

## 5. Solving the SPCP

Given the system

$$x(k+1) = Ax(k) + B_1e(k) + B_2w(k) + B_3v(k)$$

$$y(k) = C_1x(k) + D_{11}e(k) + D_{12}w(k)$$

$$\tilde{z}(k) = \tilde{C}_2x(k) + \tilde{D}_{21}e(k) + \tilde{D}_{22}\tilde{w}(k) + \tilde{D}_{23}\tilde{v}(k)$$

$$\tilde{\phi}(k) = \tilde{C}_3x(k) + \tilde{D}_{31}e(k) + \tilde{D}_{32}\tilde{w}(k) + \tilde{D}_{33}\tilde{v}(k)$$

$$\tilde{\Psi}(k) = \tilde{C}_4x(k) + \tilde{D}_{41}e(k) + \tilde{D}_{42}\tilde{w}(k) + \tilde{D}_{43}\tilde{v}(k)$$

Minimize performance index

$$\begin{aligned} J(v, k) &= \sum_{j=0}^{N-1} \hat{z}^T(k+j|k) \Gamma(j) \hat{z}(k+j|k) \\ &= \tilde{z}^T(k) \bar{\Gamma} \tilde{z}(k) \end{aligned}$$

subject to the constraints  $\tilde{\phi}(k) = 0$  and  $\tilde{\Psi}(k) \leq \tilde{\Psi}(k)$ .

# Solving the SPCP

SPCP:

$$J(v, k) = \tilde{z}^T(k) \bar{\Gamma} \tilde{z}(k) \quad (1)$$

$$\tilde{\phi}(k) = 0 \quad (2)$$

$$\tilde{\psi}(k) \leq \tilde{\Psi}(k) \quad (3)$$

Three subproblems:

1. Unconstrained: minimize (1).
2. Equality constrained: minimize (1) s.t. (2).
3. Full SPCP: minimize (1) s.t. (2) & (3).

## Unconstrained SPCP

Given the system:

$$x(k+1) = Ax(k) + B_1e(k) + B_2w(k) + B_3v(k)$$

$$y(k) = C_1x(k) + D_{11}e(k) + D_{12}w(k)$$

$$\tilde{z}(k) = \tilde{C}_2x(k) + \tilde{D}_{21}e(k) + \tilde{D}_{22}\tilde{w}(k) + \tilde{D}_{23}\tilde{v}(k)$$

Minimize performance index

$$J(v, k) = \tilde{z}^T(k) \bar{\Gamma} \tilde{z}(k)$$

## Unconstrained SPCP

Use prediction:

$$\tilde{z}(k) = \tilde{z}_0(k) + \tilde{D}_{23} \tilde{v}(k)$$

Then

$$\begin{aligned} J(v, k) &= \tilde{z}^T(k) \bar{\Gamma} \tilde{z}(k) \\ &= \left( \tilde{z}_0^T + \tilde{v}^T \tilde{D}_{23}^T \right) \bar{\Gamma} \left( \tilde{z}_0 + \tilde{D}_{23} \tilde{v} \right) \\ &= \tilde{v}^T \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \tilde{v} + \tilde{v}^T \tilde{D}_{23}^T \bar{\Gamma} \tilde{z}_0 + \tilde{z}_0^T \bar{\Gamma} \tilde{D}_{23} \tilde{v} + \tilde{z}_0^T \bar{\Gamma} \tilde{z}_0 \\ &= \tilde{v}^T \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \tilde{v} + 2 \tilde{v}^T \tilde{D}_{23}^T \bar{\Gamma} \tilde{z}_0 + \tilde{z}_0^T \bar{\Gamma} \tilde{z}_0 \\ &= \frac{1}{2} \tilde{v}^T H \tilde{v} + \tilde{v}^T f + c \end{aligned}$$

where we used  $\tilde{z}_0^T \bar{\Gamma} \tilde{D}_{23} \tilde{v} + \tilde{v}^T \tilde{D}_{23}^T \bar{\Gamma} \tilde{z}_0 = 2 \tilde{v}^T \tilde{D}_{23}^T \bar{\Gamma} \tilde{z}_0$

## Unconstrained SPCP

Optimization problem:

$$\min_{\tilde{v}} J(\tilde{v}, k) = \min_{\tilde{v}} \frac{1}{2} \tilde{v}^T H \tilde{v} + \tilde{v}^T f + c$$
$$\frac{\partial J}{\partial \tilde{v}} = H \tilde{v}(k) + f(k) = 0$$

Let  $H$  be invertible, then:

$$\tilde{v}(k) = -H^{-1} f(k)$$
$$= - \left( 2\tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \right)^{-1} 2\tilde{D}_{23}^T \bar{\Gamma} \tilde{z}_0(k)$$

## Unconstrained SPCP

Controller implementation: Let  $E_v = [ I \ 0 \ \dots \ 0 ]$   
and remind that  $\tilde{z}_0(k) = \tilde{C}_2 x(k) + \tilde{D}_{21} e(k) + \tilde{D}_{22} \tilde{w}(k)$

The control signal  $v(k)$  becomes:

$$\begin{aligned} v(k) &= E_v \tilde{v}(k) \\ &= -E_v H^{-1} f(k) \\ &= -E_v \left( \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \right)^{-1} \tilde{D}_{23}^T \bar{\Gamma} \tilde{z}_0(k) \\ &= -E_v \left( \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \right)^{-1} \tilde{D}_{23}^T \bar{\Gamma} \tilde{C}_2 x(k) - E_v \left( \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \right)^{-1} \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{21} e(k) \\ &\quad - E_v \left( \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \right)^{-1} \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{22} \tilde{w}(k) \end{aligned}$$

## Unconstrained SPCP

Controller implementation: Let  $E_v = [ I \ 0 \ \dots \ 0 ]$   
and remind that  $\tilde{z}_0(k) = \tilde{C}_2 x(k) + \tilde{D}_{21} e(k) + \tilde{D}_{22} \tilde{w}(k)$

The control signal  $v(k)$  becomes:

$$\begin{aligned} v(k) &= E_v \tilde{v}(k) \\ &= -E_v H^{-1} f(k) \\ &= -E_v \left( \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \right)^{-1} \tilde{D}_{23}^T \bar{\Gamma} \tilde{z}_0(k) \\ &= -E_v \left( \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \right)^{-1} \tilde{D}_{23}^T \bar{\Gamma} \tilde{C}_2 x(k) - E_v \left( \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \right)^{-1} \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{21} e(k) \\ &\quad - E_v \left( \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \right)^{-1} \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{22} \tilde{w}(k) \\ &= -F x(k) + D_e e(k) + D_w \tilde{w}(k) \end{aligned}$$

## Equality constrained optimization

Equality constrained standard problem:

$$\min_{\tilde{v}} J = \min_{\tilde{v}} \tilde{z}^T(k) \bar{\Gamma} \tilde{z}(k)$$

subject to the constraint:

$$\begin{aligned} \tilde{\phi}(k) &= \underbrace{\tilde{C}_3 x(k) + \tilde{D}_{31} e(k) + \tilde{D}_{32} \tilde{w}(k)}_{\tilde{\phi}_E(k)} + \tilde{D}_{33} \tilde{v}(k) \\ &= \tilde{\phi}_E(k) + \tilde{D}_{33} \tilde{v}(k) = 0 \end{aligned}$$

Step 1: Elimination

Step 2: Solve problem analytically

## Step 1: Elimination

Singular value decomposition:

$$\tilde{D}_{33} = U \begin{bmatrix} \Sigma & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

and define

$$\tilde{D}_{33}^r = V_1 \Sigma^{-1} U^T \quad \tilde{D}_{33}^\perp = V_2$$

then

$$\tilde{D}_{33} \tilde{D}_{33}^r = U \Sigma V_1^T V_1 \Sigma^{-1} U^T = I$$

$$\tilde{D}_{33} \tilde{D}_{33}^\perp = U \Sigma V_1^T V_2 = 0$$

Now  $\tilde{v}(k)$  is chosen as

$$\tilde{v}(k) = -\tilde{D}_{33}^r \tilde{\phi}_E(k) + \tilde{D}_{33}^\perp \tilde{\mu}(k) = \tilde{v}_E(k) + \tilde{D}_{33}^\perp \tilde{\mu}(k)$$

$$\tilde{v}(k) = -\tilde{D}_{33}^r \tilde{\phi}_E(k) + \tilde{D}_{33}^\perp \tilde{\mu}(k) = \tilde{v}_E(k) + \tilde{D}_{33}^\perp \tilde{\mu}(k)$$

Substitution:

$$\begin{aligned} \tilde{\phi}(k) &= \tilde{\phi}_E(k) + \tilde{D}_{33} \tilde{v}(k) = \tilde{\phi}_E(k) - \tilde{D}_{33} \tilde{D}_{33}^r \tilde{\phi}_E(k) + \tilde{D}_{33} \tilde{D}_{33}^\perp \tilde{\mu}(k) \\ &= \tilde{\phi}_E(k) - I \tilde{\phi}_E(k) + 0 \tilde{\mu}(k) = 0 \end{aligned}$$

Substitution:

$$\begin{aligned} \tilde{z}(k) &= \tilde{z}_0(k) + \tilde{D}_{23} \tilde{v}(k) = \tilde{z}_0(k) + \tilde{D}_{23} \tilde{v}_E(k) + \tilde{D}_{23} \tilde{D}_{33}^\perp \tilde{\mu}(k) \\ &= \tilde{z}_E(k) + \tilde{D}_{23} \tilde{D}_{33}^\perp \tilde{\mu}(k) \end{aligned}$$

Then

$$J(v, k) = \tilde{z}^T(k) \bar{\Gamma} \tilde{z}(k) = \frac{1}{2} \tilde{\mu}^T H \tilde{\mu} + \tilde{\mu}^T f + c$$

## Step 2: Analytic solution

$$J(v, k) = \frac{1}{2} \tilde{\mu}^T H \tilde{\mu} + \tilde{\mu}^T f + c$$

Let invertible  $H$ , then:

$$\tilde{\mu}(k) = -H^{-1} f(k) = -\left( \tilde{D}_{33}^{\perp T} \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \tilde{D}_{33}^{\perp} \right)^{-1} \tilde{D}_{33}^{\perp T} \tilde{D}_{23}^T \bar{\Gamma} \left( \tilde{z}_0(k) + \tilde{D}_{23} \tilde{v}_E \right)$$

Use

$$\begin{aligned} \tilde{v}(k) &= -\tilde{D}_{33}^r \tilde{\phi}_E(k) + \tilde{D}_{33}^{\perp} \tilde{\mu}(k) \\ v(k) &= E_v \tilde{v}(k) \end{aligned}$$

then

$$v(k) = -F x(k) + D_e e(k) + D_w \tilde{w}(k)$$

This is also an LTI-controller !!!

# Full SPCP

Given the system:

$$x(k+1) = Ax(k) + B_1e(k) + B_2w(k) + B_3v(k)$$

$$y(k) = C_1x(k) + D_{11}e(k) + D_{12}w(k)$$

$$\tilde{z}(k) = \tilde{C}_2x(k) + \tilde{D}_{21}e(k) + \tilde{D}_{22}\tilde{w}(k) + \tilde{D}_{23}\tilde{v}(k)$$

$$\tilde{\phi}(k) = \tilde{C}_3x(k) + \tilde{D}_{31}e(k) + \tilde{D}_{32}\tilde{w}(k) + \tilde{D}_{33}\tilde{v}(k)$$

$$\tilde{\psi}(k) = \tilde{C}_4x(k) + \tilde{D}_{41}e(k) + \tilde{D}_{42}\tilde{w}(k) + \tilde{D}_{43}\tilde{v}(k)$$

Minimize performance index

$$J(v, k) = \tilde{z}^T(k) \bar{\Gamma} \tilde{z}(k)$$

subject to the constraints

$$\tilde{\phi}(k) = 0 \quad \tilde{\psi}(k) \leq \tilde{\Psi}(k)$$

## Full SPCP

Equality constraint is satisfied for

$$\tilde{v}(k) = \tilde{v}_E(k) + \tilde{D}_{33}^\perp \tilde{\mu}(k)$$

Choose optimization vector  $\tilde{\mu}(k)$  as:

$$\tilde{\mu}(k) = \tilde{\mu}_E(k) + \tilde{\mu}_I(k)$$

where  $\tilde{\mu}_E(k) = -H^{-1}f(k)$  is equality constrained solution. Now

$$\tilde{\phi}(k) = 0$$

$$\tilde{\psi}(k) = \tilde{\psi}_E(k) + \tilde{D}_{43} \tilde{D}_{33}^{r\perp} \tilde{\mu}_I(k)$$

$$J(v, k) = \frac{1}{2} \tilde{\mu}_I^T H \tilde{\mu}_I$$

Inequality constraint:

$$\tilde{\Psi}(k) = \tilde{\Psi}_E(k) + \tilde{D}_{43}\tilde{D}_{33}^{\perp}\tilde{\mu}_I(k) \leq \tilde{\Psi}$$

Now let

$$A_{\Psi} = \tilde{D}_{43}\tilde{D}_{33}^{\perp}$$
$$b_{\Psi}(k) = \tilde{\Psi}_E(k) - \tilde{\Psi}(k)$$

This results in the inequality constraint:

$$A_{\Psi}\tilde{\mu}_I(k) + b_{\Psi}(k) \leq 0$$

## Full SPCP

Minimization problem:

$$\min_{\tilde{\mu}_I} \frac{1}{2} \tilde{\mu}_I^T(k) H \tilde{\mu}_I(k)$$

subject to

$$A_\Psi \tilde{\mu}_I(k) + b_\Psi(k) \leq 0$$

This a Quadratic Programming problem.

Solution found in finite number of steps.

The full SPCP results is a nonlinear controller

## Infinite horizon SPCP

Steady state  $z_{ss} = 0$ .

Otherwise: NO SOLUTION !!!

- Steady-state behavior
- Unconstrained: LQ-controller.
- Equality constraints: Structuring of input and get finite degrees of freedom in optimization.
- Equality and inequality constraints: Use finite degrees of freedom in optimization to solve the problem.

## Steady state $(v_{SS}, x_{SS}, w_{SS}, z_{SS})$

The quadruple  $(v_{SS}, x_{SS}, w_{SS}, z_{SS})$  is a steady state if there holds:

$$x_{SS} = Ax_{SS} + B_2 w_{SS} + B_3 v_{SS}$$

$$z_{SS} = C_2 x_{SS} + D_{22} w_{SS} + D_{23} v_{SS}$$

For infinite horizon

$$J = \sum_{j=0}^{\infty} \hat{z}^T(k+j|k) \hat{z}(k+j|k)$$

we need  $(v_{SS}, x_{SS}, w_{SS}, z_{SS}) = (v_{SS}, x_{SS}, w_{SS}, \mathbf{0})$

For infinite horizon:

$$\begin{aligned}x_{ss} &= Ax_{ss} + B_2 w_{ss} + B_3 v_{ss} \\z_{ss} = 0 &= C_2 x_{ss} + D_{22} w_{ss} + D_{23} v_{ss}\end{aligned}$$

Define:

$$M_{ss} = \begin{bmatrix} I - A & -B_3 \\ -C_2 & -D_{23} \end{bmatrix}$$

then:

$$M_{ss} \begin{bmatrix} x_{ss} \\ v_{ss} \end{bmatrix} = \begin{bmatrix} B_2 \\ D_{22} \end{bmatrix} w_{ss}$$

Singular value decomposition:

$$M_{ss} = \begin{bmatrix} U_{M1} & U_{M2} \end{bmatrix} \begin{bmatrix} \Sigma_M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{M1}^T \\ V_{M2}^T \end{bmatrix}$$

Necessary and sufficient condition:

$$U_{M2}^T \begin{bmatrix} B_2 \\ D_{22} \end{bmatrix} = 0$$

The solution is given by:

$$\begin{bmatrix} x_{ss} \\ v_{ss} \end{bmatrix} = V_{M1} \Sigma_M^{-1} U_{M1}^T \begin{bmatrix} B_2 \\ D_{22} \end{bmatrix} w_{ss} + V_{M2} \alpha_M$$

## Example

Consider IO-system:

$$x_o(k+1) = 0.5x_o(k) + 2u(k)$$

$$y(k) = 2x_o(k)$$

GPC-criterion:

$$J(k) = \sum (y(k+1) - r(k+1))^2 + (\Delta u(k))^2$$

IIO-system:

$$x(k) = \begin{bmatrix} y(k-1) \\ \Delta x_o(k) \end{bmatrix}, \quad v(k) = \Delta u(k), \quad w(k) = r(k+1)$$

where

$$x(k+1) = \begin{bmatrix} 1 & 2 \\ 0 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} v(k)$$

$$y(k) = \begin{bmatrix} 1 & 2 \end{bmatrix} x(k)$$

$$y(k+1) = \begin{bmatrix} 1 & 3 \end{bmatrix} x(k) + 6v(k)$$

$$z(k) = \begin{bmatrix} y(k+1) - r(k+1) \\ v(k) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} w(k) + \begin{bmatrix} 6 \\ 1 \end{bmatrix} v(k)$$

We compute:

$$M_{ss} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0.5 & -2 \\ -1 & -3 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} B_2 \\ D_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

and

$$U_2^T = \frac{1}{9} \begin{bmatrix} 1 & 4 & 0 & 8 \end{bmatrix}$$

Note that

$$U_2^T \begin{bmatrix} B_2 \\ D_{22} \end{bmatrix} = 0$$

Consider the criterion:

$$J(k) = \sum (y(k+1) - r(k+1))^2 + (u(k))^2$$

Now we find for  $x(k) = x_o(k)$ ,  $v(k) = u(k)$ ,  $w(k) = r(k+1)$ :

$$y(k+1) = x(k) + 4v(k)$$

$$\begin{aligned} z(k) &= \begin{bmatrix} y(k+1) - r(k+1) \\ v(k) \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(k) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} w(k) + \begin{bmatrix} 4 \\ 1 \end{bmatrix} v(k) \end{aligned}$$

We compute:

$$M_{SS} = \begin{bmatrix} 0.5 & 2 \\ 1 & 4 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} B_2 \\ D_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

and

$$U_2^T = \sqrt{\frac{1}{69}} \begin{bmatrix} -2 & 1 & 8 \end{bmatrix}$$

Note that

$$U_2^T \begin{bmatrix} B_2 \\ D_{22} \end{bmatrix} = -\sqrt{\frac{1}{69}} \neq 0$$

## Steady-state:

Let  $w_{ss} = D_{ssw}\tilde{w}(k)$  and define

$$\begin{bmatrix} D_{ssx} \\ D_{ssv} \end{bmatrix} = V_{M1}\Sigma_M^{-1}U_{M1}^T \begin{bmatrix} B_2 \\ D_{22} \end{bmatrix} D_{ssw}$$

$$D_{ssy} = C_1D_{ssx} + D_{12}D_{ssw} + D_{13}D_{ssv}$$

Then:

$$\begin{bmatrix} x_{ss} \\ v_{ss} \\ w_{ss} \\ y_{ss} \end{bmatrix} = \begin{bmatrix} D_{ssx} \\ D_{ssv} \\ D_{ssw} \\ D_{ssy} \end{bmatrix} \tilde{w}(k)$$

## Unconstrained infinite horizon SPCP

Let  $z_{ss} = 0$  and let  $w(k+j) = 0$  and  $\Gamma(j) = I$  for all  $j \geq 0$ .

Performance index:

$$\begin{aligned} J(v, k) &= \sum_{j=0}^{\infty} \hat{z}^T(k+j|k) \hat{z}(k+j|k) \\ &= \sum_{j=0}^{\infty} \left( \hat{x}^T(k+j|k) C_2^T + \hat{e}^T(k+j|k) D_{21}^T \right) \left( C_2 \hat{x}(k+j|k) + D_{21} \hat{e}(k+j|k) \right) + \\ &\quad + 2 \left( \hat{x}^T(k+j|k) C_2^T + \hat{e}^T(k+j|k) D_{21}^T \right) D_{23} v(k+j) \\ &\quad + v^T(k+j) D_{23}^T D_{23} v(k+j) \end{aligned}$$

Now define

$$\bar{v}(k) = v(k) + (D_{23}^T D_{23})^{-1} D_{23}^T \left( C_2 \hat{x}(k|k) + D_{21} \hat{e}(k|k) \right)$$

$$\bar{x}(k + j|k) = \begin{bmatrix} \hat{x}_\Delta(k + j|k) \\ \hat{e}(k + j|k) \end{bmatrix}$$

Then the performance index becomes

$$J(v, k) = \sum_{j=0}^{\infty} \bar{x}^T(k + j|k) \bar{Q} \bar{x}(k + j|k) + \bar{v}^T(k + j) \bar{R} \bar{v}(k + j)$$

and for  $j \geq 0$ :

$$\bar{x}(k + j + 1|k) = \bar{A} \bar{x}(k + j|k) + \bar{B} \bar{v}(k + j|k)$$

$$z(k + j|k) = \bar{C} \bar{x}(k + j|k) + \bar{D} \bar{v}(k + j|k)$$

where

$$\bar{A} = \begin{bmatrix} A - B_3(D_{23}^T D_{23})^{-1} D_{23}^T C_2 & B_1 - B_3(D_{23}^T D_{23})^{-1} D_{23}^T D_{12} \\ 0 & 0 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} B_3 \\ 0 \end{bmatrix}$$

$$\bar{C} = (I - D_{23}(D_{23}^T D_{23})^{-1} D_{23}^T) \begin{bmatrix} C_2 & D_{21} \end{bmatrix}$$

$$\bar{D} = D_{23}$$

$$\bar{Q} = \bar{C}^T \bar{C} = \begin{bmatrix} C_2 & D_{21} \end{bmatrix}^T (I - D_{23}(D_{23}^T D_{23})^{-1} D_{23}^T) \begin{bmatrix} C_2 & D_{21} \end{bmatrix}$$

$$\bar{R} = \bar{D}^T \bar{D} = D_{23}^T D_{23}$$

We have a standard LQG problem and the optimal control signal  $\bar{v}$  is given by

$$\bar{v}(k) = -(\bar{B}^T P \bar{B} + \bar{R})^{-1} \bar{B}^T P \bar{A} \bar{x}(k)$$

where  $P$  is the smallest positive semi-definite solution of the discrete time Riccati equation

$$P = \bar{A}^T P \bar{A} - \bar{A}^T P \bar{B} (\bar{B}^T P \bar{B} + \bar{R})^{-1} \bar{B}^T P \bar{A} + \bar{Q}$$

which exists due to stabilizability of  $(A, B_3)$  and invertibility of  $\bar{R}$ .

After substitution we obtain a linear controller:

$$v(k) = -F x(k) + D_w w(k) + D_e e(k|k)$$

## Infinite horizon SPCP with control horizon

Consider a system

$$x(k+1) = Ax(k) + B_1 e(k) + B_2 w(k) + B_3 v(k)$$

$$y(k) = C_1 x(k) + D_{11} e(k) + D_{12} w(k)$$

$$z(k) = C_2 x(k) + D_{21} e(k) + D_{22} w(k) + D_{23} v(k)$$

Minimize  $J(v, k) = \sum_{j=0}^{\infty} \hat{z}^T(k+j|k) \Gamma(j) \hat{z}(k+j|k)$

subject to  $v(k+j|k) = 0$ , for  $j \geq N_c$  Control horizon constraint

$$\begin{aligned} \psi(k+j) &= C_4 x(k+j) + D_{41} e(k+j) + D_{42} w(k+j) + D_{43} v(k+j) \\ &\leq \Psi, \text{ for } j \geq 0 \end{aligned}$$

Inequality constraints

For simplicity: Consider  $e(k) = 0$ ,  $w(k) = 0$ , and  $A$  is stable.

The performance index can be split in two parts:

$$J(v, k) = \sum_{j=0}^{\infty} \hat{z}^T(k+j|k) \Gamma(j) \hat{z}(k+j|k) = J_1(v, k) + J_2(v, k)$$

where

$$J_1(v, k) = \sum_{j=0}^{N_c-1} \hat{z}^T(k+j|k) \Gamma(j) \hat{z}(k+j|k)$$

$$J_2(v, k) = \sum_{j=N_c}^{\infty} \hat{z}^T(k+j|k) \Gamma_{ss} \hat{z}(k+j|k)$$

We will consider the derivation of criterion  $J_2$  before we derive criterion  $J_1$ .

## Derivation of $J_2$ :

For  $j \geq N_c$  this system is autonomous. This means that  $\hat{z}(k + j|k)$  can be computed for all  $j \geq N_c$  if initial state  $\hat{x}(k + N_c)$  is known. This state  $\hat{x}(k + N_c|k)$  can be found by successive substitution:

$$\begin{aligned}\hat{x}(k + N_c|k) &= A^{N_c} x(k|k) + \begin{bmatrix} A^{N_c-1} B_3 & \cdots & B_3 \end{bmatrix} \tilde{v}(k) \\ &= \tilde{C}_{N_c,3} x(k|k) + \tilde{D}_{N_c,33} \tilde{v}(k)\end{aligned}$$

The prediction  $\hat{z}(k + j|k)$  for  $j \geq N_c$  is obtained by successive substitution, resulting in:

$$\hat{z}(k + j|k) = C_2 A^{j-N_c} \hat{x}(k + N_c|k)$$

then

$$\begin{aligned} J_2(v, k) &= \sum_{j=N_c}^{\infty} \hat{z}^T(k+j|k) \Gamma_{ss} \hat{z}(k+j|k) \\ &= \sum_{j=N_c}^{\infty} \hat{x}^T(k+N_c|k) (A^T)^{j-N_c} C_2^T \Gamma_{ss} C_2 A^{j-N_c} \hat{x}(k+N_c|k) \\ &= \hat{x}^T(k+N_c|k) \bar{M} \hat{x}(k+N_c|k) \end{aligned}$$

where

$$\bar{M} = \sum_{j=N_c}^{\infty} (A^T)^{j-N_c} C_2^T \Gamma_{ss} C_2 A^{j-N_c}$$

The matrix  $\bar{M}$  is solution of the discrete time Lyapunov equation

$$\bar{A}_s^T \bar{M} \bar{A}_s - \bar{M} + T_s^T C_2^T \Gamma_{ss} C_2 T_s = 0$$

With

$$\hat{x}(k + N_c | k) = \tilde{C}_{N_c,3} x(k|k) + \tilde{D}_{N_c,33} \tilde{v}(k)$$

we derive

$$\begin{aligned} J_2(v, k) &= \hat{x}^T(k + N_c | k) \bar{M} \hat{x}(k + N_c | k) \\ &= \frac{1}{2} \tilde{v}^T(k) H_2 \tilde{v}(k) + \tilde{v}^T(k) f_2(k) + c_2(k) \end{aligned}$$

where

$$\begin{aligned} H_2 &= 2\tilde{D}_{N_c,33}^T \bar{M} \tilde{D}_{N_c,33} \\ f_2(k) &= 2\tilde{D}_{N_c,33}^T \bar{M} \tilde{C}_{N_c,3} x(k|k) \\ c_2(k) &= x^T(k|k) \tilde{C}_{N_c,3}^T \bar{M} \tilde{C}_{N_c,3} x(k|k) \end{aligned}$$

## Derivation of $J_1$ :

Define  $\tilde{z}(k)$  for horizon  $N_c$ :

$$\tilde{z}(k) = \begin{bmatrix} \hat{z}^T(k|k) & \hat{z}^T(k+1|k) & \dots & \hat{z}^T(k+N_c-1|k) \end{bmatrix}$$

Using the results of chapter 3 we derive:

$$\tilde{z}(k) = \tilde{C}_{N_c,2} x(k|k) + \tilde{D}_{N_c,23} \tilde{v}(k)$$

where  $\tilde{C}_{N_c,2}$  and  $\tilde{D}_{N_c,23}$  are given by

$$\tilde{C}_{N_c,2} = \begin{bmatrix} C_2 \\ C_2 A \\ \vdots \\ C_2 A^{N_c-1} \end{bmatrix}, \quad \tilde{D}_{N_c,23} = \begin{bmatrix} D_{23} & 0 & \dots & 0 \\ C_2 B_3 & D_{23} & \dots & 0 \\ \vdots & \ddots & \ddots & 0 \\ C_2 A^{N_c-2} B_3 & \dots & D_{23} \end{bmatrix}$$

Now  $J_1$  becomes

$$\begin{aligned} J_1(v, k) &= \sum_{j=0}^{N_c-1} \hat{z}^T(k+j|k) \Gamma(j) \hat{z}(k+j|k) \\ &= \tilde{z}^T(k) \bar{\Gamma}_{N_c} \tilde{z}(k) \\ &= \frac{1}{2} \tilde{v}^T(k) H_1 \tilde{v}(k) + \tilde{v}^T(k) f_1(k) + c_1(k) \end{aligned}$$

where

$$\begin{aligned} H_1 &= 2\tilde{D}_{N_c,23}^T \bar{\Gamma}_{N_c} \tilde{D}_{N_c,23} \\ f_1(k) &= 2\tilde{D}_{N_c,23}^T \bar{\Gamma}_{N_c} \tilde{C}_{N_c,2} x(k|k) \\ c_1(k) &= x^T(k|k) \tilde{C}_{N_c,2}^T \bar{\Gamma}_{N_c} \tilde{C}_{N_c,2} x(k|k) \end{aligned}$$

## Minimization of $J_1 + J_2$ :

Combining the results we obtain the problem of minimizing

$$\begin{aligned} J_1 + J_2 &= \frac{1}{2} \tilde{v}^T(k) (H_1 + H_2) \tilde{v}(k) + \tilde{v}^T(k) (f_1(k) + f_2(k)) + c_1(k) + c_2(k) \\ &= \frac{1}{2} \tilde{v}^T(k) H \tilde{v}(k) + \tilde{v}^T(k) f(k) + c(k) \end{aligned}$$

In the absence of inequality constraints, the optimal  $v(k)$  is found for

$$\begin{aligned} v(k) &= -E_v (H_1 + H_2)^{-1} (f_1(k) + f_2(k)) \\ &= -E_v H^{-1} f(k) \\ &= -E_v \left( \tilde{D}_{N_c,23}^T \bar{\Gamma}_{N_c} \tilde{D}_{N_c,23} + \tilde{D}_{N_c,33}^T \bar{M} \tilde{D}_{N_c,33} \right)^{-1} \\ &\quad \times \left( \tilde{D}_{N_c,23}^T \bar{\Gamma}_{N_c} \tilde{C}_{N_c,2} + \tilde{D}_{N_c,33}^T \bar{M} \tilde{C}_{N_c,3} \right) x(k|k) \\ &= -F x(k|k) \end{aligned}$$

If  $e(k) \neq 0$  and  $w(k) \neq 0$  we obtain

$$v(k) = -F x(k|k) + D_e e(k) + D_w \tilde{w}(k)$$

where

$$F = E_v \Lambda \left( \tilde{D}_{N_c,23}^T \bar{\Gamma}_{N_c} \tilde{C}_{N_c,2} + \tilde{D}_{N_c,33}^T \bar{M} \tilde{C}_{N_c,3} \right)$$

$$D_e = E_v \Lambda \left( \tilde{D}_{N_c,23}^T \bar{\Gamma}_{N_c} \tilde{D}_{N_c,21} + \tilde{D}_{N_c,33}^T \bar{M} \tilde{D}_{N_c,31} \right)$$

$$D_w = E_v \Lambda \left( \tilde{D}_{N_c,23}^T \bar{\Gamma}_{N_c} \tilde{D}_{N_c,22} + \tilde{D}_{N_c,33}^T \bar{M} \tilde{D}_{N_c,32} \right)$$

$$\Lambda = \left( \tilde{D}_{N_c,23}^T \bar{\Gamma}_{N_c} \tilde{D}_{N_c,23} + \tilde{D}_{N_c,33}^T \bar{M} \tilde{D}_{N_c,33} \right)^{-1}$$

## Unstable poles:

The prediction for  $j \geq N_c$  resulted in:

$$\hat{z}(k+j|k) = C_2 A^{j-N_c} \hat{x}(k+N_c|k)$$

If  $A$  has unstable poles:

$$A = \begin{bmatrix} T_s & T_u \end{bmatrix} \begin{bmatrix} \bar{A}_s & 0 \\ 0 & \bar{A}_u \end{bmatrix} \begin{bmatrix} \bar{T}_s \\ \bar{T}_u \end{bmatrix}$$

where  $\bar{A}_s$  is stable and  $\bar{A}_u$  is unstable.

$$x_{unstable}(k+N_c|k) = T_u x(k+N_c|k) = 0$$

This is an equality constraint.

## Inequality constraints:

For constraints there exists  $n_\Psi$  such that

$$\text{If } \psi(k + j|k) \leq \Psi(k + j|k) \text{ for } j = 0, \dots, N_c + n_\Psi$$

$$\text{then } \psi(k + j|k) \leq \Psi(k + j|k) \text{ for } j > N_c + n_\Psi$$

The infinite horizon predictive control problem turns out to be a Quadratic Programming (QP) problem.

# Quadratic programming (QP)

Various algorithms to solve the quadratic programming problem:

## 1. The modified simplex method:

Most efficient for small and medium-sized problems.

The algorithm will find the optimum in a finite number of steps.

## 2. The interior point method:

For large-sized quadratic programming problems.

Disadvantage: the optimum can only be approximated. Bounds for approximation can be derived (Nesterov & Nemirovsky, 1994).

## 3. Other convex optimization methods:

For example: cutting plane algorithm (Boyd & Barratt, 1991 / Demyanov & Vasilev, 1985) or the ellipsoid algorithm (Boyd & Barratt, 1991 / Grötschel, Lovász & Schrijver, 1988)

## Implementation of LTI SPCP 1

$$v(k) = -F x(k) + D_e e(k) + D_w \tilde{w}(k)$$

State  $x(k)$  and true noise signal  $e(k)$  are unknown.

Introduce

- controller state  $x_c(k)$
- noise estimate  $e_c(k)$ .

Substitution gives controller equations:

$$x_c(k+1) = A x_c(k) + B_1 e_c(k) + B_2 w(k) + B_3 v(k)$$

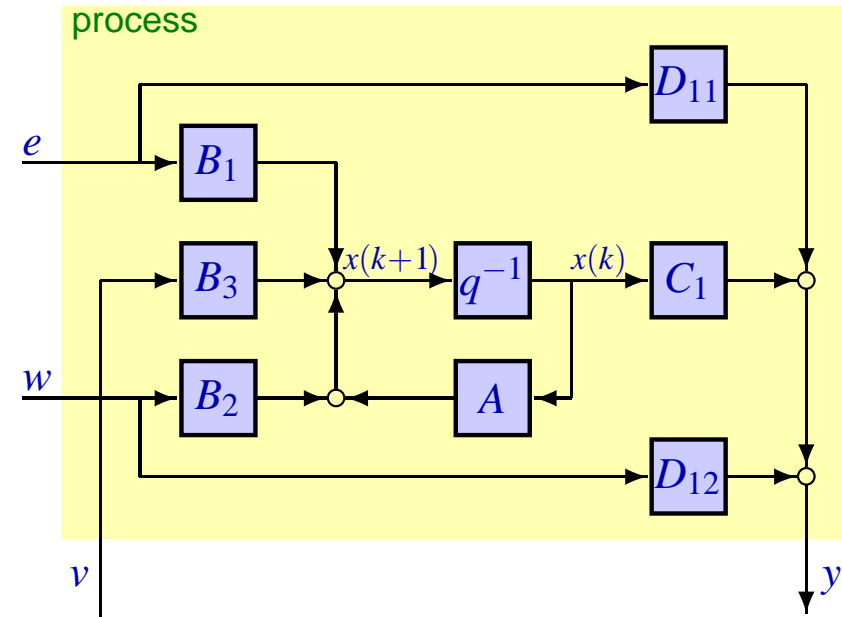
$$e_c(k) = D_{11}^{-1} \left( y(k) - C_1 x_c(k) - D_{12} w(k) \right)$$

$$v(k) = -F x_c(k) + D_e e_c(k) + D_w \tilde{w}(k)$$

## Implementation of LTI SPCP 2

$$x(k+1) = Ax(k) + B_1e(k) + B_2w(k) + B_3v(k)$$

$$y(k) = C_1x(k) + D_{11}e(k) + D_{12}w(k)$$

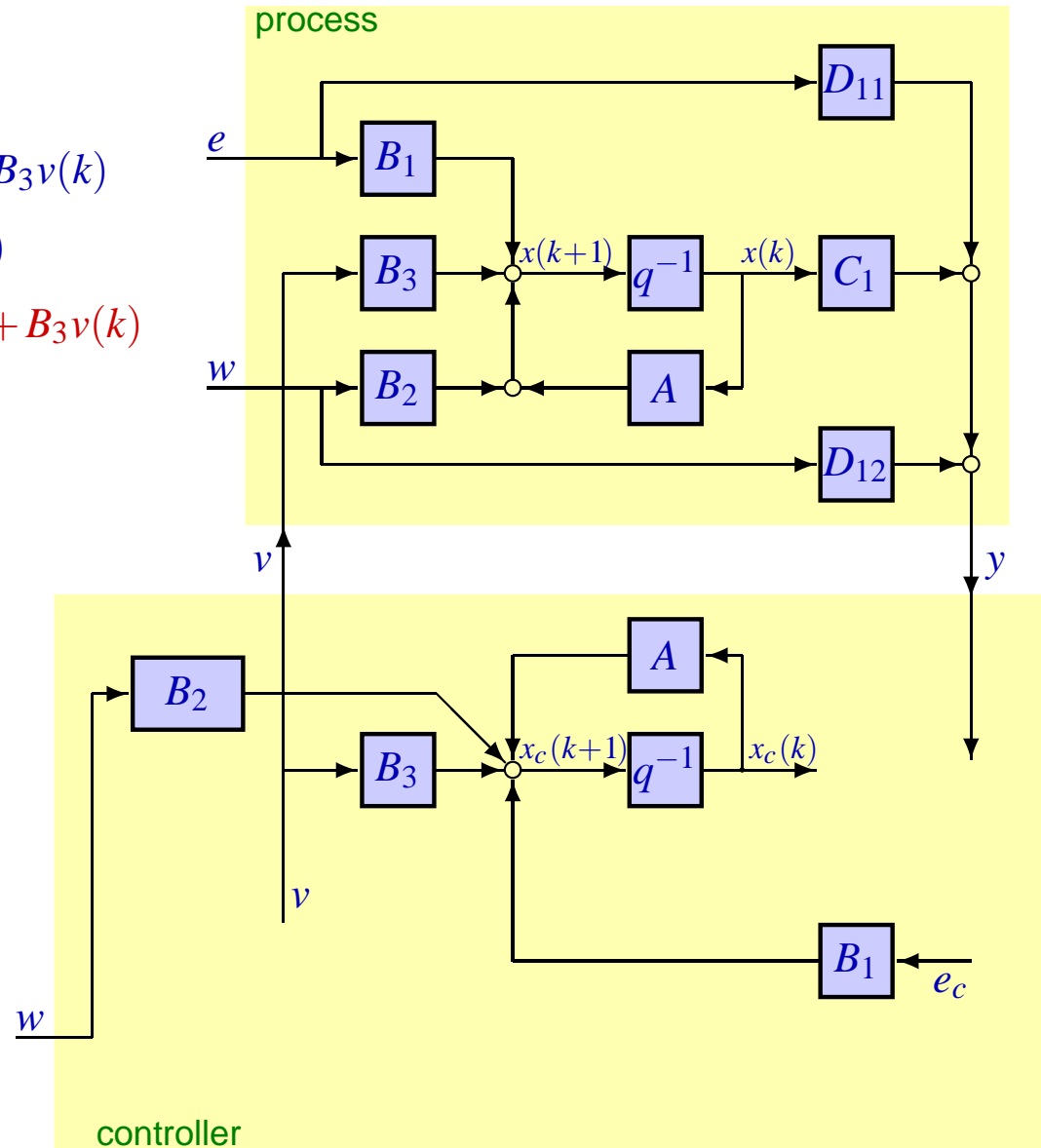


# Implementation of LTI SPCP 3

$$x(k+1) = Ax(k) + B_1e(k) + B_2w(k) + B_3v(k)$$

$$y(k) = C_1x(k) + D_{11}e(k) + D_{12}w(k)$$

$$x_c(k+1) = Ax_c(k) + B_1e_c(k) + B_2w(k) + B_3v(k)$$



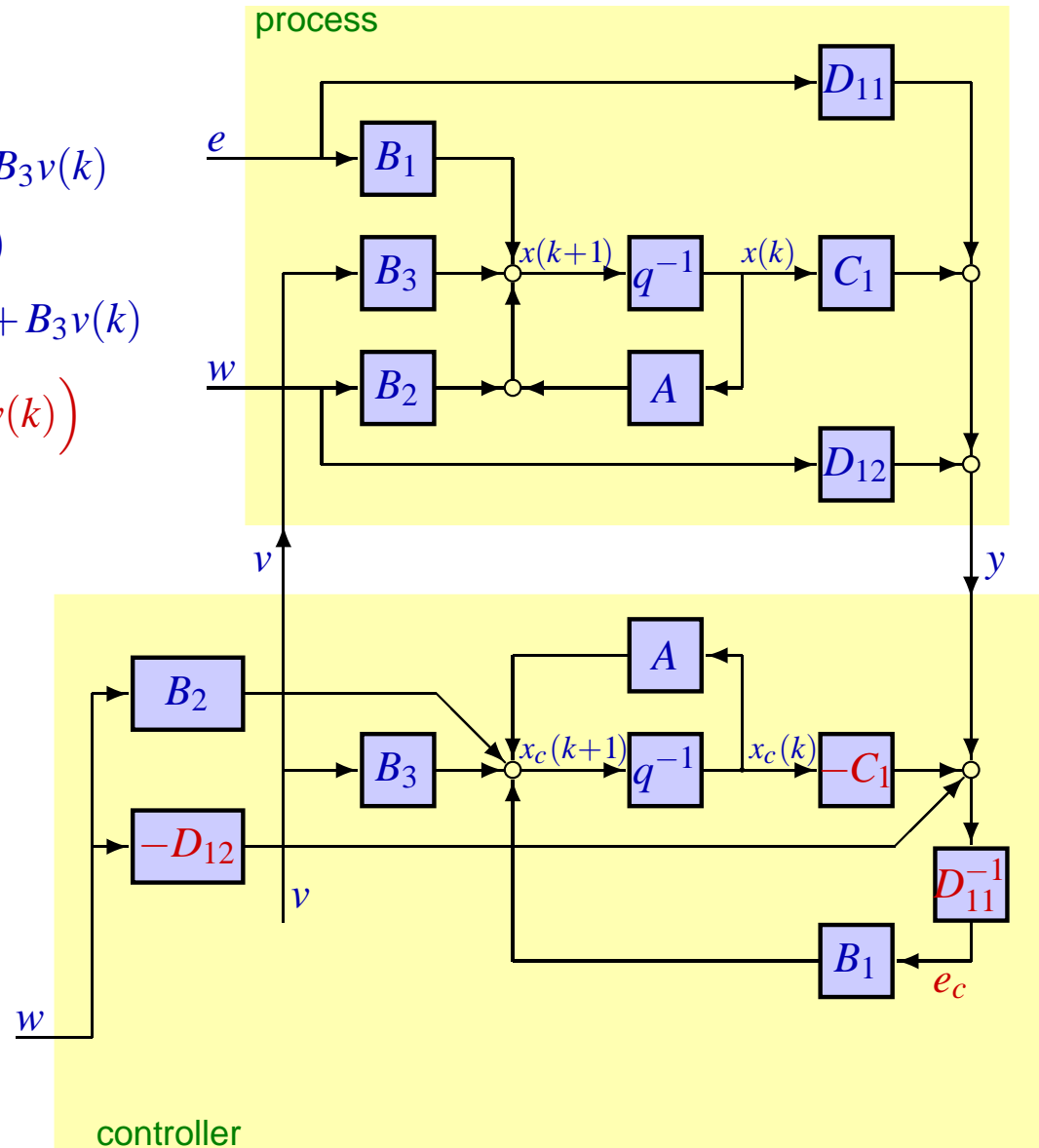
# Implementation of LTI SPCP 4

$$x(k+1) = Ax(k) + B_1e(k) + B_2w(k) + B_3v(k)$$

$$y(k) = C_1x(k) + D_{11}e(k) + D_{12}w(k)$$

$$x_c(k+1) = Ax_c(k) + B_1e_c(k) + B_2w(k) + B_3v(k)$$

$$e_c(k) = D_{11}^{-1} \left( y(k) - C_1x_c(k) - D_{12}w(k) \right)$$



# Implementation of LTI SPCP 5

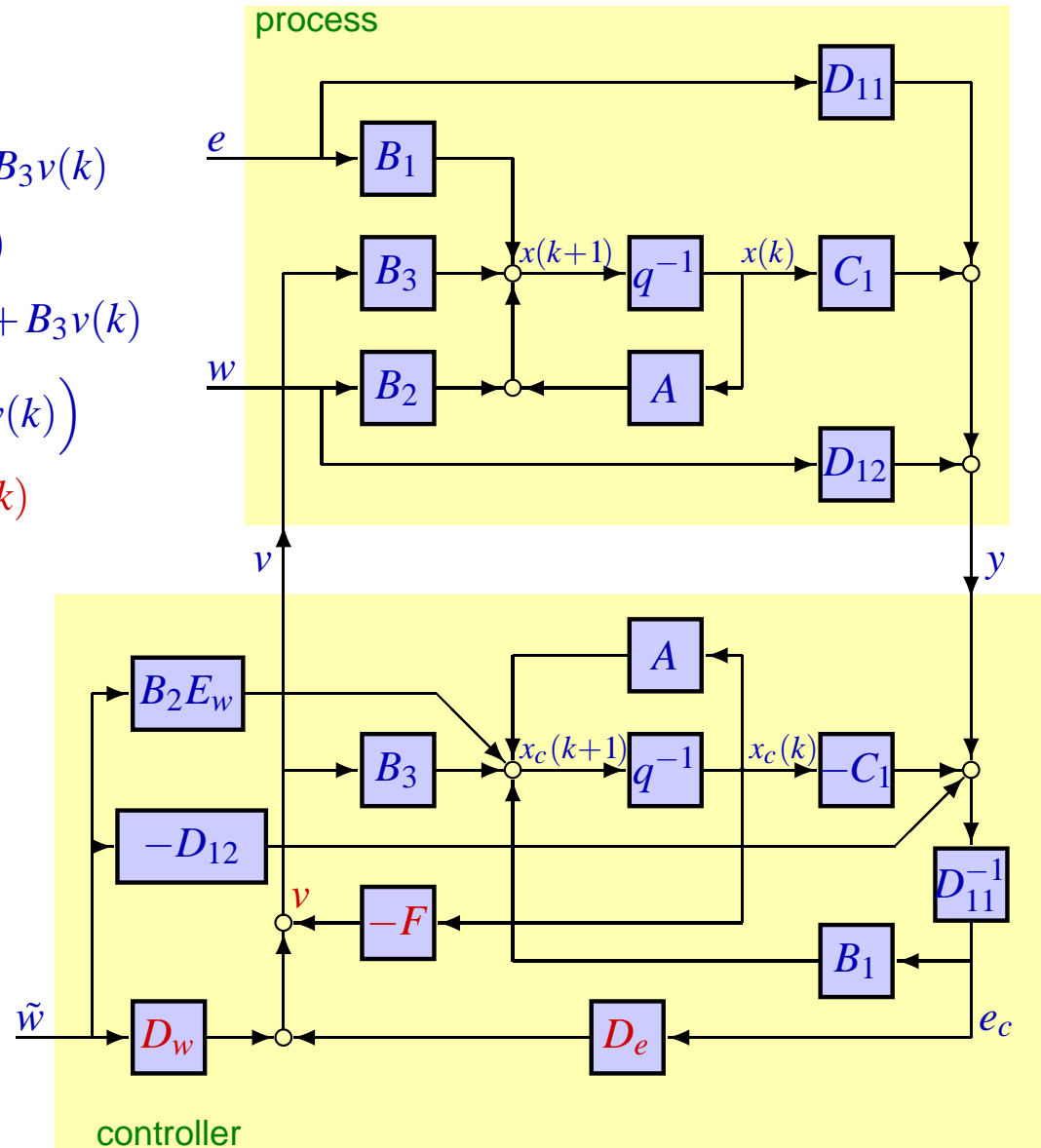
$$x(k+1) = Ax(k) + B_1e(k) + B_2w(k) + B_3v(k)$$

$$y(k) = C_1x(k) + D_{11}e(k) + D_{12}w(k)$$

$$x_c(k+1) = Ax_c(k) + B_1e_c(k) + B_2w(k) + B_3v(k)$$

$$e_c(k) = D_{11}^{-1} \left( y(k) - C_1x_c(k) - D_{12}w(k) \right)$$

$$v(k) = -Fx_c(k) + D_e e_c(k) + D_w \tilde{w}(k)$$



## Implementation of full SPCP

Minimization problem:

$$\begin{aligned} J(v, k) &= \tilde{z}^T(k) \bar{\Gamma} \tilde{z}(k) \\ &= \frac{1}{2} \tilde{\mu}_I^T H \tilde{\mu}_I + c_I \end{aligned}$$

subject to

$$A_\Psi \tilde{\mu}_I(k) + b_\Psi(k) \leq 0$$

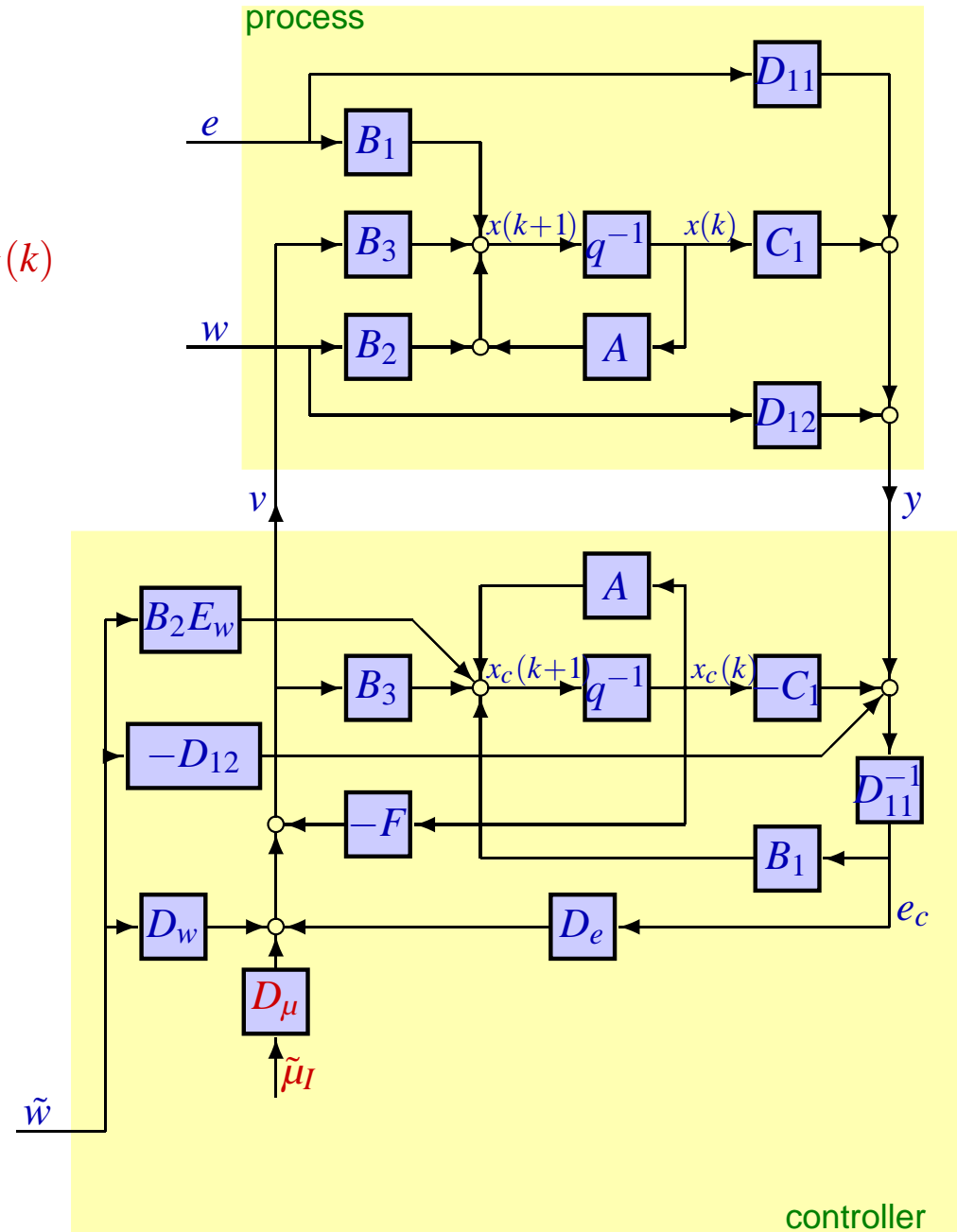
The full SPCP results is a nonlinear controller

$$v(k) = -F x_c(k) + D_e e_c(k) + D_w \tilde{w}(k) + D_\mu \tilde{\mu}_I(k)$$

where

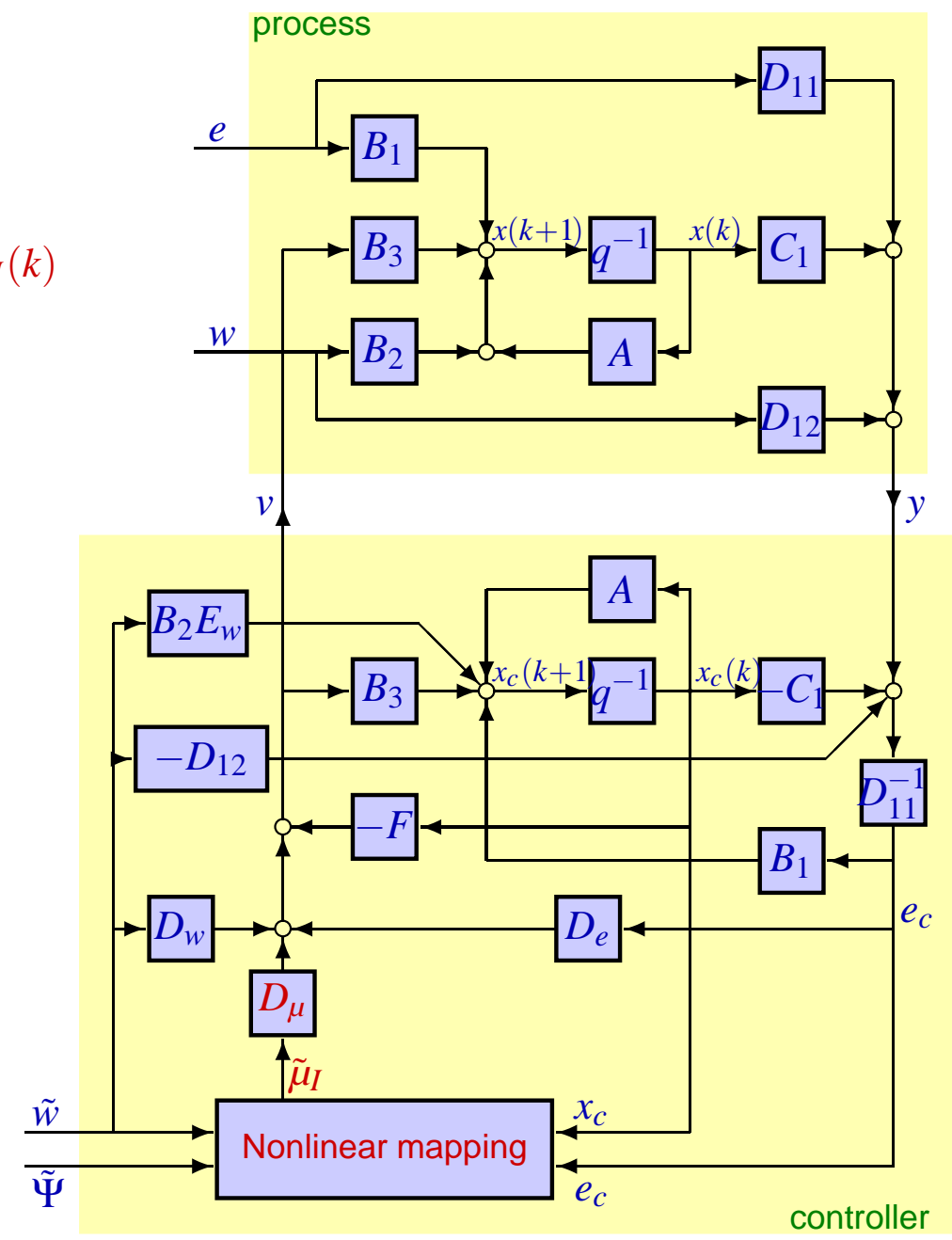
$$\tilde{\mu}_I(k) = h(x_c(k), e_c(k), \tilde{w}(k), \tilde{\Psi}(k))$$

$$v(k) = -Fx_c(k) + D_e e_c(k) + D_w \tilde{w}(k) + D_\mu \tilde{\mu}_I(k)$$



$$v(k) = -Fx_c(k) + D_e e_c(k) + D_w \tilde{w}(k) + D_\mu \tilde{\mu}_I(k)$$

$$\tilde{\mu}_I(k) = h(x_c(k), e_c(k), \tilde{w}(k), \tilde{\Psi}(k))$$



## Analytic solution to full SPCP

Analytic solution for the mapping

$$\tilde{\mu}_I(k) = h(x_c(k), e_c(k), \tilde{w}(k), \tilde{\Psi}(k))$$

The function  $h$  is a continuous and piecewise affine function of  $x_c(k)$ ,  $e_c(k)$ ,  $\tilde{w}(k)$ , and  $\tilde{\Psi}(k)$ .

Partitions the state space into polyhedral sets and compute the coefficients of the affine function for every set. The result is a search tree.

## Feasibility

$$\min_v J \quad \text{subject to} \quad \tilde{\Psi} \leq \tilde{\Psi}$$

if inequality constraints too stringent



no feasible solution to QP problem

Three algorithms to handle infeasibility:

- soft-constraint approach
- minimal time approach
- constraint prioritization

## Soft-constraint algorithm

Based on penalty-function description.

New problem becomes:

$$\min_{v, \alpha} J + c \alpha^2$$

subject to

$$\tilde{\psi} \leq \tilde{\Psi} + \alpha$$

$$\alpha \geq 0$$

$$c \gg 0$$

## Minimal time algorithm

Based on minimum duration of infeasibility.

Compute minimal time  $j_{mt}$ :

$$\min_{v, j_{mt}} j_{mt} \quad \text{subject to}$$

$$\psi(k+j) \leq \Psi(k+j) \quad \text{for } j > j_{mt}$$

and optimize new problem:

$$\min_v J \quad \text{subject to}$$

$$\psi(k+j) \leq \Psi(k+j) \quad \text{for } j > j_{mt}$$

## Constraint prioritization

New algorithm:

1. order constraints from lowest to highest priority.
2. solve optimization problem.  
in case of feasible solution: goto step 4  
in case of infeasibility: goto step 3
3. drop constraint with lowest priority, goto step 2
4. implement solution

Solve a sequence of quadratic programming problems

Algorithm minimizes the violations in priority order.

## Example: elevator

Triple integrator chain

$$\ddot{y}(t) = u(t) + \varepsilon(t)$$

$$\varepsilon(t) = \ddot{e}(t) + 0.5\dot{e}(t) + e(t)$$

where  $e(t)$  is a continuous time ZMWN signal.

There are constraints on  $y$ ,  $\dot{y}$ ,  $\ddot{y}$  and  $\ddot{y}$ .

Define the state

$$x(t) = \begin{bmatrix} \ddot{y}(t) \\ \dot{y}(t) \\ y(t) \end{bmatrix}^T$$

Continuous-time state space description:

$$\dot{x}_1(t) = u(t) + 0.1e(t)$$

$$\dot{x}_2(t) = x_1(t) + e(t)$$

$$\dot{x}_3(t) = x_2(t) + 0.5e(t)$$

$$y(t) = x_3(t) + e(t)$$

## Example: elevator

Matrix form:

$$\dot{x}(t) = A_c x(t) + K_c e(t) + B_c u(t)$$

$$y(t) = C_c x(t) + e(t)$$

where

$$A_c = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$K_c = \begin{bmatrix} 0.1 \\ 1 \\ 0.5 \end{bmatrix} \quad C_c = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

## Example: elevator

Zero-order hold transformation:

$$\begin{aligned}x_o(k+1) &= A_o x_o(k) + K_o e(k) + B_o u(k) \\ y(k) &= C_o x_o(k) + e(k)\end{aligned}$$

where

$$\begin{aligned}A_o &= \begin{bmatrix} 1 & 0 & 0 \\ T & 1 & 0 \\ T^2/2 & T & 1 \end{bmatrix} & B_o &= \begin{bmatrix} T \\ T^2/2 \\ T^3/6 \end{bmatrix} \\ K_o &= \begin{bmatrix} 0.1T \\ T + T^2/20 \\ T/2 + T^2/6 + T^3/60 \end{bmatrix} & C_o &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

## Example: elevator

Constraints:

$$|\ddot{y}(t)| \leq 0.4 \quad (\text{jerk})$$

$$|\dot{y}(t)| \leq 0.3 \quad (\text{acceleration})$$

$$|y(t)| \leq 0.4 \quad (\text{speed})$$

$$y(t) \leq 1.01 \quad (\text{overshoot})$$

translated in:

$$\begin{aligned}u(k+j-1) &\leq 0.4 && \text{(positive jerk)} \\-u(k+j-1) &\leq 0.4 && \text{(negative jerk)} \\ \hat{x}_1(k+j) &\leq 0.3 && \text{(positive acceleration)} \\-\hat{x}_1(k+j) &\leq 0.3 && \text{(negative acceleration)} \\ \hat{x}_2(k+j) &\leq 0.4 && \text{(positive speed)} \\-\hat{x}_2(k+j) &\leq 0.4 && \text{(negative speed)} \\ \hat{x}_3(k+j) &\leq 1.01 && \text{(overshoot)}\end{aligned}$$

for all  $j = 1, \dots, N$ .

## Example: elevator

Sampling-time  $T = 0.1$

Prediction & control horizon  $N = N_c = 30$ .

Minimum cost-horizon  $N_m = 1$ .

Reference signal  $r(k) = 1$  for all  $k > 0$

Weighting parameters  $P(q) = 1, \lambda = 0.1$ .

The GPC performance index (for an IO model):

$$J = \sum_{j=1}^{30} \left| \hat{y}(k+j|k) - r(k+j|k) \right|^2 + \\ + \lambda^2 u^2(k+j-1)$$

subject to the above linear constraints.

# Example: elevator

