Model Reduction

4th Lecture

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Contents of today

Main focus: Nonlinear state-space balancing.

- Reference to lecture 2.
- State space based extension of Hankel singular values and corresponding order reduction.
  Input-normal/output-diagonal form and balanced form.
- Use for model reduction.
- Relation with nonlinear Hankel operator?

Problem setting

Consider an input-output system Σ:

$$u \longrightarrow \boxed{\Sigma} \longrightarrow y$$

and a time-invariant asymptotically stable state space realization

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \qquad (*)$$

Assume that $(*)$ is a valid state space realization of Σ about x_0.

Problem setting *(continued)*

Questions:

- When is $(*)$ of minimal order?
- Is there a relation with the Hankel operator?
- Does there exist a good lower order approximation of $(*)$?

Related (partly unsolved) problems:

- What is “good” if we consider complexity, error bounds, (easier) controller design, performance measures, computation times, numerical stability, **etc.**?

2nd Lecture 12 September

- Linear systems as a paradigm
- The controllability and observability function
- Observability
- Strong accessibility
- Minimality
- **Not done yet:** Introduction to state space balanced realizations

Energy functions: Gramian extensions

Controllability function:

$$L_c(x_0) = \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0,\, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \| u(t) \|^2 \, dt$$

Observability function:

$$L_o(x_0) = \frac{1}{2}\int_{0}^{\infty} \| y(t) \|^2 \, dt, \qquad \begin{array}{c} x(0)=x_0 \\ u(\tau)=0,\ 0\le\tau<\infty \end{array}$$

Introduction to state space balanced realizations

Example:

$$f(x) = -\frac{1}{625}\begin{pmatrix} 625x_1 + 112x_1^3 + 552x_1^2x_2 + 639x_1x_2^2 + 216x_2^3 \\ 384x_1^3 + 625x_2 + 464x_1^2x_2 + 48x_1x_2^2 - 63x_2^3 \end{pmatrix},$$

$$g(x) = \begin{pmatrix} \frac{3\sqrt{2}}{5} & \frac{4\sqrt{2}}{25}\sqrt{25 + 7x_1^2 + 48x_1x_2 - 7x_2^2} \\ \frac{-4\sqrt{2}}{5} & \frac{3\sqrt{2}}{25}\sqrt{25 + 7x_1^2 + 48x_1x_2 - 7x_2^2} \end{pmatrix},$$

$$h(x) = \begin{pmatrix} \frac{2}{5}(3x_1 - 4x_2) \\ \frac{\sqrt{2}}{25}(4x_1 + 3x_2)\sqrt{25 + 16x_1^2 + 24x_1x_2 + 9x_2^2} \end{pmatrix}.$$

Introduction to state space balanced realizations

This system is zero-state observable, and f is asymptotically stable.
Solving the Hamilton-Jacobi equations we obtain:

$$L_c(x) = \frac{1}{2}x^T x, \quad L_o(x) = \frac{1}{2}x^T \begin{pmatrix} m_{11}(x) & m_{12}(x) \\ m_{21}(x) & m_{22}(x) \end{pmatrix} x,$$

$$\begin{aligned} m_{11}(x) &= \tfrac{2}{625}(425 + 72x_1^2 - 192x_1x_2 + 128x_2^2) \\ m_{12}(x) = m_{21}(x) &= \tfrac{12}{625}(-25 + 9x_1^2 - 24x_1x_2 + 16x_2^2) \\ m_{22}(x) &= \tfrac{1}{625}(1025 + 81x_1^2 - 216x_1x_2 + 144x_2^2). \end{aligned}$$

Introduction to state space balanced realizations

System has the **input-normal form**.

The eigenvalues of $M(x)$ are:

$$\lambda_1(x) = \tfrac{1}{25}(25 + 9x_1^2 - 24x_1x_2 + 16x_2^2) = 1 + (\tfrac{1}{5}(3x_1 - 4x_2))^2$$
$$\lambda_2(x) = 2.$$

The neighborhood V of 0 where the **number of distinct eigenvalues** is constant, is

$$V = \{x|(3x_1 - 4x_2)^2 < 25\},$$

i.e., $\lambda_1(x) < 2$ for $x \in V$, thus $\lambda_2(x) > \lambda_1(x)$.

Introduction to state space balanced realizations

The unitary matrix of eigenvectors is

$$T(x) = T = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}.$$

Thus, **coordinate transformation** to diagonalize $M(x)$ is

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \nu(x) = T^T x = \frac{1}{5}\begin{pmatrix} 3x_1 - 4x_2 \\ 4x_1 + 3x_2 \end{pmatrix}, \quad x \in V.$$

Consider coordinates $\psi(z) = \nu^{-1}(z)$ for $z \in W$, where

$$W = \psi^{-1}(V) = \nu(V) = \{z|z_1^2 < 1\}.$$

Introduction to state space balanced realizations

In new coordinates controllability and observability functions become for $z \in W$

$$\tilde{L}_c(z) = \frac{1}{2}z^T z, \qquad \tilde{L}_o(z) = \frac{1}{2}z^T \begin{pmatrix} 2 & 0 \\ 0 & 1 + z_1^2 \end{pmatrix} z.$$

The so-called **singular value functions** are $\tau_1(z) = 2$ and $\tau_2(z) = 1 + z_1^2$.

Coordinates $z \in W$ transform system into input-normal/output-diagonal form.

Introduction to state space balanced realizations

Input-normal/output-diagonal form:

$$\begin{cases} \dot{z}_1 = -z_1 + z_1 z_2^2 + u_1\sqrt{2} \\ \dot{z}_2 = -z_2 - z_2^3 + u_2\sqrt{2 - 2z_1^2 + 2z_2^2} \end{cases}$$

$$\begin{cases} y_1 = 2z_1 \\ y_2 = \sqrt{2}z_2 \end{cases}$$

Additional coordinate transformation to bring system in **balanced** form! **Only** on the axes!

Introduction to state space balanced realizations

Additional coordinate transformation $\bar{z} = \eta(z)$ as follows:

$$\begin{cases} \bar{z}_1 = 2^{\frac{1}{4}} z_1 \\ \bar{z}_2 = z_2 \end{cases}, \qquad \bar{z} \in \bar{W} = \nu(W) = \{\bar{z} | \bar{z}_1^2 < 2^{\frac{1}{2}}\}.$$

Now controllability and observability functions are in **balanced form**, i.e.,

$$\check{L}_c(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} 2^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix} \bar{z}, \; \check{L}_o(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} 2^{\frac{1}{2}} & 0 \\ 0 & 1 + 2^{-\frac{1}{2}}\bar{z}_1^2 \end{pmatrix} \bar{z}.$$

Introduction to state space balanced realizations

Then at coordinate axes:

$$\check{L}_c(\bar{z}_1, 0) = \frac{1}{2\sqrt{2}}\bar{z}_1^2, \qquad \check{L}_c(0, \bar{z}_2) = \frac{1}{2}\bar{z}_2^2,$$
$$\check{L}_o(\bar{z}_1, 0) = \frac{\sqrt{2}}{2}\bar{z}_1^2, \qquad \check{L}_o(0, \bar{z}_2) = \frac{1}{2}\bar{z}_2^2.$$

i.e., **less input energy** is needed to reach states $(\bar{z}_1, 0)$ than to reach the states $(0, \bar{z}_2)$, and states $(\bar{z}_1, 0)$ generate a **larger output energy** than the states $(0, \bar{z}_2)$.

Hence, on neighborhood $\bar{W}$ $\bar{z}_1$ is a more important state component than $\bar{z}_2$.

Introduction to state space balanced realizations

For **model reduction** set $\bar{z}_2 = 0$.
Reduced order system:

$$\begin{cases} \dot{\tilde{z}} = -\tilde{z} + 2^{\frac{3}{4}} u_1 \\ \tilde{y} = 2^{\frac{3}{4}} \tilde{z} \end{cases}, \qquad \tilde{z} \in \tilde{W} = \{\tilde{z} | \tilde{z} < 2^{\frac{1}{2}}\}.$$

Controllability and observability function

$$\tilde{L}_c(\tilde{z}) = \frac{1}{2\sqrt{2}}\tilde{z}^2 = 2^{-\frac{3}{2}}\tilde{z}^2, \quad \text{and} \quad \tilde{L}_o(\tilde{z}) = \frac{\sqrt{2}}{2}\tilde{z}^2 = 2^{-\frac{1}{2}}\tilde{z}^2, \qquad \tilde{z} \in \tilde{W},$$

Reduced system is asymptotically stable on $\tilde{W}$.

Overview

- **"Semi-quadratic" forms**
- Kato's result
- Input-normal/output-diagonal form
- Balanced form
- Model reduction
- Relation with minimality
- Hankel considerations

Semi-quadratic forms

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

Standing assumptions:

- $f(x)$ is as. stab. on some neighborhood Y of 0.
- System is zero-state observable on Y.
- L_o and L_c exist and are smooth on Y.
- $\dfrac{\partial^2 L_c}{\partial x^2}(0) > 0$ and $\dfrac{\partial^2 L_o}{\partial x^2}(0) > 0$.

Semi-quadratic forms *(continued)*

Version of the Fundamental Theorem of Integral Calculus:

Let L be smooth function in convex neighborhood V of 0 in $\mathbb{R}^n$, with $L(0) = 0$. Then

$$L(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i a_i(x_1, \ldots, x_n)$$

for some suitable smooth functions a_i defined on V, with $a_i(0) = \frac{\partial L}{\partial x_i}(0)$.

Semi-quadratic forms *(continued)*

Construction given by proof, i.e.,

$$\begin{aligned} L(x_1, \ldots, x_n) &= \int_0^1 \frac{\partial L(tx_1, \ldots, tx_n)}{\partial t} dt \\ &= \int_0^1 \sum_{i=1}^{n} \frac{\partial L}{\partial x_i}(tx_1, \ldots, tx_n) x_i dt \end{aligned}$$

Thus, $a_i(x_1, \ldots, x_n) = \int_0^1 \frac{\partial L}{\partial x_i}(tx_1, \ldots, tx_n) dt$.

Apply twice, then **"semi-quadratic"** form!

Next result uses Morse's lemma for **input-normal** form.

Semi-quadratic form *(continued)*

Lemma: There exists coordinate transf. $x = \phi(\bar{x})$, $\phi(0) = 0$ (defined on neighborhood of 0), such that in $\bar{x} = \phi^{-1}(x)$ function $L_c(x)$ is of the form

$$L_c(\phi(\bar{x})) = \frac{1}{2}\bar{x}^T \bar{x}.$$

In $\bar{x} = \phi^{-1}(x)$ we can write $L_o(x)$ in the form

$$L_o(\phi(\bar{x})) = \frac{1}{2}\bar{x}^T M(\bar{x})\bar{x} \quad where \quad M(0) = \frac{\partial^2 L_o}{\partial x^2}(0),$$

with $M(\bar{x})$ an $n \times n$ symmetric matrix such that its entries are smooth functions of $\bar{x}$.

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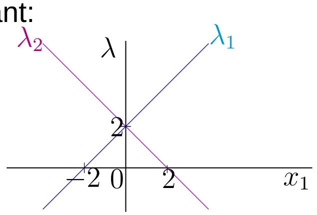
Kato's result

Lemma: If there exists a neighborhood V of 0 where the number of distinct eigenvalues of $M(\bar{x})$ is constant for $\bar{x} \in V$, then on V the eigenvalues $\lambda_i(\bar{x})$, $i = 1, \ldots, n$, are smooth functions of $\bar{x}$, as well as the associated normalized eigenvectors.

Ex.: # distinct eig. val. is not constant:

$$M(\bar{x}) = \begin{pmatrix} 2 + x_1 & 0 \\ 0 & 2 - x_1 \end{pmatrix}$$



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Input-normal/output-diagonal form

Theorem: Assume condition of Kato's result is fulfilled. On neighborhood U of 0 there exists ψ, $x = \psi(z)$, $\psi(0) = 0$, s.t. for $z \in W := \psi^{-1}(U)$ L_c and L_o are of form

$$\tilde{L}_c(z) := L_c(\psi(z)) = \frac{1}{2} z^T z,$$

$$\tilde{L}_o(z) := L_o(\psi(z)) = \frac{1}{2} z^T \begin{pmatrix} \tau_1(z) & & 0 \\ & \ddots & \\ 0 & & \tau_n(z) \end{pmatrix} z.$$

Here $\tau_1(z) \geq \cdots \geq \tau_n(z)$ are smooth functions of z, called the **singular value functions** of system.

Input-normal/output-diagonal form *(continued)*

Proof (constructive, i.e., algorithm for diagonalization): From previous lemma there exists $x = \phi(\bar{x})$, $\phi(0) = 0$ such that in $\bar{x}$ L_c and L_o are of input-normal/"semi-quadratic form. By Kato's result, on V eigenvalues of $M(\bar{x})$ and normalized eigenvectors are smooth functions of $\bar{x}$. Thus $M(\bar{x})$ is smoothly diagonalizable on V.

Since $M(\bar{x})$ can be chosen symmetric, we can write $M(\bar{x}) = T(\bar{x})\Lambda(\bar{x})T^T(\bar{x})$ where

$\Lambda(\bar{x}) = \mathrm{diag}(\lambda_1(\bar{x}), \dots, \lambda_n(\bar{x}))$, with $\lambda_1(\bar{x}) \leq \cdots \leq \lambda_n(\bar{x})$.

Input-normal/output-diagonal form *(continued)*

$\lambda_i(\bar{x}), i = 1, \dots, n$, are eig. val. of $M(\bar{x})$ and $T(\bar{x})$ is corresponding orthogonal matrix of normalized eigenvectors, i.e. $T^T(\bar{x})T(\bar{x}) = I$, $\bar{x} \in V$.
Thus

$$L_o(\phi(\bar{x})) = \tfrac{1}{2}\bar{x}^T T(\bar{x})\Lambda(\bar{x})T^T(\bar{x})\bar{x}, \quad \bar{x} \in V.$$

and

$$L_c(\phi(\bar{x})) = \frac{1}{2}\bar{x}^T\bar{x}$$

Input-normal/output-diagonal form *(continued)*

Define $z = \nu(\bar{x}) := T^T(\bar{x})\bar{x}$. Then

$$L_o(\phi(\nu^{-1}(z))) = \tfrac{1}{2}z^T\Lambda(\nu^{-1}(z))z, \quad z \in W := \nu(V).$$

$$L_c(\phi(\nu^{-1}(z))) = \frac{1}{2}z^T T^T(\bar{x})T(\bar{x})z = \frac{1}{2}z^T z.$$

Define $\tau_i(z) := \lambda_i(\nu^{-1}(z))$, $i = 1, \dots, n$,

$$\psi := \phi \circ \nu^{-1},$$

and $U := \phi^{-1}(V)$, then proof is completed.

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Balanced form

- Recall for linear systems in balanced form: $\Sigma = P = Q$.
- Input-normal/output-diagonal means $P = I$, $Q = \Sigma^2$.

Hence, nonlinear system is not really in balanced form yet!

To make model reduction stage easier, **no** aim for full balance, but only for **balance on the coordinate axes**, i.e., aim for extension of

$$L_c(0,\dots,x_i,\dots,0) = \frac{1}{2}x_i^2\sigma_i^{-1}$$
$$L_o(0,\dots,x_i,\dots,0) = \frac{1}{2}x_i^2\sigma_i.$$

Balanced form *(continued)*

- Take $\bar{z}_i = \eta_i(z_i) := \tau_i(0,\dots,0,z_i,0,\dots 0)^{\frac{1}{4}} z_i$, $i = 1,\dots,n$. Since $\tilde{L}_o(z) > 0$, $\tau_i(0,\dots,0,z_i,0,\dots,0) > 0$, for $z_i \neq 0$.
- Define $\check{L}_c(\bar{z}) := \tilde{L}_c(\eta^{-1}(\bar{z}))$, $\check{L}_o(\bar{z}) := \tilde{L}_o(\eta^{-1}(\bar{z}))$. Then

$$\check{L}_c(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} \sigma_1(\bar{z}_1)^{-1} & & 0 \\ & \ddots & \\ 0 & & \sigma_n(\bar{z}_n)^{-1} \end{pmatrix} \bar{z},$$

$$\check{L}_o(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} \sigma_1(\bar{z}_1)^{-1}\tau_1(\eta^{-1}(\bar{z})) & & 0 \\ & \ddots & \\ 0 & & \sigma_n(\bar{z}_n)^{-1}\tau_n(\eta^{-1}(\bar{z})) \end{pmatrix} \bar{z},$$

with $\sigma_i(\bar{z}_i) = \tau_i(0,\dots,0,\eta_i^{-1}(\bar{z}_i),0,\dots,0)^{\frac{1}{2}}$, $i = 1,\dots,n$.

Balanced form *(continued)*

In $\bar{z}$ coordinates

$$\check{L}_c(0,\dots,0,\bar{z}_i,0,\dots,0) = \frac{1}{2}\bar{z}_i^2\sigma_i(\bar{z}_i)^{-1}$$
$$\check{L}_o(0,\dots,0,\bar{z}_i,0,\dots,0) = \frac{1}{2}\bar{z}_i^2\sigma_i(\bar{z}_i)$$

Linearization of the complete nonlinear procedure results in the linear balancing procedure.

Linearized system:

$$\dot{\hat{z}} = \bar{A}\hat{z} + \bar{B}u$$
$$y = \bar{C}\hat{z}$$

where $\bar{A} = \frac{\partial \bar{f}}{\partial \bar{z}}(0)$, $\bar{B} = \bar{g}(0)$ and $\bar{C} = \frac{\partial \bar{h}}{\partial \bar{z}}(0)$.

Balanced form *(continued)*

Based on the linearization, we have the following result:

Theorem: Assume linearized system is **minimal** and A is **asymptotically stable**. If Hankel singular values, σ_i, $i = 1,\dots,n$, of linear system satisfy $\sigma_i \neq \sigma_j$ for $i \neq j$, $i,j = 1,\dots,n$, then locally the nonlinear system may be brought into balanced form.

Uses the fact that $\sigma_i \neq \sigma_j$ implies that there exists neighborhood V of 0 such that # of distinct eigenvalues of the matrix $M(\bar{x})$ is constant and equal to n (Kato's result).

Overview

- "Semi-quadratic" forms
- Kato's result
- Input-normal/output-diagonal form
- Balanced form
- **Model reduction**
- Relation with minimality
- Hankel considerations

Model reduction

Assume that

$$\tau_k(z) > \tau_{k+1}(z) \qquad \left(\Rightarrow \sigma(\bar{z}_k)^{-1}\tau_k(\eta^{-1}(\bar{z})) > \sigma(\bar{z}_{k+1})^{-1}\tau_{k+1}(\eta^{-1}(\bar{z}))\right).$$

and that nonlinear system is in balanced form.

Partition system as follows:

$$\bar{f}(\bar{z}) = \begin{pmatrix} \bar{f}_a(\bar{z}^a, \bar{z}^b) \\ \bar{f}_b(\bar{z}^a, \bar{z}^b) \end{pmatrix}, \quad \bar{g}(\bar{z}) = \begin{pmatrix} \bar{g}_a(\bar{z}^a, \bar{z}^b) \\ \bar{g}_b(\bar{z}^a, \bar{z}^b) \end{pmatrix}, \quad \bar{h}(\bar{z}) = \bar{h}(\bar{z}^a, \bar{z}^b)$$

where $\bar{z}^a = (\bar{z}_1, \ldots, \bar{z}_k)$ and $\bar{z}^b = (\bar{z}_{k+1}, \ldots, \bar{z}_n)$.

Model reduction *(continued)*

Truncation, then $\bar{z}^b = 0$.

$$\frac{\partial \check{L}_o}{\partial \bar{z}^a}(\bar{z}^a, 0)\bar{f}_a(\bar{z}^a, 0) + \frac{\partial \check{L}_o}{\partial \bar{z}^b}(\bar{z}^a, 0)\bar{f}_b(\bar{z}^a, 0) + \frac{1}{2}\bar{h}^T(\bar{z}^a, 0)\bar{h}(\bar{z}^a, 0) = 0$$

$$\frac{\partial \check{L}_c}{\partial \bar{z}^a}(\bar{z}^a, 0)\bar{f}_a(\bar{z}^a, 0) + \frac{1}{2}\frac{\partial \check{L}_c}{\partial \bar{z}^a}(\bar{z}^a, 0)\bar{g}_a(\bar{z}^a, 0)\bar{g}_a^T(\bar{z}^a, 0)\frac{\partial^T \check{L}_c}{\partial \bar{z}^a}(\bar{z}^a, 0) = 0$$

From the Hamilton-Jacobi equations if follows that the **controllability** function of the reduced order system is equal to $\check{L}_c(\bar{z}^a, 0)$. For observability new condition necessary.

Model reduction *(continued)*

Theorem: If $\frac{\partial \check{L}_0}{\partial \bar{z}^b}(\bar{z}^a, 0)\bar{f}_b(\bar{z}^a, 0) = 0$ for $(\bar{z}^a, 0) \in \bar{W}$ then the **observability** function of the reduced order system is given by $\check{L}_o(\bar{z}^a, 0)$. Furthermore, the reduced order system is in balanced form having singular value functions $\tau_1(z^a, 0) \geq \cdots \geq \tau_k(z^a, 0)$, for $(z^a, 0) = \eta^{-1}(\bar{z}^a, 0)$.

From linearization arguments we know that

Theorem: The subsystems $\left(\bar{f}_a(\bar{z}^a, 0), \bar{g}_a(\bar{z}^a, 0), \bar{h}(\bar{z}^a, 0)\right)$ and $\left(\bar{f}_b(0, \bar{z}^b), \bar{g}_b(0, \bar{z}^b), \bar{h}(0, \bar{z}^b)\right)$, respectively, are both **locally asymptotically stable**.

Model reduction *(continued)*

Theorem: If $\bar{f}_b(\bar{z}^a, 0) = 0$ and $\check{L}_o$ is proper (for each $c > 0$ the set $\{x \in M | 0 \leq \check{L}_o(x) \leq c\}$ is compact) on $\bar{W}$, then the reduced system is asymptotically stable on $(\bar{z}^a, 0) \in \bar{W}$.

Proof: Full order system is zero-state observable. Thus

$$\bar{h}(\bar{z}^a(t), 0) = 0,\ \forall t \geq 0 \Rightarrow (\bar{z}^a(t), 0) = (0, 0),\ \forall t \geq 0.$$

Since $\bar{f}_b(\bar{z}^a, 0) = 0$, reduced system is zero-state observable. Hence with

$\tilde{L}_o(\tilde{z}) = \frac{1}{2}\tilde{z}^T \mathrm{diag}\left(\sigma_1(\tilde{z}_1)^{-1}\tau_1(\eta^{-1}(\tilde{z}, 0)), \ldots, \sigma_k(\tilde{z}_k)^{-1}\tau_k(\eta^{-1}(\tilde{z}, 0))\right)\tilde{z}.$

$\tilde{L}_o(\tilde{z}) > 0$ for $\tilde{z} \in \tilde{W}, \tilde{z} \neq 0$. Thus, by previous lecture, reduced order system as. stable.

Overview

- "Semi-quadratic" forms
- Kato's result
- Input-normal/output-diagonal form
- Balanced form
- Model reduction
- **Relation with minimality**
- Hankel considerations

Relation with minimality

$L_o(x^1, 0, 0, 0)$ and $L_c(x^1, 0, 0, 0)$ can be transformed to input-normal/output-diagonal form, i.e., there exists $x^1 = \psi(z)$, $\psi(0) = 0$, $(\psi^{-1}(x^1), 0, 0, 0) \in Y$, such that

$$L_c(\psi(z), 0, 0, 0) = \frac{1}{2}z^T z$$

$$L_o(\psi(z), 0, 0, 0) = \frac{1}{2}z^T \begin{pmatrix} \tau_1(z) & & 0 \\ & \ddots & \\ 0 & & \tau_{n_1}(z) \end{pmatrix} z.$$

Thus x^1-part can be **balanced**, with singular value functions $\tau_1(z) \geq \cdots \geq \tau_{n_1}(z)$.

Relation with minimality *(continued)*

Also consider x^2, then there exists $(z^1, z^2) = \phi^{-1}(x^1, x^2)$ s.t.

$$L_c(\phi(z^1, z^2), 0, 0) = \frac{1}{2}z^{1T}z^1 + \frac{1}{2}z^{2T}z^2$$

$$L_o(\phi(z^1, z^2), 0, 0) = \frac{1}{2}\begin{pmatrix} z^{1T} & z^{2T} \end{pmatrix} M(z^1, z^2) \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}.$$

If Kato's condition is fulfilled, we may diagonalize $M(z^1, z^2)$.

Then functions on the diagonal are $\bar{\tau}_1(z^1, z^2) \geq \cdots \geq \bar{\tau}_{n_1+n_2}(z^1, z^2)$, where $\bar{\tau}_i(z^1, 0) = \tau_i(z)$, $i = 1, \ldots, n_1$, and $\bar{\tau}_j(0, x^2) = 0$, $j = n_1 + 1, \ldots, n_1 + n_2$.

Relation with minimality *(continued)*

In accordance with the linear case, where **unobservable** part yields **zero** 'Hankel singular values'.

Note that it is **not** possible to transform the whole system in input-normal/output-diagonal form, since $L_c(0, 0, x^3, x^4)$ is infinite.

Still in accordance with the linear theory, since here we are dealing with the **'inverse of the controllability Gramian'**.

Hence the part of the system that is not strongly accessible yields an '**inverse of the controllability Gramian**' that is **infinite**, and thus a '**controllability Gramian**' that is **zero**.

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- **Hankel considerations**

Hankel considerations

- Note that this notion of balanced form for nonlinear systems is not unique, due to non-uniqueness in semi-quadratic form!
- Restricted form of balancing, more related to Hankel operator and similarity invariants are obtained.
- In case semi-quadratic form is given, then unique up to orthogonal transformation related to multiplicity of singular value function.
- Other possibility is "block" diagonalization based on ordering of eigenvalues of matrix $M(\bar{x})$.

Hankel considerations *(continued)*

- Hankel norm for **linear** systems

$$\begin{aligned}\|\Sigma\|_H^2 &= \max_{u \in L_{2+}} \frac{\|\mathcal{H}(u)\|^2}{\|u\|^2} = \max_{u \in L_{2+}} \frac{\langle u, \mathcal{H}^*\mathcal{H}(u)\rangle}{\langle u, u\rangle} \\ &= \max_x \frac{x^T Q\, x}{x^T P^{-1} x} = \lambda_{\max}(\mathcal{H}^*\mathcal{H}) = \lambda_{\max}(PQ) = \sigma_1^2\end{aligned}$$

- Hankel norm for **nonlinear** systems

$$\begin{aligned}\|\Sigma\|_H^2 &= \max_{u \in L_{2+}} \frac{\|\mathcal{H}(u)\|^2}{\|u\|^2} = \max_{u \in L_{2+}} \frac{\langle u, \mathcal{H}^*(\mathcal{H}(u), u)\rangle}{\langle u, u\rangle} \\ &= \max_x \frac{L_o(x)}{L_c(x)} = \textbf{???}\end{aligned}$$

Hankel considerations *(continued)*

How to determine **???**

- For relation with Hankel operator and Hankel norm, balanced state-space form does not suffice.
- By considering both eigenstructure of
  - ⋆ **differential adjoint** $(d\mathcal{H}(\cdot))^* \, (\mathcal{H}(\cdot))$ and
  - ⋆ **full nonlinear Hilbert adjoint** $\mathcal{H}^*(\mathcal{H}(u), u)$,

  characterization based on sort of **parametrization** that is related to the input value yields form that fill in the **???**, i.e., give explicit expression for Hankel norm.

Hankel considerations *(continued)*

- Appropriate assumptions, then there exists $x = \Phi(z)$ s.t.

$$\begin{aligned} L_c(\Phi(z)) &= \frac{1}{2} z^T z \\ L_o(\Phi(z)) &= \frac{1}{2} z^T \mathrm{diag}(\tau_1(z), \ldots, \tau_n(z)) z \end{aligned}$$

$$\frac{\partial L_c(\Phi(z))}{\partial z_i} = 0 \Longleftrightarrow \frac{\partial L_o(\Phi(z))}{\partial z_i} = 0$$

$$\tau_i(0, \ldots, 0, z_i, 0, \ldots 0) = \rho_i^2(z_i), \qquad \|\Sigma\|_H = \max_{z_1} \{\rho_1(z_1)\}$$

Now **unique** and balanced structure preserving model reduction tool on coordinate axes!

Remarks

- Notes and slides are on the web (from Wednesday on): **http://www.dcsc.tudelft.nl/~jscherpen/disccursus.html**
- State-space balancing can be found in Chapter 6.
- Extensive treatment of Hankel considerations and “unique” balancing tool can be found in Chapter 5 (adjoint systems and operators) and 7 (balanced form).
- Please recall that I prefer to receive the exercises by mid-October. Let me know if that is a problem!