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Model reduction for nonlinear control systems

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2. Linear systems as a paradigm

In the theory of continuous-time linear systems, the system Hankel operator plays an important role in a number of realization problems. For example, when viewed as a mapping from past inputs to future outputs, it plays a direct role in the abstract definition of *state*. It also plays a central role in minimality theory, in model reduction problems, and related to these, in linear identification methods. Specifically, the Hankel operator supplies a set of similarity invariants, the so called Hankel singular values, which can be used to quantify the importance of each state in the corresponding input-output system. The Hankel operator can also be factored into the composition of an observability and controllability operator, from which Gramian matrices can be defined and the notion of a balanced realization follows. The Hankel singular values are most easily computed in a state space setting using the product of the Gramian matrices, though intrinsically they depend only on the given input-output mapping.

In this chapter we briefly review the well known linear system definitions of the system Hankel matrix; the Hankel operator; the controllability and observability operators, Gramians and functions; the balanced realization and the corresponding model reduction procedure. The purpose of this review is twofold, namely to establish notation, and to motivate the approach taken in the nonlinear case, i.e., this chapter clarifies the line of thinking for the next chapters.

2.1 The Hankel, controllability and observability operator

Consider a continuous-time, causal linear input-output system $S : u \rightarrow y$, with input space $u \in U$ and output space $y \in Y$, and with impulse response $H(t)$. Let

$$H(t) = \sum_{k=0}^{\infty} H_{k+1} \frac{t^k}{k!}, \quad t \geq 0 \quad (2.1)$$

denote its Taylor series expansion about $t = 0$ where $H_k \in \mathbb{R}^{p \times m}$ for each k . The system Hankel matrix is defined as $\hat{\mathcal{H}} = [\hat{\mathcal{H}}_{i,j}]$ where $\hat{\mathcal{H}}_{i,j} = H_{i+j-1}$

for $i, j \geq 1$. If S is also BIBO stable then the system Hankel operator is the well defined mapping

$$\begin{aligned} \mathcal{H} : L_2^m[0, +\infty) &\rightarrow L_2^p[0, +\infty) \\ : \hat{u} &\rightarrow \hat{y}(t) = \int_0^\infty H(t + \tau)\hat{u}(\tau) d\tau. \end{aligned} \quad (2.2)$$

mapping the past inputs to the future outputs. If we define the *time flipping* operator as

$$\begin{aligned} \mathcal{F} : L_2^m[0, +\infty) &\rightarrow L_2^m \\ : \hat{u} &\rightarrow u(t) = \begin{cases} \hat{u}(-t) & : t < 0 \\ 0 & : t \geq 0. \end{cases} \end{aligned}$$

then clearly $\mathcal{H}(\hat{u}) = (S \circ \mathcal{F})(\hat{u})$.

When \mathcal{H} is known to be a compact operator, then its (Hilbert) adjoint operator, \mathcal{H}^* , is also compact, and the composition $\mathcal{H}^*\mathcal{H}$, is a self-adjoint compact operator with a well defined spectral decomposition:

$$\mathcal{H}^*\mathcal{H} = \sum_{i=1}^{\infty} \sigma_i^2 \langle \cdot, \psi_i \rangle_{L_2} \psi_i, \quad \sigma_i \geq 0, \quad (2.3)$$

$$\langle \psi_i, \psi_j \rangle_{L_2} = \delta_{ij}, \quad \langle \psi_i, (\mathcal{H}^*\mathcal{H})(\psi_i) \rangle_{L_2} = \sigma_i^2. \quad (2.4)$$

where σ_i^2 is an eigenvalue of $\mathcal{H}^*\mathcal{H}$ with corresponding eigenvector ψ_i , ordered as $\sigma_1 \geq \dots \geq \sigma_n > 0$, and called the *Hankel singular values* for the input-output system Σ .

Let (A, B, C) be a state space realization of S with dimension n , i.e., consider a linear system:

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx \end{aligned} \quad (2.5)$$

where $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$. We assume throughout this chapter that (2.5) is *stable and minimal*, i.e., controllable and observable. Any such realization induces a factorization of the system Hankel matrix into the form $\hat{\mathcal{H}} = \hat{\mathcal{O}}\hat{\mathcal{C}}$, where $\hat{\mathcal{O}}$ and $\hat{\mathcal{C}}$ are the (extended) observability and controllability matrices. If the realization is asymptotically stable then the Hankel operator can be written as the composition of uniquely determined observability and controllability operators; that is, $\mathcal{H} = \mathcal{O}\mathcal{C}$, where the controllability and observability operators are defined as

$$\begin{aligned} \mathcal{C} : L_2^m[0, +\infty) &\rightarrow \mathbb{R}^n : \hat{u} \rightarrow \int_0^\infty e^{At}B\hat{u}(t) dt \\ \mathcal{O} : \mathbb{R}^n &\rightarrow L_2^p[0, +\infty) : x \rightarrow \hat{y}(t) = Ce^{At}x. \end{aligned}$$

Since \mathcal{C} and \mathcal{O} have a finite dimensional range and domain, respectively, they are compact operators; and the composition $\mathcal{O}\mathcal{C}$ is also a compact

operator. From the definition of the (Hilbert) adjoint operator, it is easily shown that \mathcal{C} and \mathcal{O} have corresponding adjoints

$$\begin{aligned} \mathcal{C}^* : \mathbb{R}^n &\rightarrow L_2^m[0, +\infty) : x \rightarrow B^T e^{A^T t} x \\ \mathcal{O}^* : L_2^p[0, +\infty) &\rightarrow \mathbb{R}^n : y \rightarrow \int_0^\infty e^{A^T t} C^T y(t) dt. \end{aligned}$$

2.2 Balanced state space realizations

The above input-output setting can be related with the well-known Gramians that are related to the state space realization. In order to do so, we consider the energy functions given in the following definition.

Definition 2.2.1. *The controllability and observability functions of a smooth state-space system are defined as*

$$L_c(x_0) = \min_{\substack{u \in L_2(-\infty, 0) \\ x(-\infty) = 0, x(0) = x_0}} \frac{1}{2} \int_{-\infty}^0 \|u(t)\|^2 dt \quad (2.6)$$

and

$$L_o(x_0) = \frac{1}{2} \int_0^\infty \|y(t)\|^2 dt, \quad x(0) = x_0, \quad u(t) \equiv 0, \quad 0 \leq t < \infty, \quad (2.7)$$

respectively. \triangle

The value of the controllability function at x_0 is the minimum amount of input energy required to reach the state x_0 from the zero state, and the value of the observability function at x_0 is the amount of output energy generated by the state x_0 . The following results are well known:

Theorem 2.2.1. *Consider the system (2.5). Then $L_c(x_0) = \frac{1}{2} x_0^T P^{-1} x_0$ and $L_o(x_0) = \frac{1}{2} x_0^T Q x_0$, where $P = \int_0^\infty e^{At} B B^T e^{A^T t} dt$ is the controllability Gramian and $Q = \int_0^\infty e^{A^T t} C^T C e^{At} dt$ is the observability Gramian. Furthermore P and Q are symmetric and positive definite, and are unique solutions of the Lyapunov equations*

$$AP + PA^T = -BB^T \quad (2.8)$$

and

$$A^T Q + QA = -C^T C, \quad (2.9)$$

respectively. \triangle

From the form of the Gramians in this theorem it follows immediately that for any $x_1, x_2 \in \mathbb{R}^n$:

$$\begin{aligned} \langle x_1, \mathcal{C}\mathcal{C}^* x_2 \rangle &= x_1^T \int_0^\infty e^{At} B B^T e^{A^T t} dt x_2 \\ &= x_1^T P x_2 \end{aligned} \quad (2.10)$$

$$\begin{aligned} \langle x_1, \mathcal{O}^* \mathcal{O} x_2 \rangle &= x_1^T \int_0^\infty e^{A^T t} C^T C e^{At} dt x_2 \\ &= x_1^T Q x_2. \end{aligned} \quad (2.11)$$

and the relation with the energy functions is given as

$$L_c(x) = \frac{1}{2} x^T P^{-1} x = \frac{1}{2} \langle x, (\mathcal{C}\mathcal{C}^*)^{-1} x \rangle \quad (2.12)$$

$$L_o(x) = \frac{1}{2} x^T Q x = \frac{1}{2} \langle x, (\mathcal{O}^* \mathcal{O}) x \rangle. \quad (2.13)$$

The following (balancing) theorem is originally due to Moore [53].

Theorem 2.2.2. *The eigenvalues of QP are similarity invariants, i.e., they do not depend on the choice of the state space coordinates. There exists a state space representation where*

$$\Sigma := Q = P = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{pmatrix} \quad (2.14)$$

with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$ the square roots of the eigenvalues of QP . Such representations are called balanced, and the system is in balanced form. Furthermore, the σ_i 's, $i=1, \dots, n$, equal the Hankel singular values, i.e., the singular values of the Hankel operator (2.2). \triangle

Two other representations, that may be obtained from (2.14) by coordinate transformations $x = \Sigma^{-\frac{1}{2}} \bar{x}$ and $x = \Sigma^{\frac{1}{2}} \tilde{x}$, respectively, follow easily from the above theorem.

Definition 2.2.2. *A state space representation is an input-normal/output-diagonal representation if $P = I$ and $Q = \Sigma^2$, where Σ is given by (2.14). Furthermore, it is an output-normal/input-diagonal representation if $P = \Sigma^2$ and $Q = I$. \triangle*

The largest Hankel singular value is equal to the Hankel norm of the system, i.e.,

$$\|G\|_H^2 = \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{L_o(x)}{L_c(x)} = \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{x^T Q x}{x^T P^{-1} x} = \max_{\substack{\bar{x} \in \mathbb{R}^n \\ \bar{x} \neq 0}} \frac{\bar{x}^T \Sigma^2 \bar{x}}{\bar{x}^T \bar{x}} = \sigma_1^2, \quad (2.15)$$

where $G = C(sI - A)^{-1} B$ is the transfer matrix of the system. This gives a characterization of the largest Hankel singular value. The other Hankel

singular values may be characterized inductively in a similar way; we refer to Courant and Hilbert [14] and Gantmacher [28].

So far, we have assumed the state-space representation to be minimal. However, if we consider non-minimal state-space realizations, i.e., the system is not controllable and/or observable, then we obtain σ'_i 's that are zero, corresponding to the non-controllable or non-observable part of the system, and thus, to the non-minimal part of the system. Related to this observation, we have that the minimal realization of a linear input-output system has a dimension n that is equal to the Hankel rank, or in other words, it equals the rank of the Hankel matrix. A well-known result related to the latter is the following theorem.

Theorem 2.2.3. *If (A, B, C) is asymptotically stable, then the realization is minimal if and only if $P > 0$ and $Q > 0$.* \triangle

2.3 Model reduction

Once the state space system is in balanced form, an order reduction procedure based on this form may be applied. Thus, in order to proceed, we assume that the system (2.5) is in balanced form. Then the controllability and observability function are $\bar{L}_c(\bar{x}_0) = \frac{1}{2}\bar{x}_0^T \Sigma^{-1} \bar{x}_0$ and $\bar{L}_o(\bar{x}_0) = \frac{1}{2}\bar{x}_0^T \Sigma \bar{x}_0$, respectively. For small σ_i the amount of control energy required to reach the state $\tilde{x} = (0, \dots, 0, x_i, 0, \dots, 0)$ is large while the output energy generated by this state \tilde{x} is small. Hence, if $\sigma_k \gg \sigma_{k+1}$, the state components x_{k+1} to x_n are far less important from this energy point of view and may be removed to reduce the number of state components of the model. We partition the system (2.5) in a corresponding way as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = (C_1 \quad C_2), \quad (2.16)$$

$$x^1 = (x_1, \dots, x_k)^T, \quad x^2 = (x_{k+1}, \dots, x_n)^T, \quad \Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix},$$

where $\Sigma_1 = \text{diag}(\sigma_1, \dots, \sigma_k)$ and $\Sigma_2 = \text{diag}(\sigma_{k+1}, \dots, \sigma_n)$.

Theorem 2.3.1. *Both subsystems (A_{ii}, B_i, C_i) , $i = 1, 2$, are again in balanced form, and their controllability and observability Gramians are equal to Σ_i , $i = 1, 2$.* \triangle

The following result has been proved by Pernebo and Silverman [65].

Theorem 2.3.2. *Assume that $\sigma_k > \sigma_{k+1}$. Then both subsystems (A_{ii}, B_i, C_i) , $i = 1, 2$, are asymptotically stable.* \triangle

The subsystem (A_{11}, B_1, C_1) may be used as an approximation of the full order system (2.5). The optimality of this approximation in the Hankel and \mathcal{H}_∞ -norm has been studied by Glover [29], and an upper bound for the error is given. The \mathcal{H}_∞ -norm of $G(s) = C(sI - A)^{-1}B$ is defined as

$$\|G\|_\infty = \sup_{\omega \in \mathbb{R}} \lambda_{\max}^{\frac{1}{2}}(G(-j\omega)^T G(j\omega)),$$

where $\lambda_{\max}^{\frac{1}{2}}(G(-j\omega)^T G(j\omega))$ is the square root of the maximum eigenvalue of $G(-j\omega)^T G(j\omega)$. Denote the transfer matrix of the reduced order system (A_{11}, B_1, C_1) by $\tilde{G}(s) = C_1(sI - A_{11})^{-1}B_1$.

Theorem 2.3.3. $\|G - \tilde{G}\|_H \leq \|G - \tilde{G}\|_\infty \leq 2(\sigma_{k+1} + \dots + \sigma_n)$. \triangle

Hence, if we remove the state components x_{k+1}, \dots, x_n that correspond to small Hankel singular values $\sigma_{k+1}, \dots, \sigma_n$ (small compared to the rest of the singular values, i.e., $\sigma_k \gg \sigma_{k+1}$), then the error is small, and the reduced order system (A_{11}, B_1, C_1) constitutes a good approximation in terms of the Hankel norm to the full order system.

The model reduction method that we gave above consists of simply truncating the model. It is also possible to reduce the model in a different way. Instead of setting $x^2 = (x_{k+1}, \dots, x_n) = 0$ we approximate the system by setting $\dot{x}^2 = 0$ (thus interpreting x^2 as a very fast stable state, which may be approximated by a constant function of x^1 and u). The resulting algebraic equation can be solved for x^2 as (note that A_{22}^{-1} exists by Theorem 2.3.2)

$$x^2 = -A_{22}^{-1}(A_{21}x^1 + B_2u).$$

Substitution in (2.5) leads to a reduced order model $(\hat{A}, \hat{B}, \hat{C})$ defined as

$$\begin{aligned} \hat{A} &:= A_{11} - A_{12}A_{22}^{-1}A_{21} \\ \hat{B} &:= B_1 - A_{12}A_{22}^{-1}B_2 \\ \hat{C} &:= C_1 - C_2A_{22}^{-1}A_{21} \end{aligned}$$

The system $(\hat{A}, \hat{B}, \hat{C})$ also gives an approximation to the full order system (2.5). Theorems 2.3.1, 2.3.2 and 2.3.3 also hold if we replace the system (A_{11}, B_1, C_1) by $(\hat{A}, \hat{B}, \hat{C})$.

2.4 Notes and references

Huge amount of reference, see e.g., [90],[42],[44], [73], [45],[53], [65], [29], [18], [37]. A recent course book is [62]. This list is by far not complete! For overview paper on linear model reduction methods, see web-site of Antoulas, i.e., <http://www-ece.rice.edu/~aca/>

2.5 LQG balancing

A major drawback of the original balancing method as described in Section 2.2 is that it only applies to stable systems. Furthermore, the method emphasizes the (open-loop) input-output characteristics of the system, while it is a priori not clear if it yields good approximations in closed-loop configurations. In this section we treat LQG balancing for linear systems, which was introduced by Jonckheere and Silverman [42, 43] (see also Verriest [82]). In Opdenacker and Jonckheere [63] this concept is further developed. LQG balancing was introduced with the aim of finding a model reduction method for a system (not necessarily stable) together with its corresponding LQG compensator. LQG balancing has been treated from another point of view in Weiland [85]. First we give a review of the formulation of Jonckheere and Silverman [43] and Opdenacker and Jonckheere [63].

LQG compensation is formulated for a minimal state-space system

$$\begin{aligned}\dot{x} &= Ax + Bu + Bd \\ y &= Cx + v\end{aligned}\tag{2.17}$$

where $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and d and v are independent Gaussian white noise processes with covariance functions $I\delta(t - \tau)$. The criterion

$$J(x_0, u(\cdot)) = E \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [x^T(t)C^T Cx(t) + u^T(t)u(t)]dt\tag{2.18}$$

is required to be minimized. The resulting optimal compensator is given by

$$\begin{aligned}\dot{z} &= Az + Bu + SC^T(y - Cz) \\ u &= -B^T Pz.\end{aligned}\tag{2.19}$$

Here S is the stabilizing solution (i.e. $\sigma(A - SC^T C) \subset \mathbb{C}^-$) to the Filter Algebraic Riccati Equation (FARE)

$$AS + SA^T + BB^T - SC^T CS = 0,\tag{2.20}$$

and P is the stabilizing solution (i.e. $\sigma(A - BB^T P) \subset \mathbb{C}^-$) to the Control Algebraic Riccati Equation (CARE)

$$A^T P + PA + C^T C - PBB^T P = 0.\tag{2.21}$$

Theorem 2.5.1. ([43, 63]) *The eigenvalues of PS are similarity invariants and there exists a state space representation where*

$$M := P = S = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix}\tag{2.22}$$

with $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n > 0$. This is called a LQG balanced representation, or LQG balanced form. \triangle

Jonckheere & Silverman [43] and Opdenacker & Jonckheere [63] argue that if $\mu_k \gg \mu_{k+1}$ then the state components x_1 up to x_k are more difficult both to control and to filter than x_{k+1} up to x_n and a synthesis based only on the state components x_1 up to x_k probably retains the essential properties of the system in a closed-loop configuration. Corresponding to the partitioning of the state in the first k components and the last $n - k$ components, the partitioning of the matrices is done as in (2.16), and the reduced order system is

$$\begin{aligned}\dot{x} &= A_{11}x + B_1u + B_1d \\ y &= C_1x + v\end{aligned}\tag{2.23}$$

Theorem 2.5.2. ([43, 63]) *Assume $\mu_k > \mu_{k+1}$. Then (A_{11}, B_1, C_1) is minimal, the reduced order system (2.23) is again LQG balanced and the optimal compensator for system (2.23) is the reduced order optimal compensator of the full order system (2.17). \triangle*

As explained in Section 2.2, the original idea of balancing stable linear systems, as introduced by Moore [53], considers the Hankel singular values σ_i , $i = 1, \dots, n$, which are a measure for the importance of a state component in a balanced representation. This balancing technique is based on the input energy which is necessary to reach this state component and the output energy which is generated by this state component. A similar kind of reasoning, using a different pair of energy functions, may be used to achieve the similarity invariants μ_i , $i = 1, \dots, n$, as above, see Weiland [85]. To follow this reasoning, we consider the minimal system (2.17) *without noise*, i.e.

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}\tag{2.24}$$

where $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$. We define the energy functions

$$\begin{aligned}K^-(x_0) &:= \min_{\substack{u \in L_2(-\infty, 0) \\ x(-\infty) = 0, x(0) = x_0}} \frac{1}{2} \int_{-\infty}^0 (\|y(t)\|^2 + \|u(t)\|^2) dt \\ K^+(x_0) &:= \min_{\substack{u \in L_2(0, \infty) \\ x(\infty) = 0, x(0) = x_0}} \frac{1}{2} \int_0^{\infty} (\|y(t)\|^2 + \|u(t)\|^2) dt\end{aligned}$$

$K^-(x_0)$ is called the *past energy* and $K^+(x_0)$ the *future energy* of the system in the state x_0 .

Theorem 2.5.3. ([85]) *$K^-(x_0) = \frac{1}{2}x_0^T S^{-1}x_0$ and $K^+(x_0) = \frac{1}{2}x_0^T Px_0$, where S and P are the stabilizing solutions of the FARE and the CARE, the equations (2.20) and (2.21), respectively. \triangle*

For the LQG balanced representation of Theorem 2.5.1 the past and future energy function are $K^-(x_0) = \frac{1}{2}x_0^T M^{-1}x_0$ and $K^+(x_0) = \frac{1}{2}x_0^T Mx_0$, respectively, where M is diagonal. The importance of the state $\tilde{x} = (0, \dots, 0, x_i, 0, \dots, 0)$ in terms of past and future energy may be measured by the similarity invariant μ_i . For large μ_i the influence of the state \tilde{x} on the future energy is large while the influence on the past energy is small. Hence, if $\mu_k \gg \mu_{k+1}$, the state components x_{k+1} to x_n are ‘not important’ from this energy point of view and may be removed to reduce the number of state components of the model.

2.6 Balancing of the normalized coprime representation

In Meyer [50] and Ober and McFarlane [61] balancing of the normalized coprime representation of a linear system is treated. Balancing of the normalized coprime representation was introduced with the aim of finding a model reduction method for unstable linear systems. In [50] balancing of the normalized *right* coprime factorization is treated, while in [61] balancing of the normalized *left* coprime factorization is treated. Here we give a brief review on this subject.

We consider the system (2.24) and its transfer function $G(s) = C(sI - A)^{-1}B$. Furthermore, we consider the stabilizing solution P to the CARE (2.21) and the stabilizing solution S to the FARE (2.20), leading to the stable matrices $\hat{A} := A - BB^T P$ and $\tilde{A} := A - SC^T C$. First we treat normalized right and then normalized left coprime factorizations.

We may write any transfer matrix $G(s) = C(sI - A)^{-1}B$ as a right fraction $G(s) = N(s)D(s)^{-1}$ of stable transfer matrices $N(s)$ and $D(s)$. If we choose (e.g. Nett *et al.* [58])

$$\begin{aligned} N(s) &:= C(sI - \hat{A})^{-1}B \\ D(s) &:= I - B^T P(sI - \hat{A})^{-1}B \end{aligned}$$

then the *factorization* is *right coprime*, i.e., $N(s)$ and $D(s)$ have no common zeros at the same place in the closed right half plane. A state space realization of the transfer matrix (the so called graph operator)

$$\begin{pmatrix} N(s) \\ D(s) \end{pmatrix}$$

is

$$\begin{aligned} \dot{x} &= (A - BB^T P)x + Bw \\ \begin{pmatrix} y \\ u \end{pmatrix} &= \begin{pmatrix} C \\ -B^T P \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} w \end{aligned} \quad (2.25)$$

with w a (fictitious) input variable. Furthermore, we are able to find stable transfer matrices $U(s)$ and $V(s)$, such that the Bezout identity

$$U(s)N(s) + V(s)D(s) = I. \quad (2.26)$$

is fulfilled. Indeed, take $U(s) = B^T P(sI - \tilde{A})^{-1} S C^T$ and $V(s) = I + B^T P(sI - \tilde{A})^{-1} B$ (see e.g. Vidyasagar [83], Nett *et al.* [58]). The fact that we are able to find a stable left inverse of the graph operator, i.e., we can find the solutions $U(s)$ and $V(s)$ to the Bezout identity (2.26), is equivalent to the factorization being right coprime. Furthermore, the graph operator is inner, i.e.,

$$\left\| \begin{pmatrix} N \\ D \end{pmatrix} w \right\|_2 = \|w\|_2,$$

or

$$N(-s)^T N(s) + D(-s)^T D(s) = I.$$

Therefore, the factorization is called *normalized*. It is easily checked that the observability Gramian of the system (2.25) is P . Denote its controllability Gramian by R .

In a similar way we may write the transfer matrix $G(s)$ as a left fraction $G(s) = \tilde{D}(s)^{-1} \tilde{N}(s)$ of stable transfer matrices $\tilde{D}(s)$ and $\tilde{N}(s)$. If we choose (e.g. Nett *et al.* [58])

$$\begin{aligned} \tilde{N}(s) &:= C(sI - \tilde{A})^{-1} B \\ \tilde{D}(s) &= C(sI - \tilde{A})^{-1} S C^T - I \end{aligned}$$

then this is a *left factorization*. Obviously $\hat{y}(s) = G(s)\hat{u}(s)$ is equivalent with $0 = \tilde{N}(s)\hat{u}(s) - \tilde{D}(s)\hat{y}(s)$. Moreover, a state space realization of the transfer matrix

$$\begin{pmatrix} \tilde{N}(s) & \tilde{D}(s) \end{pmatrix}$$

is

$$\begin{aligned} \dot{x} &= (A - S C^T C)x + (B \quad S C^T) \tilde{w} \\ z &= Cx + (0 \quad -I) \tilde{w}. \end{aligned} \quad (2.27)$$

If we take

$$\tilde{w} = \begin{pmatrix} u \\ y \end{pmatrix}$$

as the input variable, then the dynamics resulting from setting $z = 0$ in (2.27) is a state space representation of $G(s)$. We are able to find stable transfer matrices such that the Bezout Identity is fulfilled, i.e. there exist stable transfer matrices $\tilde{U}(s)$ and $\tilde{V}(s)$, such that

$$\tilde{N}(s)\tilde{U}(s) + \tilde{D}(s)\tilde{V}(s) = I. \quad (2.28)$$

Indeed, we may take $\tilde{U}(s) = B^T P(sI - \hat{A})^{-1} S C^T$ and $\tilde{V}(s) = I + C(sI - \hat{A})^{-1} S C^T$, (see e.g. Vidyasagar [83], Nett *et al.* [58]). This proves that the factorization is *left coprime*. Furthermore, $(\tilde{N}(s) \quad \tilde{D}(s))$ is co-inner, i.e.,

$$\tilde{N}(s)\tilde{N}(-s)^T + \tilde{D}(s)\tilde{D}(-s)^T = I,$$

which means that the factorization is *normalized*. Hence, $(\tilde{N}(s) \quad \tilde{D}(s))$ represents the normalized left coprime factorization of system (2.24). The system (2.27) has as controllability Gramian the positive definite matrix S and we denote its observability Gramian by the matrix Q . Note that the right factorization

$$\begin{pmatrix} N(s) \\ D(s) \end{pmatrix}$$

can be seen as an *image* representation of $G(s)$, while the left factorization

$$(\tilde{N}(s) \quad \tilde{D}(s))$$

can be regarded as a *kernel* representation of $G(s)$.

The following result does not seem to have been stated explicitly before.

Theorem 2.6.1. *The Hankel singular values of the right and left factorization (2.25) and (2.27), respectively, are the same.*

Proof It follows from the Lyapunov equations (2.8) and (2.9) for the systems (2.25) and (2.27), that $R = (I + SP)^{-1}S$ and $Q = (I + PS)^{-1}P$. Now, it is easily obtained that PR and SQ have the same eigenvalues. ■

The Hankel singular values of (2.25) (and, hence, of (2.27)) are called the *graph Hankel singular values* of the original system (2.24). These singular values have the following property:

Theorem 2.6.2. **([50, 61])** *The graph Hankel singular values of system (2.24) are strictly less than one.* △

Denote the graph Hankel singular values by τ_i , $i = 1, \dots, n$, and assume $\tau_1 \geq \dots \geq \tau_n$. The relation between τ_i , $i = 1, \dots, n$, and the similarity invariants μ_i , $i = 1, \dots, n$, of Theorem 2.5.1 is given by the following theorem:

Theorem 2.6.3. **([61, 85])** $\mu_i = \tau_i(1 - \tau_i^2)^{-\frac{1}{2}}$ for $i = 1, \dots, n$. △

This implies that the reduced model that is obtained by model reduction based on balancing the (left or right) normalized coprime factorization is the same as the reduced model that is obtained by model reduction based on LQG balancing. Consider the normalized right coprime representation (2.25) and assume that it is in balanced form with

$$A := P = R = \begin{pmatrix} \tau_1 & & 0 \\ & \ddots & \\ 0 & & \tau_n \end{pmatrix}.$$

Furthermore, assume that $\tau_k > \tau_{k+1}$ and define correspondingly $\Lambda =: \text{diag}\{\Lambda_1, \Lambda_2\}$. It follows (Meyer [50]) that reducing the order of (2.25) by truncating the system to the first k state components (the partitioning is done corresponding to (2.16)), again gives a normalized right coprime representation.

Theorem 2.6.4. ([50]) *The reduced order system of (2.25) is of the form*

$$\left((A_{11} - B_1 B_1^T D_1), B_1, \begin{pmatrix} C_1 \\ -B_1^T D_1 \end{pmatrix}, \begin{pmatrix} 0 \\ I \end{pmatrix} \right),$$

with controllability and observability Gramian Λ_1 . This system is the normalized right coprime representation of the system (A_{11}, B_1, C_1) , which is minimal. \triangle

2.7 \mathcal{H}_∞ balancing

\mathcal{H}_∞ balancing for linear systems has been introduced by Mustafa and Glover [54, 55, 56]. A first motivation for \mathcal{H}_∞ balancing is similar to the main motivation for LQG balancing, i.e., model reduction of the system and a corresponding \mathcal{H}_∞ compensator. The set of invariants defined in [54, 55, 56] are related to invariants defined by Weiland [85]. First we review the formulation of Mustafa and Glover [54, 55, 56].

The normalized \mathcal{H}_∞ control problem (see also Doyle *et al.* [16]) on which the \mathcal{H}_∞ balancing method is based, is the minimum entropy problem associated with the system

$$\begin{aligned} \dot{x} &= Ax + Bu + Bd \\ y &= Cx + v \\ z &= \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} Cx \\ u \end{pmatrix}. \end{aligned} \tag{2.29}$$

Here $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $z \in \mathbb{R}^{m+p}$, and d and v are independent Gaussian white noise processes with covariance functions $I\delta(t - \tau)$. By the form of system (2.29) we call the \mathcal{H}_∞ control problem that is formulated normalized. Let $w := (d^T, v^T)$ and assume that the system (A, B, C) is minimal. G is the transfer matrix of the system (A, B, C) . It is easily shown that the closed-loop transfer matrix H from w to z is

$$H = \begin{pmatrix} (I - GK)^{-1}G & (I - GK)^{-1}GK \\ K(I - GK)^{-1}G & K(I - GK)^{-1} \end{pmatrix}$$

where K is the transfer matrix of the controller. For notational simplicity the Laplace transform variable s is suppressed. The block diagram of the closed-loop system is given in Figure 2.1. The \mathcal{H}_∞ constraint is $\|H\|_\infty < \gamma$.

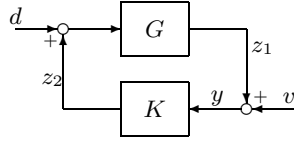


Fig. 2.1. Block diagram of the closed-loop system.

Here $\gamma \in \mathbb{R}$ and, as before in Section 2.2,

$$\|H\|_\infty = \sup_{\omega \in \mathbb{R}} \lambda_{\max}^{\frac{1}{2}}(H(-j\omega)^T H(j\omega))$$

The optimal \mathcal{H}_∞ norm is $\gamma_0 := \inf\{\|H\|_\infty \mid K \text{ stabilizes } G\}$. We assume $\gamma > \gamma_0$ (the strictly suboptimal case). The entropy that has to be minimized is given by

$$I(\gamma) = -\frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln \left| \det \left(I - \frac{1}{\gamma^2} H(-j\omega)^T H(j\omega) \right) \right| d\omega \quad (2.30)$$

The normalized \mathcal{H}_∞ problem is to find a controller K that stabilizes G and minimizes the entropy $I(\gamma)$, over the class of stabilized closed-loop transfer matrices H satisfying $\|H\|_\infty < \gamma$, where $\gamma > \gamma_0$. Take

$$L := \left(Y_\infty^{-1} - \frac{1}{\gamma^2} X_\infty \right)^{-1}, \quad (2.31)$$

where Y_∞ is the unique positive definite stabilizing solution (i.e., $\sigma(A - (1 - \gamma^{-2})C^T C Y_\infty) \subset \mathbb{C}^-$) to the \mathcal{H}_∞ Filter Algebraic Riccati Equation (HFARE)

$$A Y_\infty + Y_\infty A^T + B B^T - \left(1 - \frac{1}{\gamma^2}\right) Y_\infty C^T C Y_\infty = 0, \quad (2.32)$$

and where X_∞ is the unique positive definite stabilizing solution (i.e., $\sigma(A - (1 - \gamma^{-2})B B^T X_\infty) \subset \mathbb{C}^-$) to the \mathcal{H}_∞ Control Algebraic Riccati Equation (HCARE)

$$A^T X_\infty + X_\infty A + C^T C - \left(1 - \frac{1}{\gamma^2}\right) X_\infty B B^T X_\infty = 0. \quad (2.33)$$

There exists a controller K that solves the normalized \mathcal{H}_∞ problem if and only if $\lambda_{\max}(X_\infty Y_\infty) < \gamma^2$ (the coupling condition), and the resulting so called central controller (Doyle *et al.* [16]) takes the form

$$\begin{aligned} \dot{\hat{x}} &= A \hat{x} - \left(1 - \frac{1}{\gamma^2}\right) B B^T X_\infty \hat{x} + L C^T (y - C \hat{x}) \\ u &= -B^T X_\infty \hat{x}. \end{aligned} \quad (2.34)$$

This controller does not exactly have the form that is given by Mustafa and Glover in [55], but in [30] they prove that the controller (2.34) also solves the problem.

Theorem 2.7.1. ([55]) *The eigenvalues of $X_\infty Y_\infty$ are similarity invariants and there exists a state space representation where*

$$N := X_\infty = Y_\infty = \begin{pmatrix} \vartheta_1 & & 0 \\ & \ddots & \\ 0 & & \vartheta_n \end{pmatrix} \quad (2.35)$$

with $\vartheta_1 \geq \vartheta_2 \geq \dots \geq \vartheta_n > 0$. This state space representation is called a \mathcal{H}_∞ balanced representation or \mathcal{H}_∞ balanced form and the ϑ_i 's are called the \mathcal{H}_∞ characteristic values. \triangle

The following theorem states some properties of ϑ_i , $i = 1, \dots, n$, where μ_i , $i = 1, \dots, n$, are as in Theorem 2.5.1, and σ_i , $i = 1, \dots, n$, are as in Theorem 2.2.2.

Theorem 2.7.2. ([54])

- a. $\gamma > \vartheta_i \geq \mu_i$, $i = 1, \dots, n$.
- b. Each ϑ_i is a non-increasing function of γ .
- c. Each ϑ_i is a continuous function of γ .
- d. $\lim_{\gamma \rightarrow \infty} \vartheta_i = \mu_i$, $i = 1, \dots, n$.
- e. If $\gamma = 1$, then $\vartheta_i = \sigma_i$, $i = 1, \dots, n$. \triangle

In Mustafa and Glover [54, 55, 56] it is argued that if $\vartheta_k \gg \vartheta_{k+1}$, then the state components x_1 up to x_k are more difficult both to control and to filter in an \mathcal{H}_∞ sense than x_{k+1} up to x_n . Corresponding to the partitioning of the state in the first k components and the last $n - k$ components, the partitioning of the matrices is done as in (2.16), and the resulting reduced order system is of the form

$$\begin{aligned} \dot{x} &= A_{11}x + B_1u + B_1d \\ y &= C_1x + v \\ z &= \begin{pmatrix} C_1x \\ u \end{pmatrix} \end{aligned} \quad (2.36)$$

Theorem 2.7.3. ([55]) *The reduced order system (2.36) is again \mathcal{H}_∞ balanced and the normalized \mathcal{H}_∞ controller for system (2.36) is the reduced order normalized \mathcal{H}_∞ controller of the full order system (2.29). \triangle*

For $\gamma > 1$ it is also possible to reduce the system (2.29) via the normalized coprime factors of the scaled transfer matrix $G(s) = C(sI - A)^{-1}B$, i.e., the transfer matrix $\beta G(s)$, where $\beta := \sqrt{1 - \gamma^{-2}}$. This gives us the same reduced order system. In fact, this comes down to the same relation as the relation between balancing of the normalized coprime factorization and LQG balancing. For details about this we refer to Mustafa and Glover [55].

In Section 2.5 we gave an alternative way to consider LQG balancing that may be derived from Weiland [85]. A similar kind of reasoning, using a different pair of energy functions may be used to achieve the \mathcal{H}_∞ characteristic values ϑ_i , $i = 1, \dots, n$, for $\gamma \neq 1$. To explain this alternative way we consider the minimal system (A, B, C) , as in (2.24). For $\gamma \neq 1$, we define Q^- as

$$Q^-(x_0) := \min_{\substack{u \in L_2(-\infty, 0) \\ x(-\infty) = 0, x(0) = x_0}} \frac{1}{2} \int_{-\infty}^0 ((1 - \gamma^{-2}) \|y(t)\|^2 + \|u(t)\|^2) dt.$$

Furthermore, if $\gamma > 1$ we define Q^+ as

$$Q^+(x_0) := \min_{\substack{u \in L_2(0, \infty) \\ x(\infty) = 0, x(0) = x_0}} \frac{1}{2} \int_0^\infty (\|y(t)\|^2 + \frac{1}{1 - \gamma^{-2}} \|u(t)\|^2) dt,$$

while if $\gamma < 1$, we define Q^+ as (instead of minimizing over u , we maximize over u)

$$Q^+(x_0) := \max_{\substack{u \in L_2(0, \infty) \\ x(\infty) = 0, x(0) = x_0}} \frac{1}{2} \int_0^\infty (\|y(t)\|^2 + \frac{1}{1 - \gamma^{-2}} \|u(t)\|^2) dt.$$

$Q^-(x_0)$ is called the \mathcal{H}_∞ -past energy and $Q^+(x_0)$ the \mathcal{H}_∞ -future energy of the system in the state x_0 . Note that for $\gamma \rightarrow \infty$ we have $Q^+ = K^+$ and $Q^- = K^-$, where K^+ and K^- are as defined in Section 2.5, and that for $\gamma \rightarrow 1$ we have $Q^+ = L_o$ and $Q^- = L_c$, where L_o and L_c are as defined in Section 2.2.

Theorem 2.7.4. $Q^-(x_0) = \frac{1}{2}x_0^T Y_\infty^{-1}x_0$ and $Q^+(x_0) = \frac{1}{2}x_0^T X_\infty x_0$, where Y_∞ and X_∞ are the stabilizing positive definite solutions of (2.32) and (2.33), respectively. \triangle

This means that if $\gamma \neq 1$ then for the \mathcal{H}_∞ balanced representation the \mathcal{H}_∞ -past and \mathcal{H}_∞ -future energy function are $Q^-(x_0) = \frac{1}{2}x_0^T N^{-1}x_0$ and $Q^+(x_0) = \frac{1}{2}x_0^T N x_0$, respectively, with N as in (2.35). The importance of the state $\tilde{x} = (0, \dots, 0, x_i, 0, \dots, 0)$ in terms of \mathcal{H}_∞ -past and \mathcal{H}_∞ -future energy may be measured by the \mathcal{H}_∞ characteristic values ϑ_i . For large ϑ_i the influence of the state \tilde{x} on the \mathcal{H}_∞ -future energy is large while the influence on the \mathcal{H}_∞ -past energy is small. Hence if $\vartheta_k \gg \vartheta_{k+1}$, the state components x_{k+1} to x_n are not important from this energy point of view and may be removed to reduce the number of state components of the model.

2.8 Conservative mechanical systems

Model reduction for linear conservative mechanical systems by balancing of the associated gradient system has been introduced by Van der Schaft

and Oeloff [72]. The systems that are considered are only Lyapunov (not asymptotically) stable, and thus balancing as treated in Section 2.2 does not immediately apply. It is possible to apply model reduction by considering balancing for unstable linear systems, as treated in Section 2.6, but this in general does not preserve the structure of the mechanical system, which may be a drawback from a modelling and control point of view. The model reduction method that is treated here does preserve the structure of the mechanical system. The key is to associate with the conservative mechanical system a gradient system which is asymptotically stable, and to apply the balancing method of Section 2.2 to this associated gradient system. We review the formulation of Van der Schaft and Oeloff [72].

We consider linear conservative mechanical systems with collocated sensors and actuators corresponding to outputs y and inputs u . In Hamiltonian form such system may be written as

$$\begin{aligned} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} &= \begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ B \end{pmatrix} u \\ y &= B^T q \end{aligned} \quad (2.37)$$

where $P = P^T > 0$, $Q = Q^T > 0$. Here $q = (q_1, \dots, q_n)$ is the vector of generalized configuration coordinates and $p = (p_1, \dots, p_n)$ is the vector of generalized momenta. The quantities $\frac{1}{2}p^T P p$ and $\frac{1}{2}q^T Q q$ are the kinetic and potential energy, respectively, of the system. Naturally $P > 0$, but $Q > 0$ is an assumption. This implies that the poles of the system are all on the imaginary axis and not equal to zero, and that the system is Lyapunov stable (see Appendix C), with Lyapunov function being the total energy $\frac{1}{2}p^T P p + \frac{1}{2}q^T Q q$. The transfer matrix $F_H(s)$ of system (2.37) is

$$F_H(s) = B^T (Is^2 + PQ)^{-1} PB$$

and satisfies the symmetry properties $F_H(-s) = F_H(s) = F_H^T(-s)$. The associated *gradient system* of (2.37) is defined by

$$\begin{aligned} \dot{x} &= -PQx + PBu \\ y &= B^T x \end{aligned} \quad (2.38)$$

with inner product P^{-1} and potential function $\frac{1}{2}x^T Q x$ (see also Brockett [9]). For a detailed study of gradient systems we refer to the literature (e.g. Brockett [9], Van der Schaft [69], see also Chapter 6). The gradient system is asymptotically stable and its transfer matrix $F_G(s)$ is related to $F_H(s)$ by $F_G(s^2) = F_H(s)$.

Theorem 2.8.1. ([68, 69]) *The Hamiltonian system (2.37) is controllable if and only if its associated gradient system (2.38) is controllable. Furthermore, the Hamiltonian system is controllable if and only if the Hamiltonian system is observable. The same is valid for the gradient system. \triangle*

If we assume that the Hamiltonian system (2.37) is minimal then we can apply the balancing theory of Section 2.2 to the gradient system (2.38), since this system is asymptotically stable. The controllability Gramian W and the observability Gramian M of (2.38) are the unique symmetric solutions to the Lyapunov equations

$$(-PQ)W + W(-PQ)^T = -(PB)(PB)^T \quad (2.39)$$

and

$$(-PQ)^T M + M(-PQ) = -BB^T, \quad (2.40)$$

respectively. To bring the gradient system into balanced form we need a nonsingular coordinate transformation $x = S\bar{x}$. This coordinate transformation induces the symplectic transformation $q = S\bar{q}$, $\bar{p} = S^T p$, that transforms the Hamiltonian system (2.37) into a Hamiltonian system with as associated gradient system the transformed system (2.38), i.e., (2.38) in the new coordinates \bar{x} .

We assume throughout this section that the gradient system (2.38) is in balanced form, i.e.,

$$\Sigma := W = M = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{pmatrix},$$

with $\sigma_1 \geq \dots \geq \sigma_n > 0$. Then the Hamiltonian system (2.37) is a *pseudo balanced representation*.

Theorem 2.8.2. ([72]) *For the pseudo balanced representation $P = I$. \triangle*

Suppose that $\sigma_k \gg \sigma_{k+1}$ and partition the system correspondingly as follows:

$$x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

Consider the reduced order gradient system (2.38):

$$\begin{aligned} \dot{x}^1 &= -Q_{11}x^1 + B_1 u \\ y &= B_1^T x^1 \end{aligned} \quad (2.41)$$

and the corresponding reduced order Hamiltonian system (2.37):

$$\begin{aligned} \begin{pmatrix} \dot{q}^1 \\ \dot{p}^1 \end{pmatrix} &= \begin{pmatrix} 0 & I_k \\ -Q_{11} & 0 \end{pmatrix} \begin{pmatrix} q^1 \\ p^1 \end{pmatrix} + \begin{pmatrix} 0 \\ B_1 \end{pmatrix} u \\ y &= B_1^T q^1 \end{aligned} \quad (2.42)$$

Theorem 2.8.3. ([72]) *System (2.41) is balanced and asymptotically stable with $Q_{11} > 0$ and system (2.42) is a pseudo balanced minimal Hamiltonian system with (2.41) as associated gradient system. \triangle*

This means that the order of a Hamiltonian system can be reduced by considering the associated gradient system. The dimension of the state space of the Hamiltonian system is twice the dimension of the state space of the associated gradient system. Hence, the order of the Hamiltonian system is reduced based on the Hankel singular values of a lower order system. A drawback of this method is that there is no theoretical interpretation available. For example there is no interpretation in terms of energy functions of the Hamiltonian system.

The pseudo balancing method may also be applied to weakly damped mechanical systems by first leaving out the damping, and adding it again to the pseudo balanced form. Weakly damped systems are asymptotically stable, but by carrying out a limiting analysis for infinitely small damping it is easily seen that the original open-loop balancing method is not feasible in this case (see e.g. Jonckheere [41]). Furthermore, the results of open-loop balancing are very sensitive to the damping parameters, which are often not precisely known.

The most common approach for model reduction of a Hamiltonian system is based on *modal analysis*. One considers the system (2.37) without inputs and outputs. It is well-known (e.g. Meirovitch [49]) that there exists a canonical transformation $\hat{q} = Tq$, $\hat{p} = T^{-T}p$, such that

$$\begin{pmatrix} \dot{\hat{q}} \\ \dot{\hat{p}} \end{pmatrix} = \begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix} \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} \quad (2.43)$$

transforms into

$$\begin{pmatrix} \dot{\hat{q}} \\ \dot{\hat{p}} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\Omega & 0 \end{pmatrix} \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix}, \quad (2.44)$$

where $\Omega = \text{diag}(\omega_1^2, \dots, \omega_n^2)$. It is clear that the dynamics of (2.44) decomposes into n independent eigenmodes with eigenfrequencies $\omega_1, \dots, \omega_n$. Model reduction is now achieved by leaving out the eigenmodes corresponding to some of the eigenfrequencies, usually the higher ones. It immediately follows that model reduction by this method preserves the structure of the Hamiltonian system. However, since modal analysis is a priori only concerned with the 'drift term', and not with the input and output matrix, the method is not really satisfying from a system and control theoretic point of view. Comparing the pseudo balancing method with the method based on modal analysis, it can be seen that the system with a system matrix of the form (2.44) is pseudo balanced if and only if the rows of the B -matrix are all orthogonal, and thus B is invertible (e.g. Van der Schaft [70]). In general, model reduction of an Hamiltonian system based on pseudo balancing results in a reduced order Hamiltonian system from which the eigenmodes do not form a subset of the set of eigenmodes of the full order Hamiltonian system. This is in analogy with model reduction of stable lin-

ear systems based on balancing, which yields poles of the reduced order system that do not form a subset of the poles of the full order system.

2.9 Conclusions

In this chapter we treated several balancing techniques for linear systems. For a stable linear system the balancing method of Section 2.2 may be applied, and used as a tool for model reduction. This method is based on the past input and future output energy functions of the stable linear system.

The method of Section 2.2 does not apply to unstable linear systems. In that case, we may apply LQG balancing as treated in Section 2.5. This method is a form of closed-loop balancing. It may be interpreted by a different pair of past and future energy functions. An alternative approach is to apply balancing as in Section 2.2 to the normalized coprime representation of the unstable system (see Section 2.6). This last method has no direct interpretation in terms of a past and future energy function of the original system, but model reduction based on balancing the normalized coprime representation yields the same reduced order model as LQG balancing.

Another form of closed-loop balancing is given by \mathcal{H}_∞ balancing, as treated in Section 2.7. This method is based on the solution of the normalized \mathcal{H}_∞ control problem, and is a tool to reduce the order of the model and the controller. \mathcal{H}_∞ balancing also may be interpreted by a pair of past and future energy functions.

Finally, we treated pseudo balancing in Section 2.8. Pseudo balancing is an open-loop balancing method that is defined for linear conservative mechanical systems (Hamiltonian systems). It is also possible to apply balancing for unstable systems (see Section 2.5 and 2.6) to the Hamiltonian system, but model reduction based on such method in general does not preserve the structure of the mechanical system. Instead, for pseudo balancing the balancing method of Section 2.2 is applied to the associated gradient system, which is asymptotically stable. Model reduction based on this method preserves the structure of the system. No interpretation of pseudo balancing is yet available in terms of a past and future energy function of the original Hamiltonian system.

As far as LQG and \mathcal{H}_∞ balancing is concerned, we did not treat the properties of a reduced order controller applied to the full order model. For studying these properties we refer to the literature (e.g. Opdenacker and Jonckheere [63], Mustafa and Glover [55]). Furthermore, we did not treat the modifications of the different balancing techniques that have been proposed. For studying these modifications we also refer to the literature (e.g. Enns [17]).

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