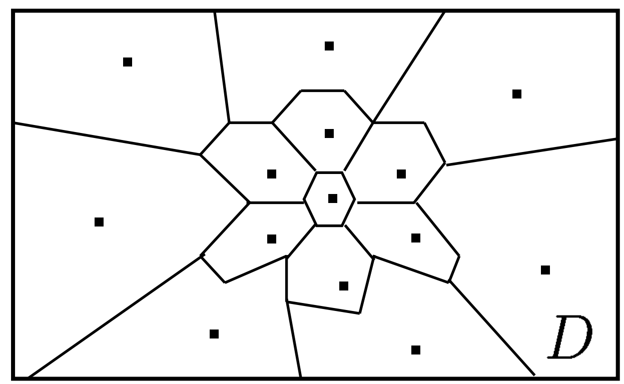
$$\dot{V} \leq -|x|\Big[|x| - 2\|PBK\||e|\Big] = -|x|^2\left[1 - 2\|PBK\|\frac{|e|}{|x|}\right]$$

only need this ratio to be small

Revised problem:

$$\boxed{H(\mathcal{Q},\mathcal{W}) := \max_{i=1,\dots,N} \max_{x \in W_i} \frac{|x - q_i|}{|x|} \to \min_{\mathcal{Q},\mathcal{W}}}$$

Logarithmic quantization:
Lower precision far away,
higher precision close to 0



Only applicable to linear systems

WEIGHTED MULTICENTER PROBLEM

$$H(\mathcal{Q},\mathcal{W}) := \max_{i=1,\dots,N} \max_{x\in W_i} \frac{|x-q_i|}{|x|} \to \min_{\mathcal{Q},\mathcal{W}}$$

on D not containing 0 (annulus)

Critical points of $H(\mathcal{Q},\mathcal{W})$ satisfy

1. $\mathcal{W}$ is the Voronoi partition $\mathcal{V}(\mathcal{Q})$ as before
2. Each q_i is the weighted center $q^*(W_i)$

   (solution of the weighted 1-center problem)

   This is the center of sphere enclosing W_i

   with smallest $\boxed{r/|c|}$

Lloyd algorithm – as before

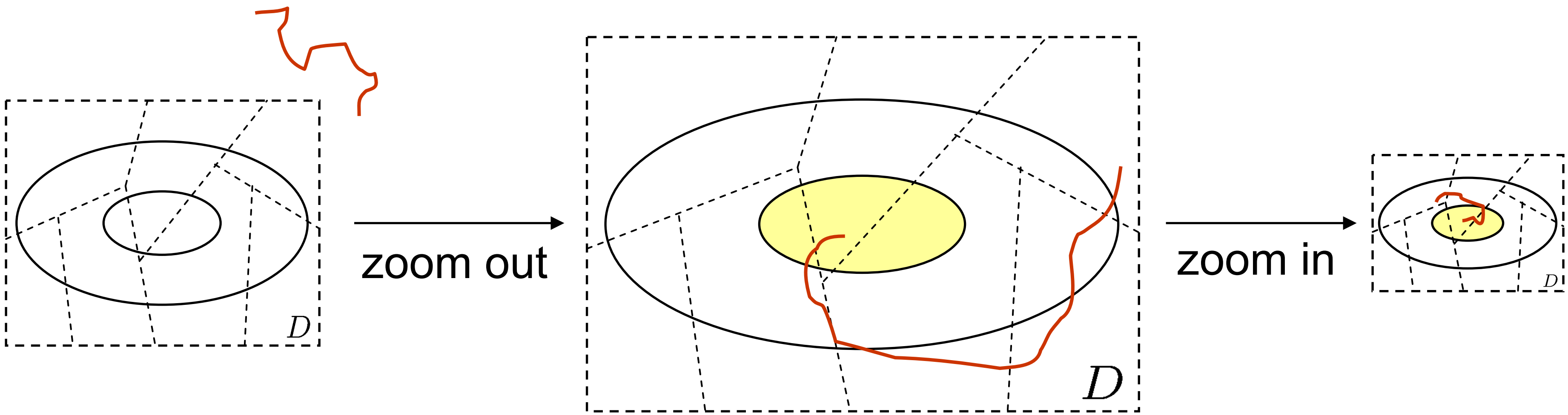
Gives 25% decrease in $|x(\infty)|$ for 2-D example

DYNAMIC QUANTIZATION: IDEA

Temperature sensor – can adjust threshold settings

Digital camera – can zoom in and out

Encoder – can change the coding mechanism



Zoom out to overcome saturation

After ultimate bound is achieved, recompute partition for smaller region

Can recover global asymptotic stability

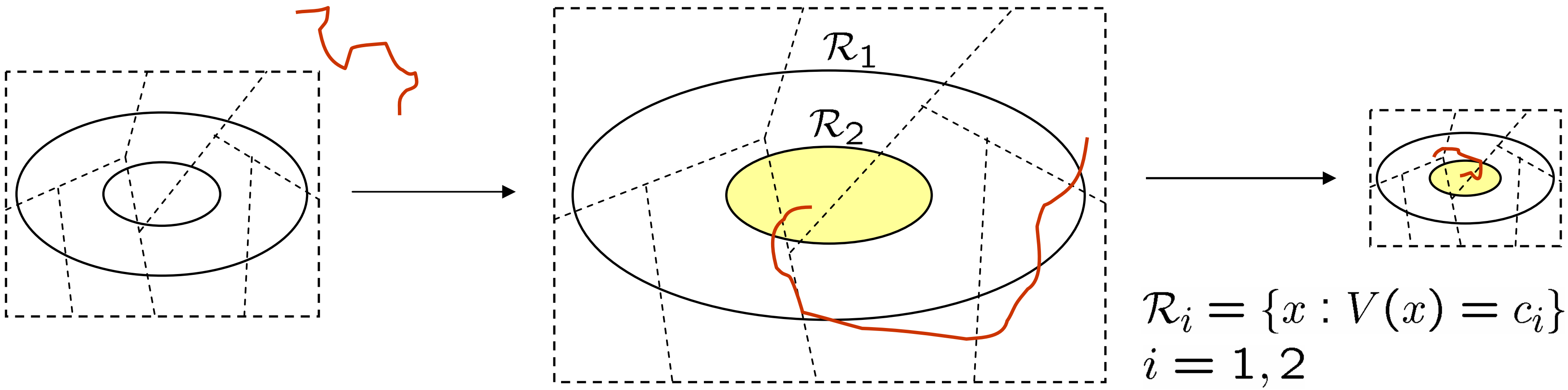
(also applies to input and output quantization)

DYNAMIC QUANTIZATION: DETAILS

$q(x/\mu)$, μ – zooming variable

Hybrid quantized control: μ is discrete state

(More realistic, easier to design and analyze, robust to time delays)



$\mathcal{R}_i = \{x : V(x) = c_i\}$
$i = 1, 2$

$x(0)$ unknown

Increase μ fast enough

until $q(x/\mu) \le M - \Delta$

$\Rightarrow x/\mu \le M \Leftrightarrow x \le M\mu$

We know: solutions starting in $\mathcal{R}_1$

enter $\mathcal{R}_2$ in finite time T

$\mu \mapsto \mu \cdot c_2/c_1$ after T units of time

dwell time

BASIC QUESTIONS

- What can we say about a given quantized system?
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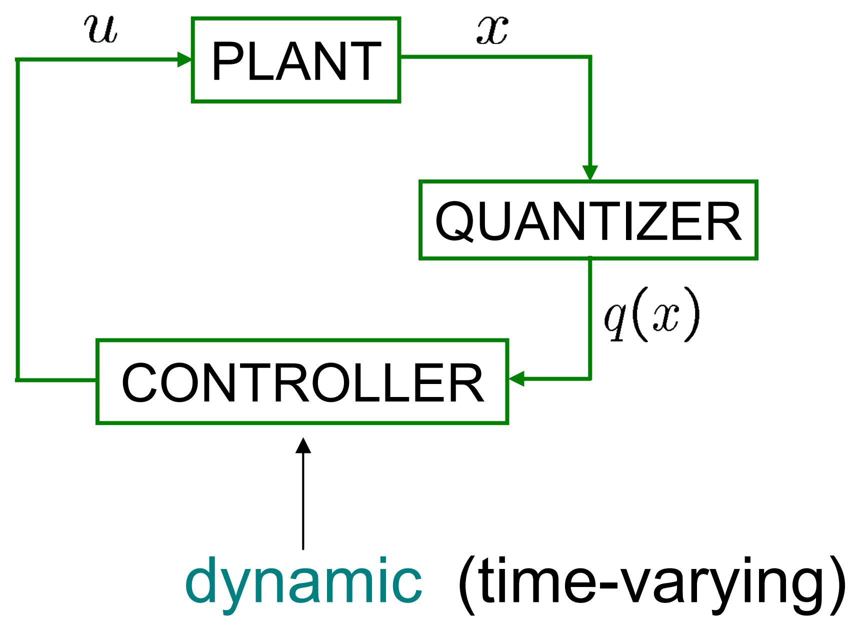
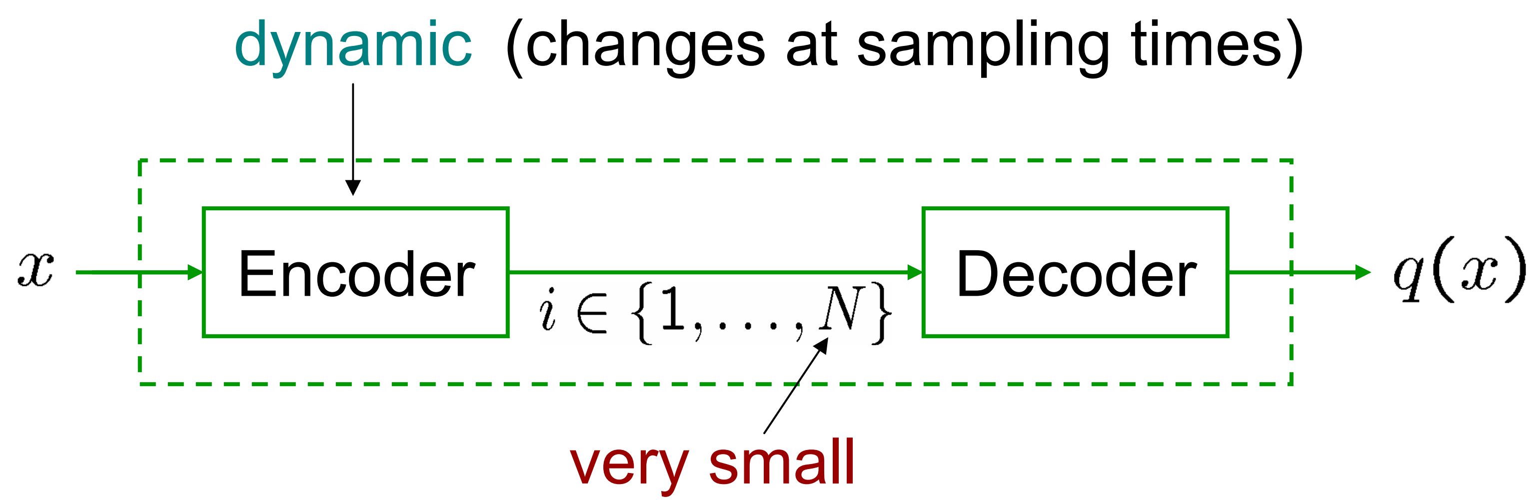
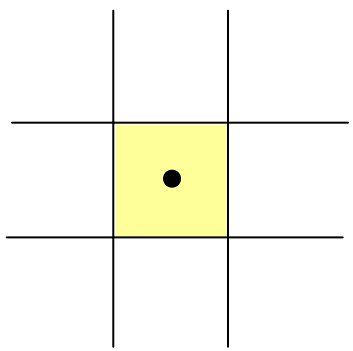
ACTIVE PROBING for INFORMATION





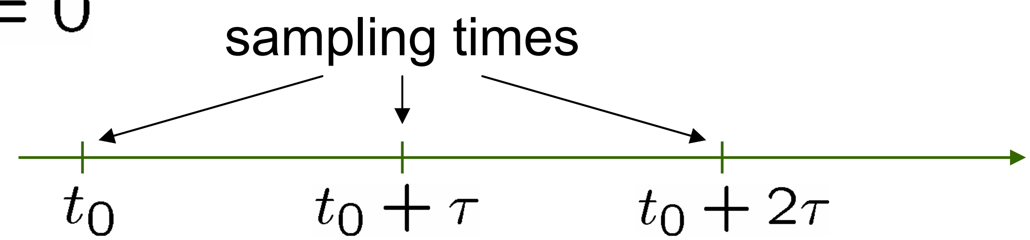
LINEAR SYSTEMS

Example: $\dot{x} = Ax + Bu,\ n = 2,\ N = 9 = 3^n$



Zoom out to get initial bound

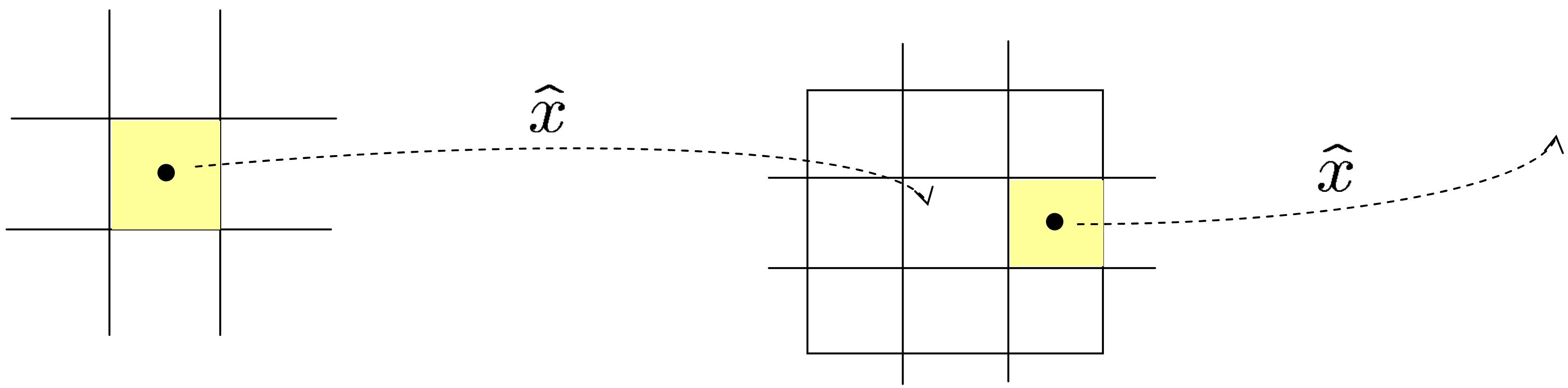
$\hat{x}(t_0) := 0$



Between sampling times, let $\dot{\hat{x}} = A\hat{x} + Bu$

LINEAR SYSTEMS

Example: $\dot{x} = Ax + Bu, \;\; n = 2, \;\; N = 9 = 3^n$



Between sampling times, let $\dot{\hat{x}} = A\hat{x} + Bu$

Consider $e := \hat{x} - x$ $\qquad \dot{x} = Ax + Bu$ $\;\Rightarrow\; \dot{e} = Ae$

The norm $\|e\|_\infty = \max\limits_{1 \le i \le n} |e_i|$:

- grows at most by the factor $\Lambda := e^{\|A\|_\infty \tau}$ in one period
- is divided by 3 at the sampling time

LINEAR SYSTEMS (continued)

$$e = \hat{x} - x$$

The norm $\|e\|_\infty$:

- grows at most by the factor $\boxed{\Lambda := e^{\|A\|_\infty \tau}}$ in one period
- is divided by 3 at each sampling time

Pick τ small enough s.t. $\boxed{\Lambda < 3} \Rightarrow e \to 0$

sampling frequency vs. open-loop instability

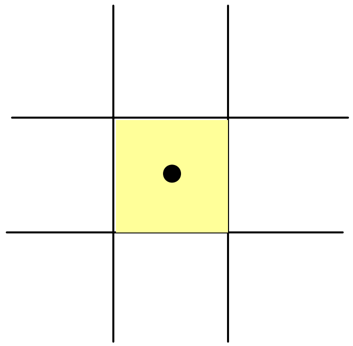
amount of static info provided by quantizer

$u(t) = K\hat{x}(t)$ where $A + BK$ is Hurwitz

$$\dot{x} = Ax + BK\hat{x} = (A+BK)x + BK\underset{\downarrow \atop 0}{e} \Rightarrow x \to 0$$

NONLINEAR SYSTEMS

Example: $\dot{x} = f(x, u), \; n = 2, \; N = 9 = 3^n$



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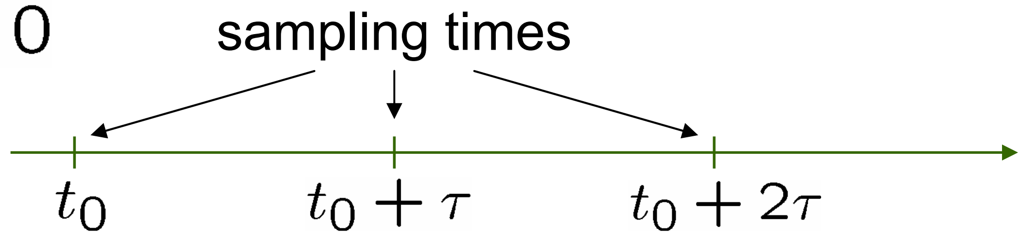
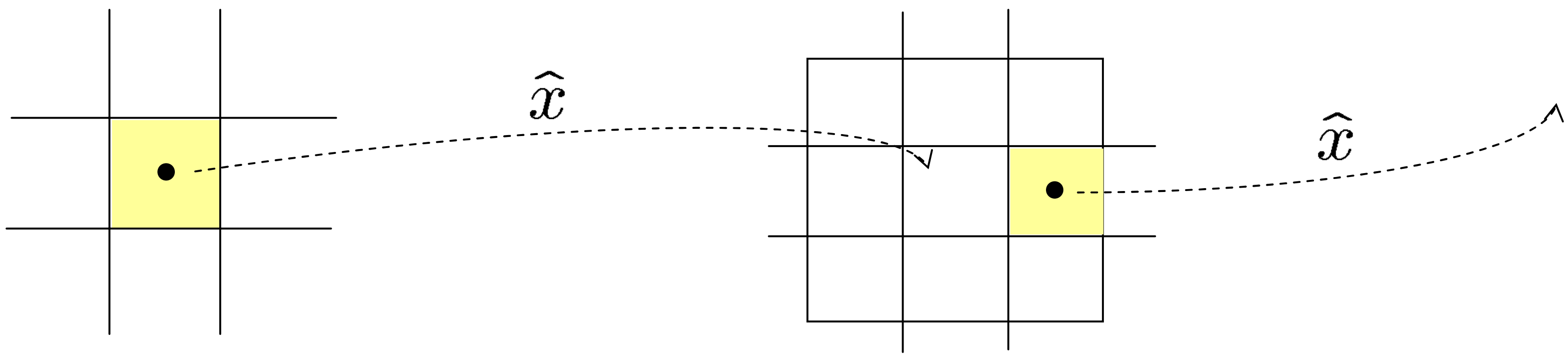


Between samplings $\dot{\hat{x}} = f(\hat{x}, u)$

NONLINEAR SYSTEMS

Example: $\dot{x} = f(x, u), \ n = 2, \ N = 9 = 3^n$



Between samplings $\quad \left.\begin{aligned} \dot{\widehat{x}} &= f(\widehat{x}, u) \\ \dot{x} &= f(x, u) \end{aligned}\right\} \Rightarrow \dot{e} = f(\widehat{x}, u) - f(x, u)$

Let $e := \widehat{x} - x$

$\|f(\widehat{x}, u) - f(x, u)\|_\infty \le L\|e\|_\infty$ where L is Lipschitz constant of f

The norm $\|e\|_\infty$:

- grows at most by the factor $\Lambda := e^{L\tau}$ in one period
- is divided by 3 at the sampling time

NONLINEAR SYSTEMS (continued)

$$e = \hat{x} - x$$

The norm $\|e\|_\infty$:

- grows at most by the factor $\Lambda := e^{L\tau}$ in one period
- is divided by 3 at each sampling time

Pick τ small enough s.t. $\Lambda < 3 \Rightarrow e \to 0$

$$u(t) = k(\hat{x}(t))$$

$$\dot{x} = f(x, k(\hat{x})) = f(x, k(x + e))$$

Need ISS w.r.t. measurement errors!

RESEARCH DIRECTIONS

- Robust control design
- Locational optimization
- Performance
- Applications

REFERENCES

Brockett & L, 2000 (IEEE TAC)
Bullo & L, 2003 (submitted)